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VOLUME 1

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.
1904

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.

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NOTICE CONCERNING THE SECOND SERIES OF "PROCEEDINGS."

THE following statement made by the President, Prof. H. Lamb, at the meeting of the Society, held on June 11th, 1903, is reported in *Proceedings*, Vol. xxxv., pp. 460, 461:—

> "The first four volumes of *Proceedings* contain the papers of eight and a half years (January, 1865 – June, 1873). Vols. v.–xxix. correspond with the Sessions 1873–1898, one with each Session. Vols. xxx.–xxxv. are edited in accordance with the rule that the 'Volumes shall contain as nearly four hundred pages as may be found convenient, provided that each volume shall begin with the Report of Proceedings at a meeting, not necessarily an Annual General Meeting.' These volumes contain the records of proceedings at meetings, followed by the papers read at the meetings, and they also contain Appendices in which are Notes and Corrections and Obituary Notices of deceased members. The Council has decided that in future the records of proceedings at meetings and matter of the kind previously placed in Appendices shall be collected at the beginnings of volumes and shall have a different pagination from that of the papers, and that the records of proceedings at meetings shall be issued for a Session at a time in the earliest completed volume after the end of the Session. . . . The volumes are to contain as nearly five hundred pages as may be found convenient."

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OBITUARY NOTICES

LIFE AND WORKS OF L. CREMONA

[This notice has been written by Prof. E. Bertini, of Pisa, through the kind offices of Prof. U. Dini. The translation has been made by Mr. C. Leudesdorf. The Council is greatly indebted to all these gentlemen for the interest which they have taken in the matter.]

A GREAT mathematician, a man eminent in mind and in character, Luigi Cremona, an honorary member of the Society since 1871, died in Rome on June 10th, 1903, of heart disease, which for more than a year had affected his health and endangered his life. To him and to other distinguished mathematicians—Battaglini, Beltrami, Betti, Brioschi, Casorati—now unhappily all gone, Italy owes the revival, in the second half of the past century, of the study of the higher mathematics, and the noteworthy advances therein which took place amid the excitement and the struggles of the national revival. But to Cremona specially belongs the merit of the new life infused into the study of pure geometry in Italy, and of the vigorous impulse which it there received in the path opened by Poncelet, Möbius, Steiner, Chasles, Plücker, Staudt, &c. The Italian geometers who followed in this path, even if not pupils of Cremona, and even at a time when his scientific activity had ceased, considered him their master, feeling that they owed their inspiration to him, and that their activity had its source in him, in his noble teaching, in his immortal works.

Luigi Cremona was born on December 7th, 1830, at Pavia. He was the eldest of four sons (one of whom, Tranquillo, became a painter of note) of Gaudenzio Cremona, an official in the communal service, by his second wife, Teresa Andreoli. He pursued his secondary and University studies in his native town, interrupting them in 1848 to take part as a volunteer in the War of Independence. His life as a soldier lasted eighteen months, and he was present at the chief battles in the heroic defence of Venetia, distinguishing himself both by his devotion to duty and by his bravery. On the completion of his University studies, in which Bordoni and Brioschi were his masters and Beltrami and Casorati his fellow-students, he gained, in 1853, his diploma as civil engineer. At Pavia he remained till 1855, first as a private tutor of mathematics; afterwards, on becoming qualified as secondary teacher, as instructor in the Liceo; and there he wrote his first two papers (*Annali delle Sc. fis. e mat.*, VI., VII.)*, suggested respectively by studies of his two masters. He was next appointed to a post at the *Ginnasio liceale* of Cremona, which he held for more than two years; and there, continuing with indefatigable zeal the geometrical studies already begun, he published his earliest researches in the subject, and laid the foundations of his future career.* In 1859 he became a teacher in the Liceo Beccaria at Milan; and in 1860 he was appointed to the Professorship of Higher Geometry in the University of Bologna, afterwards undertaking there the teaching of descriptive geometry also. In 1866 he transferred himself to Milan, where he taught graphical statics in the Polytechnic School, and higher geometry in the Normal School, which Brioschi had the intention of making a part of the Institute itself. In 1873 he received an invitation to go to Rome as Director of the School of Engineers, which he accepted. He reorganised the school, and brought it to a high state of perfection, himself undertaking the teaching of higher geometry in the Faculty of Science; and these offices he held for thirty years, until the end of his life.

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* In the references, Roman numerals refer to the volume, Arabic numerals to the series.

It is difficult to condense into a brief statement an adequate account of Cremona's great scientific work. The first steps which he took in modern geometry did not proceed from any impulse derived from University lectures; for such teaching, received with honour and success in France and Germany, did not at that time exist in the Italian Universities, although these possessed men of brilliant achievement in other subjects. They had their origin in the work of a distinguished analyst, Brioschi, who, tracing the rapid course of mathematics in its various developments, directed the attention of some of his best pupils to it, giving them books, help, and advice. To this we have the testimony of Cremona himself, who, in his fine "Introduction to the Course of Higher Geometry in the University of Bologna" (*Politechnico*, x.), publicly expresses his gratitude to Brioschi.

The chief works which showed Cremona's command of geometry, and marked him out for nomination to one of the Professorships of Higher Geometry established in Italy in 1860 (one of which—at Naples—had been worthily filled by the appointment of Battaglini), are those on twisted cubics (*Annali di Matem.*, I. 1, II. 1), and those on quadrics inscribed in a developable of the fourth class and on the conics inscribed in a developable of the fourth degree and third class (*ibid.*). In the former of these a variety of properties, metrical and descriptive, of the curve and its null-system are investigated by means of a parametric representation, which, however, Cremona at a later date discovered to be due to Möbius. In the latter are to be found a great number of metrical theorems, especially concerning the species of the quadrics and conics in question.

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* Cremona married, in 1854, Elisa Ferrari, the descendant of a family of patriots. By her he had three children: Elena, Countess Perozzi; the engineer Vittorio; and Itala, Signora Cozzolino. The death, in 1882, of his wife, an excellent woman and a good mother, was a source of great grief to Cremona, who wrote to his friends that to her, who had ensured to him peace and tranquillity of mind, taking on herself all the cares of the house and the education of the children, he owed his studies and his career. His second wife, whom he married in 1887, was Anna Maner-Müller, a lady of culture and distinction, who was his tender and affectionate companion to the last. May the thought of Cremona's virtues and the scientific eminence which he attained be some solace to his widow and his children in their affliction!

In these papers, as also in those which immediately followed them—for example, those on associated conics and quadrics, and on confocal quadrics (*Annali di Matem.*, III. 1)—the proofs are based on analytical geometry, which Cremona applied with great ability, availing himself of the latest improvements which had been introduced into its methods, making skilful use, for example, of tangential coordinates. The many solutions of questions, too, published during this period (1857–1860) in the *Nouvelles Annales de Mathématiques*, some of which amount really to the statement of new theorems (see, *e.g.*, Vol. XIX.), are generally obtained by the application to geometry of algebraic analysis.

Cremona had indeed "opened his eyes to the sun" of pure geometry, as in fact he himself declared,* and he had already, in his early works, shown the power, the elegance, and the accuracy of his mind; but he had not yet found his real sphere of action, nor had he embraced in its entirety the field of geometric investigation of his time. In those early works his researches are in the main inspired by Chasles, many of whose results they demonstrated, completed, or extended. The prevailing influence of the French school, and in particular of the distinguished mathematician just mentioned, on the early scientific work of Cremona, even if not evidenced by his own statements, would be clear from many signs. Very characteristic, for example, is his review of Staudt's *Geometrie der Lage* (*Annali di Matem.*, I. 1), in which, while praising that fine work, he deplores in incisive language the severance of descriptive from metrical properties,† and calls to mind that Chasles had found fault with the celebrated school of Monge for the same thing, though in their case it had not been carried to the same extreme point.

However, thanks to his indomitable energy and application, Cremona very soon completely mastered the methods and the researches of the

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* Cf. the commemorative notice by Veronese in Vol. XII. of the *Rendiconti della R. Accademia dei Lincei*.

† Others also made this remark at the time concerning Staudt; but at a later period the great importance of his work was recognised.

German school also, and gained command over the whole movement which was leading to the formation of a new doctrine of geometry. Contact with the work of the great foreign geometers quickly determined the bent of his mind towards the methods of pure geometry—the subject for which nature had formed him.* This may already be perceived in the "Considerazioni di Storia della Geometria" (*Politechnico*, IX.), written on the occasion of the publication of Novi's translation of Amiot's treatise. Cremona takes this opportunity to lay before his readers a remarkable historical sketch in which, among other things, an account is given of the Neapolitan geometers and of Bellavitis' method of equipollences, with its interesting applications. More clearly is it seen in the "Introduction" already quoted, in which the principal achievements of geometry in regard to the projective relations of forms of the first, second, and third kind, and of the forms thence derived, are set out in fair show, by one animated by a keen love of science, which causes him to say: "Happy shall I be if the power is given to me to inspire you young men with this thirst for knowledge, without which nothing good or great can be accomplished."†

From 1861 onwards we notice clearly a new manner in Cremona's work, both in the style of proof and in the interpretation of result. He adopted, and made full use of, those pure geometric methods entirely independent of algebra or analysis, which, through the labours of distinguished mathematicians—Poncelet, Chasles, Steiner, Staudt—were already common property; but he gave to them an impress of his own which gradually developed itself and attained greater and greater perfection in his successive works—an impress which was characterized by a singular facility and simplicity of proof, by an excellent and perfectly lucid style, and by an exquisite artistic feeling. Treating in a fundamental and in a beautiful way the quartic of the second species (*Annali di Matem.*, IV. 1), he proved the properties of it discovered by Cayley, Salmon, and Steiner, and other properties discovered by himself, as a linear construction of the curve, generalized later for a class of curves, in another paper (*ibid.*). He investigated the fundamental properties of skew cubic surfaces (*Rend. Ist. Lomb.*, II.), and afterwards (in 1862) illustrated the case observed by Cayley (*Crelle*, LX.). He enunciated several elegant properties of developables of the fifth order (also in 1862, *Comptes Rendus*, LIV.), and ended by giving a remarkable example of a null-system of a superior order, the first discovered (cf. R. Sturm, *Linien-Geometrie*, Vol. I., p. 78). On the subject of twisted cubics, with regard to which he made several researches from 1861 to 1863 (*Crelle*, LVIII., LX., LXIII., LXVIII.; *Nouvelles Annales de Math.*, I. 2; *Giornale di Matem.*, I. 1; II. 1), he developed a large number of theorems both metrical and descriptive, some of which he had already treated analytically, respecting "conjoint" points and planes, the nature of the conics inscribed in the osculating developable and of the quadrics passing through the cubic, the determination of the cubic by points and chords, &c.

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* In 1872 Cremona, in the fine and vigorous preface to his *Elementi di Geometria projettiva*, wrote: "I have laid greater stress on descriptive properties than on metrical ones; and have followed rather the methods of the *Geometrie der Lage* of Staudt than those of the *Géométrie supérieure* of Chasles. . . ."

† The programme of studies developed in this "Introduction" deals with descriptive properties, but Cremona states that he has so arranged it because these can easily be enunciated without having recourse to algebraic symbols; and he adds that, in the lectures to follow, he will pay even greater attention to metrical relations, and in support quotes certain words of Chasles.

But the work in which Cremona displayed most brilliantly his great intellectual powers and his artistic temperament, and his wide and profound knowledge of geometry, is the "Introduzione ad una teoria geometrica delle curve piane," published in 1862 (*Mem. Acc. Bologna*, XII. 1), afterwards translated into German by Curtze and into Czech by Weyr. In 1852 had appeared the first edition of the celebrated book by Salmon setting forth the theory of plane curves, treated by methods very various in character, but in the main analytical and connected with the theory of invariants of algebraic forms. The object which Cremona proposed to himself in his treatise, on the other hand, was to prove all the known results (in particular the very important ones which had merely been enunciated by Steiner) and new ones in addition, making use of pure geometry only. It may perhaps be said that the title of pure geometry in this connexion is not an absolutely correct one; for the geometrical method followed by him, though it proceeds synthetically in every case, nevertheless postulates certain fundamental algebraic theorems, whereas the name of pure geometry is usually given only to a geometric structure which is entirely independent of algebra (Staudt, Kötter, De Paolis, &c.). The book in question, the slight defects in which were to some extent amended in a paper subsequently published (*Annali di Matem.*, VI. 1), enriched the theory of polar curves with new properties, and from it was derived the greater part of what was then known of the projective geometry of plane curves, and in particular of those of the third degree. It was a work of fundamental importance which admirably furthered the end which its author had at heart—that, namely, of spreading in Italy the love for speculations of rational geometry.

The principles and methods of the "Introduzione" found excellent application in numerous and remarkable papers published within a few years (1862–65). Among these, besides those already mentioned and others of less importance, should be noted the two elegant memoirs (*Crelle*, LXIII.) on Steiner's surface and on the three-cusped hypocycloid; in which the properties discovered by Steiner, Kummer, Weierstrass, and Schröter are in a masterly manner coordinated in simple and harmonious treatment, and additional properties proved, such as the existence of a quadric passing through the four conics of contact of the singular tangent planes of Steiner's surface, and the theorems relative to certain polygons circumscribed about the hypocycloid.

The works, however, of this period which take highest rank by reason of their importance in the progress of modern geometry are the two notes on the geometrical transformation of plane figures. The first note (*Mem. Accad. di Bologna*, II. 2; 1863) gives, in addition to the general theory of such transformations, the two fundamental equations which must be satisfied by the number of points of equal multiplicity common to the curves in the one plane which correspond to straight lines in the other, and the actual construction, by means of twisted curves, of that special class of transformations of this kind known as those of De Jonquières or "isological." It is right to observe that De Jonquières had already, in 1859, alluded to these (without giving any proof) in a letter (*Comptes Rendus*, XLIX.) introductory to a memoir of his on twisted curves; which letter remained unpublished till 1864, when it was given in abstract in the *Nouvelles Annales de Math.* (III. 2), and in full in the *Giornale de Matem.* (XXIII. 1). In this letter he states that the transformations in question, interesting in themselves, become more so in their application to curves in space. We may presume that this allusion escaped Cremona's notice, and we may do so the more confidently because it was to the determination of these special transformations that no inconsiderable part of his note was devoted. Similarly, Cremona must have failed to notice the remark of Magnus, in the preface to the *Sammlung von Aufgaben und Lehrsätze aus der analytischen Geometrie* (Berlin, 1833), as to the existence of a transformation in which to straight lines correspond quartics with three double points and three simple ones—a case perhaps adduced, however, as an example of a transformation obtained by the composition or repetition of several quadric transformations.[*]

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[*] Cremona also, in the introduction to the Note in question, observes that the composition of several quadric transformations gives rise to transformations of an order higher than the second. The converse theorem—that every such transformation is the result of the composition of quadric transformations—was, at a later date, discovered almost simultaneously by Clifford, Nöther, and Rosanes (cf. Clebsch's works quoted below), and, in consequence of an objection made by Segre, proved rigorously by Castelnuovo.

Magnus, again, in Vol. VIII. of *Crelle*, went no further (as Cremona observes) than the consideration of the homographic and quadric transformations simply as one-to-one transformations between two planes; nor did Schiaparelli. But, in any case, what has been said does not deprive Cremona of the great merit of having been the first to realize the full importance of the problem of these transformations, and, above all, to give the perfectly general solution of it. The second Note (*Accad. di Bologna*, v. 2; 1864), in which De Jonquières and Magnus are quoted, is an addition of great value to the first, containing as it does the investigation of the Jacobian of a unicursal system and the theorem of the equality of the numbers of conjugate solutions, discovered by an inductive method which is not quite complete, but proved afterwards by Clebsch (*Math. Annalen*, IV.) and others.* With good reason are these transformations called "Cremonian transformations," and the geometry of the properties which remain unaltered by them "Cremonian geometry." The study of them supplied new methods of research in connexion with such subjects as involutions in a plane, the singularities of plane curves, &c.

In 1866–7 Cremona, extending the principles of the "Introduzione," published the "Preliminari ad una Teoria geometrica delle superficie" (*Accad. di Bologna*, VI. 2, VII. 2). About this time he presented to the Academy of Berlin the Memoir on cubic surfaces, afterwards published in *Crelle* (Vol. LXVIII.), which was rewarded by that Academy with one-half of the prize founded by Steiner.† These works, combined subsequently in the German translation by Curtze (1869), give a systematic and synthetic exposition of the theory of surfaces, with special application to those of the third degree, based on the properties of polar surfaces. Possibly, in the establishment of the first principles and the proof of the fundamental propositions—this criticism applies rather to the "Preliminari" than to the "Introduzione"—a somewhat more exact treatment might be desired; but the able arrangement of the matter, the simplicity and ingenuity of the proofs, and the admirable style, will always make these two treatises attractive and useful to students of geometry. In the Memoir on cubic surfaces, their representation on a plane is arrived at by help of a birational transformation of space, and is applied to the study of curves on the surface. This point was reached simultaneously (as a comparison of dates readily shows) by Cremona and Clebsch; the latter, however, deduced the representation in question from the formulæ which are the expression of Grassmann's generation of the surface by three projective pencils. The representation on a plane of Steiner's surface, and of skew cubic surfaces (*Rend. Ist. Lomb.*, IV. 1) provided Cremona, moreover, with simple methods by which he proved, in the case of the former surface, the theorem (discovered analytically by Clebsch) that its asymptotic lines are general twisted quartics of genus zero, and in the case of the latter surfaces an analogous theorem, according to which, however, the asymptotic lines are a particular class of the said quartics, viz., those with two inflexional tangents. This particular class, noticed first by Cayley, was afterwards the subject of a graceful Note by Cremona (*Rend. Ist. Lomb.*, I. 2). Thus, from the representation on a plane of skew surfaces having two rectilinear directors, Cremona deduced the new and remarkable property that the asymptotic lines of such surfaces are algebraic curves (*Annali di Matem.*, I. 2).

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* In 1872 the two Notes, combined and augmented, were translated into French by Dewulf.

† Cremona was again awarded the Steiner prize, for his researches in Geometry, in 1874.

The works just mentioned inaugurated another fertile and important period of research, devoted by Cremona to unicursal surfaces and to birational transformations of space. With regard to the first of these, a classical Memoir by Clebsch (*Math. Annalen*, I.) had already given, in addition to several particular cases, some general properties ; and a Memoir by Nöther (*ibid.*, III.) had established the general theorem, so important in its consequences, that any surface on which a rational pencil of rational curves can be drawn must be unicursal. This latter memoir Cremona followed up by two Notes (*Rend. Ist. Lomb.*, IV. 2) on the quartic surface having a double conic and on a particular case of such a surface, investigated by means of a quadric transformation of space. With regard to the other subject mentioned, that of birational transformations of space, besides the transformation of the second order (whose inverse is also of the second order), Magnus and Hesse had already considered, and Cremona had adopted in the representation of cubic surfaces, the cubic transformation which is analogous in a certain sense to the quadric transformation between two planes. In addition, Cayley had given the formulæ for the transformation of the second order whose inverse is of the third order (*Proc. Lond. Math. Soc.*, III.). After this appeared almost simultaneously (1871), and certainly independently, the works of Nöther and of Cremona on birational transformations of space, on those of the third order in particular, and on their applications to unicursal surfaces. Cremona himself expressed pleasure at this coincidence, which extended even to details (cf. the second Note in the *Rend. Ist. Lomb.*, IV. 2), and made a point of declaring that it was the beautiful researches which Nöther had previously published that had inspired him with the desire to take up the subject of birational transformations again.

Cremona's works on these subjects show considerable variety, and some of them are reproductions or amplifications of earlier ones (*Göttinger Nachrichten*, May 3, 1871; two Notes in the *Rend. Ist. Lomb.*, IV. 2; *Math. Annalen*, IV.; *Mem. Ist. Bologna*, I. 3, II. 3; *Annali di Matem.*, v. 2). In the *Nachrichten*, among particular cases of many kinds, Cremona gave the transformation of the second order whose inverse is of the fourth, which he was the first to notice; and applied it to establish the existence of a unicursal surface having one tacnode and three double points, the first example (after the cubic surfaces) of a unicursal surface which is not a monoid surface, and which does not possess a multiple line; all which was developed subsequently with greater completeness in the Bologna Memoir. Shortly afterwards, Nöther discovered a more general type of such surfaces, viz., surfaces with one tacnode only, but still unicursal. (*Göttinger Nachrichten*, June 7, 1871; cf. also *Math. Annalen*, XXIII.) These were, at a later date, the subject of special study by Cremona in a Memoir in the *Collectanea Mathematica in memoriam Domenico Chelini* (1881). In the paper published in the *Annali di Matem.*, left uncompleted, Cremona summarized, in that elegant geometric form so characteristic of him, the general results arrived at by himself and by Nöther (possibly, it may be, without attaining complete rigorousness as regards every point of the proof); he further gave all the cases of the transformation of the second order, and some of those of the third, supplying also the corresponding formulae, direct and inverse. Among these general results must be mentioned, simple as it is, the ingenious method, due to Cremona, of constructing all the unicursal systems to which a given unicursal surface belongs, and the resulting construction of the parts of the Jacobian which correspond to lines—a method consisting in a suitable breaking up of the linear system which is the image of the sections of the surface by all the other unicursal surfaces of the same order and having the same singularities.

As at Bologna, so too at Milan, Cremona gave evidence of extraordinary scientific activity. In fact, in the period just described (and still more in an interruption of it which extended from 1869 to 1871), he devoted attention also to a variety of subjects of a kind different from those at which he had hitherto worked. We may mention the fine memoir on the species of skew quartic surfaces (*Mem. Ist. Lomb.*, VIII. 2), and the papers on the birational transformations of curves (two Notes, one in collaboration with Casorati: *Rend. Ist. Lomb.*, II. 2), on Abelian Integrals (*Mem. Ist. Bologna*, x. 2), on the twenty-seven lines lying on a cubic surface (*Rend. Ist. Lomb.*, III. 2), on the lines of curvature of surfaces, especially those of quadrics (*Mem. Ist. Bologna*, I. 3), and on the bitangents of a quartic curve having a double point (in collaboration with Brioschi: *Math. Annalen*, IV.).

The publications of this period are further remarkable in that analytical considerations are in them explicitly made the foundation of the geometric treatment, and in some cases, indeed, are employed systematically throughout. Cremona, who had begun with an education entirely analytical, and had afterwards been strongly attracted towards pure geometry, was not slow to recognize that the two forces ought to be combined; and he did, in fact, so combine them in the work of research, giving the preference, however, in the summing up of results, to synthesis. To Clebsch, with whom he had long and friendly relations, he wrote in 1864: "I am fully convinced as to the mutual help which analysis and synthesis afford one another in geometry." *

Besides the paper forming part of the "Collectanea" (1881) already mentioned, Cremona wrote in 1884 two short notes on a transformation of the sixth order whose inverse is of the fifth (*Proc. London Math. Soc.*, Vol. XV.), and on the representation of the surface reciprocal to the general cubic surface (*Proc. Roy. Soc. Edin.*, Vol. XII.); but his researches on transformations in space may be said to come to an end in 1873 with his removal to Rome. First the task of reorganizing the *Scuola degli Ingegneri*, then political life, and, finally, declining health interfered with his scientific activity. He still continued, nevertheless, to work at mathematical subjects of various kinds, and to excel in them by reason of the lucidity, the elegance, and the happy coordination, which characterize all his writings.

Starting from the representation of a linear complex in ordinary space, as given by Nöther and Lie, he deduced the theory of systems of right lines of the second degree from the known properties of cubic surfaces or from those of quartic surfaces having a double conic (*Mem. Acc. Lincei*, III. 2), in the course of which he made express reference to considerations of the geometry of hyperspace. By a minute study of the configuration of fifteen right lines on a cubic surface having a double point, he established, in a simple and concise form, the theorems both old and new concerning Pascal's hexagram proved by Veronese, by means of considerations concerning homological triangles (*Mem. Acc. Lincei*, I. 3). From the determination of certain polar hexahedra with respect to a cubic surface, he deduced the reduction of its equation to a sum of six cubes and the construction of Sylvester's pentahedron (*Math.*

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* Cf. Veronese, *loc. cit.*

Annalen, XIII.). Finally, he extended not merely to ordinary space, but to hyperspace, certain results of Weyr and Darboux as to plane curves of order n passing through all the points of intersection of $n+1$ tangents to a conic.

With the work previously mentioned on the surface reciprocal to a cubic surface (1884) Cremona concluded his scientific publications.* But the Italian School of Geometry, which he had originated and fostered, continued with confidence in the path on which it had started; and while widening its field of view and the scope of its labours, and pushing its activity in many directions, still preserved unaltered the taste for that pure geometric method which the great master had implanted in it.

Never, to his last days, did Cremona lose his interest and his ardent love for science. Even in the early years of his residence at Rome, he had already recognized that the political atmosphere and his new occupations were not favourable to the study which was so dear to him. In 1877 he almost consented, at the request of Professors Betti and Dini, to transfer himself to Pisa, and he was mainly dissuaded from doing so, it seems, by the earnest entreaties of Sella. At the time he wrote to one of his pupils, who was doing his best to bring about the change, that he was grateful to him "for having lent his aid in assisting him to return to pure science." Even during the time of his physical suffering—not a short space of time—Cremona never ceased to follow the progress of science. To Professor Segre, who visited him in his study five days before the end, when the final crisis of his malady was not yet apparent, he spoke of Gauss' "Tagebuch," edited by Klein, saying that it struck him as strange that so many of the beautiful results discovered by Gauss should have remained unpublished by him. On Segre mentioning that in the eighth volume of Gauss' works were to be found many things that had been attributed to other men of science, in particular the pseudosphere,† Cremona, having taken down the volume and verified the statement, expressed surprise at the fact, and added that it had escaped his notice, though he had carefully gone through the whole of that volume. The love of teaching, and of spreading the knowledge of geometry, was no less marked in Cremona than the love of science. His numerous courses of higher geometry always included some account of the latest and most remarkable advances. In 1865–6, for example, he discussed plane cubic curves and cubic surfaces; in 1868–9, Abelian integrals, as dealt with in the book of Clebsch and Gordan (which led to his own investigations on the same subject, previously mentioned); in 1870–1, unicursal surfaces, dealing in particular with the theorem of Nöther, to which attention has already been called (*Math. Ann.*, III.); in 1873–4, Cremonian transformations in the plane and in space; in 1874–5, line geometry; and in his last years, Lie's continuous groups. His course of Descriptive Geometry at Bologna deserves mention, as also that on Graphical Statics delivered at Milan. In the former he introduced various improvements, anticipating those which were made at a later date, by Fiedler especially; and of these he also gave a sketch in the Note on Perspective published under the assumed name of Marco Uglieni (*Giornale di Matem.*, III.). In the latter course he followed the ideas of Culmann, and made noteworthy additions to them. Out of these arose the beautiful little book (translated into German and English), *Elementi del Calcolo grafico*, and the valuable memoir (translated into English), *Le figure reciproche nella Statica grafica*, based on the ingenious idea of considering the reciprocal diagrams of graphical statics as projections of two reciprocal polyhedra in a null-system.

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* In 1900 he wrote the beautiful commemorative notice of Beltrami, with whom he had long been on terms of intimate friendship.

† On p. 265 of Vol. VIII. of the *Werke*, Gauss alludes first to surfaces of revolution which are applicable, and then more particularly to that formed by the revolution of the tractrix, to which he gives the name of "Gegenstück der Kugel."

Cremona's gifts as a teacher were clearness, exactness, order, the art of presenting difficult things in a simple and intelligible form, and an excellent style. His lectures, prepared with great care, were always quiet in manner, his words always well considered and balanced, but incisive and easy to follow; and not only did his teaching sink deeply into the mind of the student and fire him with a love of geometric truth, but it inspired him with noble and high ideals. Far-reaching was the influence for good which Cremona exercised upon the organization and the curricula of the Italian secondary schools. For technical schools specially he published (1866 to 1868) a translation of Baltzer's *Elemente der Mathematik;* and along with Betti and Brioschi he supported and defended* the adoption, in the case of classical schools, of Euclid as a text-book. To him we, in Italy, owe the introduction of the teaching of projective geometry first (1871) into the technical institutes, and later (1876) into the first year's course at the Universities. For the use of the technical institutes he published the most excellent *Elementi di Geometria Projettiva* (translated into French, German, and English), in which he permitted himself the use of a variety of methods in order to produce a book which, as he himself says, should be absolutely elementary and technical.

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* Cf. a letter from Brioschi and Cremona to Houël in the *Nouvelles Annales de Math.*, VIII. 2.

His strong understanding, the integrity of his character, and the extent of his learning even in matters foreign to mathematics, caused Cremona's work to be highly appreciated in the Senate of the Kingdom of Italy, of which he was appointed a member in 1876, and of which he became Vice-President. To the Bill on higher education he proposed a remarkable amendment. On this Cremona spent much trouble, and he held to it with great firmness and considerable confidence; but his hopes were only too surely disappointed by the vicissitudes inseparable from parliamentary government. Noteworthy, too, among the many reports which he had to draw up in the office of the Senate, is the one which he compiled a few weeks before his death, in which he combated with strong arguments, and with his usual clearness and confidence, the proposal to divide the responsibility for the sections of the technical institutes and of certain higher schools between two Ministers —those of Agriculture and Instruction.

Cremona belonged for a great number of years to the Superior Council of Public Instruction, and on several occasions held the office of Vice-President. Besides this, he was Minister of Public Instruction, but only for a few weeks (from June 1 to June 29, 1898), so that the universal hope that he would be able to effect a serious reform in educational matters was doomed to disappointment.

He was a Fellow of the most distinguished academies, both Italian and foreign, a Doctor *honoris causa* of the Universities of Edinburgh, Dublin, and Christiania, a Knight of the Civil Order of Savoy; and in May, 1903, the German Emperor conferred on him, by special commission, the order *Pour le Mérite*, a distinction by which very few Italians have been honoured.

Cremona's character has been drawn for us* in a few words as follows:—"An upright, inflexible man, he possessed a strong sense of duty, and looked for the same in others. It was possible on occasion to differ from him in his judgment of men or things, but never to doubt the honesty of that judgment. On intimate acquaintance with him one recognized a man capable of impetuous admiration, of scornful contempt; qualities which revealed a nature rich in sensibility, eager for good, intolerant of all meanness." As regards his pupils, he was not sparing of open and severe rebuke in case they showed any impatience when promotion seemed to come slowly, or any idleness or slackness in study, or carelessness in their published work; but at the same time he urged them on vigorously and encouraged them heartily, giving them kind advice and ready help. He fully realized, and rightly so, that excellence of style contributes in no small degree to the success of a scientific memoir. To a pupil he wrote: "The form in which a work is drawn up, if Italian, accurate, clear, may double its value, and will certainly add to the number of its readers." And, so believing, he published works which will always be read with real intellectual pleasure, and which awaken in the reader a love for the study of geometry; and his name, an honour and a pride of the new Italy, will remain imperishable in the history of geometry, among those of the greatest of his time, not only because of the depth and importance of his researches, but also by reason of the elegant structure of his works and the harmoniousness of the conceptions and the form in which they are presented.

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* D'Ovidio: Obituary Notice in the *Atti di Torino*, XXXVIII.

JOSIAH WILLARD GIBBS

"JOSIAH WILLARD GIBBS was the son of Josiah Willard Gibbs (Y. C. 1809), the distinguished Professor of Sacred Literature in the University from 1822 to 1861, and of Mary Anna (Van Cleve) Gibbs. He was born in New Haven, Conn., on February 11th, 1839, and died on April 28th, 1903. He was prepared for college at the Hopkins Grammar School, New Haven, and entered the class July 24th, 1854. In his college course he won the Berkeley Premium for Latin Composition; 1857, Bristed Scholarship; 3rd Prize Latin Examination, 2nd term, Junior year; Berkeley Premium for Latin Composition; 1858, 1st De Forest Mathematical Prize; Clark Scholarship; Latin Oration.

"He occupied the first five years after graduation in 1858 in mathematical and other studies in New Haven. In the autumn of 1863 he became tutor in Yale, and was engaged with the duties of that position until August, 1866, when he went to Europe.

"The winter of 1866–67 he spent in Paris, and the winter of 1867–68 and the following summer, in Berlin, studying especially physics, but devoting a part of his time to mathematics. The winter of 1868–69 he passed in Heidelberg, and the next spring in France, reaching home in June, 1869. In July, 1871, he was elected Professor of Mathematical Physics in Yale."*

The prevailing interests revealed in Gibbs' work are those of a mathematician, though his facility in algebra was perhaps slight, and he was most successful when casting his arguments into graphical form. His mind was always straining towards complete general views. His direct geometrical or graphical bent is shown by the attraction which vectorial modes of notation in physical analysis exerted over him, as they had done in a more moderate degree over Maxwell, the interpreter of Faraday in this respect; his generalizing tendency was illustrated in the formal address, in which he expounded to the American Association the fascinations of the mathematical notations and operations appropriate to this subject, where he could not reach finality until his treatment had got into *n* dimensions of space. This bent towards exhaustive survey of his subject probably served Gibbs in good stead, by driving him to mathematical completeness in his exposition of thermodynamics, where others would have stopped short with the fragment of the theory that covered the physical applications then prominent or likely to arise. But his tendency to wind up the exposition and regard the account as closed when the logical fabric has been welded together, and to assign a subsidiary place to the details of such particular physical illustrations as then existed—from restraint, be it noted, not from lack of knowledge—retarded for many years the application of his methods by experimenters, to whom the behaviour of actual things is of more interest than the perfection of an abstract formulation of their relations.

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* *Yale Alumni Weekly*, May 6th, 1903. See the Royal Society's Obituary Notice by Prof. Larmor, from which the present sketch is mainly taken.

The achievement by which Prof. Gibbs will chiefly be remembered is his development, into their full scope, of the fundamental principles which regulate equilibrium and the trend of transformation in inanimate matter in bulk, that is, in general chemical and physical phenomena.

Afterwards Gibbs turned his attention to the electrical theory of light, then in the tentative stage of development, and published in 1882 three papers in which the electrical relations forming the foundation of Maxwell's theory were expounded on the most general formal basis. The medium is taken to be heterogeneous (molecular) in its smallest parts, but of an averaged homogeneous structure as regards elementary regions of dimensions comparable with a wave-length. General linear relations of a formal type between the Maxwellian vectors are assigned, involving the case of rotational media when they are not self-conjugate. The precise part of the electrical basis utilised was the universally admitted general type of formula (Neumann-Maxwell) for the kinetic energy of the (circuital) displacement-currents in the field. The object of the papers was to point out how naturally the laws of optical reflexion, and of double refraction including its dispersion, flow from the electrical ideas as contrasted with the mechanical theory of Cauchy and Green. Thus the analysis is in some respects open to the remark that the electrical foundation is refined and generalized until there is but little distinctively electrical that is left except the frame; there remains a very general scheme of formal analytical relations, not unlike Hertz's later manner of conceiving the Maxwellian electrical equations, which has to become more restricted and particularised for practical use. In a subsequent paper he contrasted Lord Kelvin's remarkable labile mechanical æther, then recently announced, with electric theory in the same general manner, again laying stress on the formal optical fitness of the system of equations which are the expression of the latter. In all this work we recognize the same penetration and skill, in the formulation and expression of the utmost generality of outlook, which he showed in pure mathematics by his partiality for the study of generalized algebras and vectorial analysis, and which in thermodynamics has largely constituted the strength of his work, though at the same time it has retarded its absorption into the general body of scientific doctrine.

After a period in which Prof. Gibbs' work was much interrupted by ill health, he again appeared before the scientific world early in 1902 as the author of a notable treatise of 207 pages octavo, entitled *Elementary Principles in Statistical Mechanics, developed with especial reference to the Rational Foundation of Thermodynamics*, which was published in connexion with the Bi-centenary of the University of Yale. Having had a principal share in evolving the ultimate form of the principles that govern physical and chemical equilibrium, when matter in bulk is considered in terms of its observable properties alone, and having taken care to state them free of all vestige of molecular theory, it was natural that his thoughts should have turned to the less definite problems opened out by the molecular hypothesis of the constitution of matter, which provides a rational base for the axioms on which Carnot, Clausius, and Thomson originally built.

Prof. Gibbs, during his lifetime, was invited to honorary membership of most of the leading learned societies and academies of both hemispheres that pursue physics and mathematics. In particular, he became a Foreign Member of the Royal Society in 1897, and was awarded its crowning distinction, the Copley Medal, in 1901; he had been elected a Foreign Member of the London Mathematical Society in 1892.

GEORGE SALMON

[For this notice the Council is indebted to Sir Robert Ball.]

THE late Provost of Trinity College, Dublin, was the third and the greatest of three consecutive Provosts, each being highly distinguished in science. Humphry Lloyd, the brilliant experimentalist, and Jellett, well known for his optical and mathematical investigations, were the immediate predecessors of him of whom we now write, and who was admitted by universal consent to be one of the most famous mathematicians of the last century.

Had Salmon never done anything else than write his *Treatise on Conic Sections*, he would have richly earned the gratitude of all mathematicians. For more than half a century the book has been the standard authority on the subject. At the time it was written it was immeasurably superior to all other treatises on curves of the second order. Descartes had invented the method of coordinates. Poncelet, in his projective geometry, had developed the most beautiful and instructive branch of the subject, and it will be remembered that Salmon handsomely acknowledged the inspiration that Poncelet had been to him. Chasles had given to us those exquisite geometrical theories by which methods of wonderful power for geometrical investigation were provided. Graves, the late Bishop of Limerick, had contributed his theorem about the arcs of confocal conics, and it may well be doubted whether within the whole range of geometrical beauty there is anything more fascinating. Boole had just commenced three epoch-making investigations concerning invariants, which have originated a vast department of the modern mathematics. Many other illustrious names could, of course, be added, from the times of the ancient Greeks down to the middle of the last century, of the discoverers of the fundamental principles of the conic sections.

It was, however, reserved for the master mind of Salmon to create a great and comprehensive treatise which should be worthy of the subject. To the materials already provided by the labours of others Salmon added largely. He made the Cartesian coordinates flexible as they had never been before. He enormously increased their power as instruments of geometrical research, by the methods of abridged notation. He dovetailed the analytical methods with the methods of pure geometry. He gave to the projective geometry the position which its importance demanded. He showed how the method of reciprocal polars and the methods of projections established recondite theorems as the immediate consequences of elementary principles. In the later editions he gave the applications of invariants and covariants, and thus laid the foundation of the applications of these beautiful conceptions to geometrical investigation. By all this original work he provided the matrix in which the results of the previous investigators were incorporated and worked into a systematic whole, and his great *Treatise on Conic Sections* was the result. The style of this work is admitted to be the model of what a mathematical work should be. The excellence of its arrangement, the abounding wealth of illustrative examples, the philosophical manner in which the reader is led to a knowledge of the broad principles of geometry and algebra, the beauty and interest of the matters treated, the suppression of needless detail, the clear distinction between what is of real importance and what is merely casual—in all these points, it is not too much to say that Salmon's *Conics* is a book of unapproached merit. It superseded every other book of the kind when it appeared, and now nearly sixty years later its supremacy still remains unchallenged. I believe it may be added that Continental mathematicians, without a dissentient voice, coincide in this estimate of the value of this work. It is not, indeed, that other books on conic sections have not appeared. They have appeared, and I suppose may be reckoned by the score. No doubt they have their uses ; but I believe every one of the authors of these books has been glad to acknowledge his indebtedness to Salmon. Indeed, this great work has been, and still is, the quarry to which for many a long day every one has resorted who wanted to write on conics. There are huge reserves still in that quarry, and at least one author has confessed to me how saddening it has been to him time after time, whenever he thought he had discovered something good in the conics, to have found out that, if it was not wrong, it was already in Salmon.

Salmon used to lecture on mathematics in Trinity College, and the writer of this note enjoyed the inestimable privilege of attending these lectures in 1858–9. He was an admirable teacher. I particularly remember the course on conics. It was presumed that we had read or were reading his book, so the lectures often took the line of showing us the improvements or extensions which he was preparing for future editions. He would kindly encourage his pupils to make their comments, and even honour them by asking for their efforts to aid him in questions which were occupying him at the moment. He used to warn us that the good things in conics were mostly " threshed out," but still there were some leavings. It was, I think, in this way that Mr. Burnside, now Prof. W.

Snow Burnside, of Trinity College, who was also one of Dr. Salmon's pupils, originated some theorems and methods which duly appeared in later editions of the *Conics*. I also attended Salmon's lectures on "Elementary Theoretical Dynamics," and never shall I forget the delight with which we received his illustrations of the use of differential equations in dynamics.

The great success of the *Conics* naturally suggested that the same process which had led to such triumphs in curves of the second degree should be applied to curves of higher degrees. Hence arose Salmon's second book, *The Higher Plane Curves.* It will, I suppose, be admitted that this book, like the *Conics,* has no rival in the same subject. That there are geometrical beauties in the subject of higher plane curves will not be denied. For example, no one can deny beauty to the theorem stating that the four tangents drawn to a cubic from a variable point on the curve have a constant anharmonic ratio. Of course there are innumerable occasions, in practical as well as theoretical mechanics, when the theory of plane curves other than conics becomes of importance. I must, however, add, though I am perhaps expressing only my own opinion, when I say that the *Higher Plane Curves* is the least interesting of Salmon's four great works. It lacks the magic of the *Conics.* It has not the importance in nature possessed by the *Quadrics,* which form so large part of the *Geometry of Three Dimensions.* Nor does it lead to such splendid theories and conceptions as do the lessons in *Modern Higher Algebra.*

The genius of Salmon has, however, lightened up the subject of the higher plane curves as probably no other writer could have done. Here, again, the admirable simplicity of his methods minimizes the difficulties in a most difficult and intricate subject. The present generation are so accustomed to the artifices rendered familiar by Salmon that we are often hardly conscious of all we owe to him. Descartes, no doubt, taught us how to represent the coordinates of a point on a curve, with the help of the rectangular axes; but it was Salmon who saw how to avail himself of the privilege that those axes might be placed anywhere. He sometimes put the origin on the curve and made the tangent one of the axes, and at once a great simplification comes into the equation of the curve. It is Salmon also who has applied and developed his abridged notation and made symbols represent ideas rather than numerical magnitudes. To Salmon also we, in great measure, owe the first great application of the theory of invariants and covariants to the study of cubics and other curves. The subject is truly a terrific one.

Throughout his life Salmon was intimately associated by correspondence with the greatest of contemporary mathematicians. Among these, perhaps, the chief was Cayley, and an indication of this remarkable friendship is given in connexion with the book now under consideration. By the time a second edition of the *Higher Plane Curves* was called for, Salmon had become a Professor of Divinity, and had thus made engagements which, in his own words, "left me no leisure to make acquaintance with recent mathematical discoveries, or even to keep up any memory of what I had previously known." In this emergency he applied to Professor Cayley to obtain advice as to the choice of some younger mathematician, who would undertake the editing of the work. To his agreeable surprise, Professor Cayley offered to do this himself. This offer was gratefully accepted, and the second edition was the joint work of Salmon and Cayley. The opening chapter is by Cayley, and the opening words of that chapter may well be quoted as showing the deep impression which the principles of the projective geometry had by this time made on the two great mathematicians:

"We have in the plane a special line, the line infinity, and on this line two special (imaginary) points, the circular points at infinity. A geometrical theorem has either no relation to the special line and points, and it is then *descriptive;* or it has a relation to them, and it is then *metrical.*"

In the third edition, 1879, Dr. Salmon also availed himself of the aid of Mr. G. L. Cathcart, Fellow of Trinity College, Dublin. Mr. Cathcart has, indeed, for many years mainly undertaken the work of editing all successive issues of Dr. Salmon's mathematical works.

It was in 1862 that Salmon published the great work known as *A Treatise on the Analytic Geometry of Three Dimensions.* Those of us who had to make some acquaintance with ellipsoids and hyperboloids before 1862 will recall the impossibility of obtaining any book on solid geometry which we, who had been reared on Salmon's *Conics*, could regard as quite satisfactory. We had to struggle as well as we could with such books as there were. We had Frost, and we had some French book of which I have forgotten the name. But the appearance of the work which we have been accustomed familiarly to call the "Surfaces," at once gave the student what he so greatly wanted. Here we find the methods adopted with much success in the conics applied to the quadrics. In the chapters on the general theory of surfaces the methods characteristic of Salmon's line of thought are specially to be noted. In the higher parts of the subject we specially remember the striking discovery of the 27 right lines on the surface of the third degree. With reference to this great geometrical advance Salmon remarks in a foot-note: "Cayley first showed that a definite number of right lines must lie on the surface; the determination of that number as above, and the discussions in Art. 458, were supplied by me."

As in the case of the other books, the " Surfaces " has grown in each successive edition. The publication of this book, no doubt, stimulated the study of the subject, and the great developments of the theoretical parts of the subject became incorporated. In the advanced parts of the subject, efforts of the geometrical imagination are called for to which no adequate response can be made with present intelligence. We may be quite satisfied with the demonstration that on the reciprocal of the general surface of the n-th order there will be a nodal curve of the degree $\frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$, but no finite intelligence can give geometrical vividness to such a statement. A large part of the second half of this book, especially in the later editions, is devoted to most elaborate investigations of this kind, which, while they are marvellous expositions of mathematical ingenuity, transcend the powers of human conception almost as much as do the theorems of four dimensional space, and more than the most important investigations of non-Euclidean geometry. And here it may be noted that of four dimensional space or non-Euclidean geometry Salmon has never treated. It is, perhaps, too much to say that he regarded such speculations with scorn, but I have heard him say, jestingly, that he reserved such themes for the next world.

Of course, in the geometrical use of the imaginaries, Salmon greatly delighted. I remember well the zest with which he perplexed us at lectures by showing how a line could be perpendicular to itself, and how on the same line (of course it is $x+\sqrt{-1}\,y=0$) the distance between any two points was zero. But, nevertheless, so far as I remember, he has not in any of his books made use of the complex variable in the way in which it is now of such importance in modern mathematics. It should also be noted that Salmon's methods depended but little upon the differential or integral calculus. The subjects that engrossed his attention, and which he made of such absorbing interest, were essentially treated by geometrical and algebraical processes. When he feels constrained to render some account of a subject like Monge's differential equation relating to curvature, he introduces it with a sort of apology as being only a concession to some presumed desire on the part of the reader. As one of his old pupils, now a distinguished mathematician, and one of Salmon's warmest admirers, once said to me, "Salmon is not a calculus man."

Those who, like myself, are old enough to remember when Salmon's *Lessons introductory to Modern Higher Algebra* first appeared, will not need to be reminded of the impression made by the appearance of that book. It was the first announcement to many mathematicians that an algebra existed which was not the theory of equations. As to the origin of the subject we may quote Salmon's own words: "What I have called modern algebra may be said to have taken its origin from a paper in the *Cambridge Mathematical Journal* for November, 1841, when Dr. Boole established the principles just stated (*i.e.* that the discriminant of a binary quantic is an invariant) and made some important applications of them. Subsequently, Professor Cayley proposed to himself the problem to determine *a priori* what function of the coefficients of a given equation possesses this property of invariance: that, when the equation is linearly transformed, the same function of the new coefficients is equal to the given function multiplied by a quantity independent of the coefficients."

Salmon was primarily attracted to this magnificent theory by its remarkable applications to the theories of conics and other curves, as well as surfaces. He thought at first that he could include in appendices to his geometrical works all that was necessary for his object. But the the interest of the subject grew upon him, and as he made further acquaintance with Cayley's famous series of memoirs on quantics, Salmon decided to prepare his well-known work on *Modern Higher Algebra*, with the dedication: "To A. Cayley, Esq., and J. J. Sylvester, Esq.: I beg to inscribe this attempt to render some of their discoveries better known, in acknowlegdment of the obligations I am under not only to their published writings, but also to their instructive correspondence."

It is in this book that Salmon has undertaken some of the most formidable calculations that any pure mathematician has ever faced. The portentous invariant E alone requires several pages for its expression. He has told us that, when the work was done, he proceeded to check it by testing whether the sum of the coefficients was zero. He was dismayed by finding a balance of several thousands, but, divining a possible error, he halved this balance, and finding this was exactly one of the two coefficients, he immediately surmised a wrong sign, which proved to be the case, and the verification was complete. The tables of symmetric functions and of eliminants, which are given in the appendix, are perhaps more generally useful than the value of E, which has, indeed, not been reprinted in the later editions.

In the latter half of Salmon's career, he was no longer the ardent mathematician. During the first part of this later period he was the distinguished Regius Professor of Divinity, and during the later part he was the great Provost. Books not a few issued from him during the years when he ceased to cultivate mathematics, but they are not books with which the Mathematical Society has concern. To Salmon was accorded the distinction of winning fame as a theologian which was only second to his fame as a mathematician. He used often to say he had forgotten all his mathematics, but till quite recently, though not, I think, quite up to the end of his life, there was one branch of calculation which had a great fascination for him. It was the determination of the number of figures in the recurring periods in the reciprocals of prime numbers. For many years, when he attended a concert (and of music he was extremely fond), or when he was attending a meeting of the Church Synod, he used to spend every spare moment in scribbling figures. Often it would be thought that he could not divide his attention between his figures and the business in hand; but when it came to be his turn to speak (he spoke on such occasions extremely well and to the point) it would be found that he had missed nothing urged by previous speakers. He used often to say that his work at these figures he regarded as frivolous or useless. He even tried to break himself of what he deemed a pernicious habit. "When I get up to 20,000, I will stop," he used to say: and so he did—but then, on having to sit out some unusually dull speech, or on the occasion of a confinement to the home for some slight ailment, his resolution would break down, and he would begin again. I think the last time I heard about it he was approaching the primes near 50,000.

In his earlier years chess was Salmon's principal recreation, and he was a first-rate player. He was one of those chosen to play against Murphy, on the celebrated occasion when Murphy challenged twelve European players simultaneously, and I believe Salmon succeeded in obtaining a draw. His love of music we have already mentioned, and he was a great reader of current literature of all kinds, especially novels. His humour was delightful, and it seasoned his gravest works. He inspired the affection, as well as the respect, of all who had the good fortune of coming into contact with him. In the University of Dublin, Salmon is venerated as the most distinguished Provost who has presided over Trinity College in the three hundred years of its history.

GEORGE HENRY STUART

[For this notice the Council is indebted to Prof. A. G. Greenhill.]

GEORGE HENRY STUART was Director of Education in Madras. He graduated at Cambridge in 1874, as Fourth Wrangler in the Mathematical Tripos, and was subsequently Fellow of Emmanuel College. He was educated at Bishop Stortford School, where, as he told us, he rose to be one of the two top boys of the school who divided all the prizes, for English, Latin and Greek Composition, History and kindred subjects; the other boy was Cecil Rhodes.

Consequently, in his official career in India, he was able to alternate the tedium of elementary mathematical instruction by lectures on English Literature, a subject in which he took great interest. At the time of his death he was engaged on an equally congenial subject in the reduction of the records of the Census.

While professing not to be a mathematician or to take much interest in the science of mathematics, Stuart was fascinated very readily in a problem requiring analytical skill; and he has published articles in the *Quarterly Journal of Mathematics* on the subject of the "Complex Multiplication of Elliptic Functions," which are cited by Halphen in his *Fonctions elliptiques*, Vol. III., as standard memoirs of reference.

He came to my assistance on one occasion in the reduction of a piece of very heavy algebraical work, required in the "Transformation of the Eleventh Order of the Elliptic Functions" (*Proc. London Math. Soc.*, Vol. XXVII., p. 471). I remember how we scrutinized his results, and I could not believe they were correct, till it dawned upon us that the order of the result could be reduced to one-fifth by means of a quintic transformation. Our excitement was great as we proceeded with the verification, and Stuart confessed that he was rewarded for all trouble; but I could never entice him afterwards into a similar piece of work—"coolie work" he called it.

His fate was the usual one of the Anglo-Indian: ambitious of attaining the highest official position, he stuck to his post in defiance of medical warning; in the other alternative, he might have taken his retirement and pension in time, and lived "to regret it."

NOTES AND CORRECTIONS.

MR. CAMPBELL sends the following note on "Continuous Groups":—

In the *Fortschritte der Mathematik* (Band xxxii., Heft 1, 377) Prof. Engel criticizes the proof of the "Third Theorem in Continuous Groups," which the author gave in the *Proceedings*, Vol. xxxiii., pp. 285–294. It is the object of the present note to point out the difficulty which has been raised, and to show how easily it may be met and how little it affects the argument.

The aim of the paper referred to was to construct a simply transitive group $X_1, \ldots, X_r$, such that

$$X_j X_i - X_i X_j \equiv c_{ji1} X_1 + \ldots + c_{jir} X_r,$$

where $c_{ji\kappa} \ldots$ is a given system of structure constants of the r-th order.

If the structure constants are such that the matrix of r^2 columns and r rows

$$\left\| \begin{matrix} c_{j1\kappa} & \cdots \\ \vdots & \cdots \\ c_{jr\kappa} & \cdots \end{matrix} \right\| \quad (j = 1, \ldots, r;\ \kappa = 1, \ldots, r)$$

is of the r-th order (that is, if not all r-rowed determinants vanish), then it is well known, and easily verified, that

$$X_i \equiv \Sigma c_{piq} x_p \frac{\partial}{\partial x_q} \quad (i = 1, \ldots, r)$$

will be a group of the r-th order and of the required structure. Its parameter group will then satisfy the conditions of the proposed problem.

If, however, the matrix is of order n, less than r, the group cannot be thus simply constructed.

The construction in the *Proceedings* was made to depend on the construction of a simply transitive group of order n whose constants were known. In constructing this group of lower order, it was assumed (§ 5) that the matrix of n^2 columns (no longer r^2 columns)

$$\left\| \begin{matrix} c_{j1\kappa} & \cdots \\ \vdots & \cdots \\ c_{jn\kappa} & \cdots \end{matrix} \right\| \quad (j = 1, \ldots, n;\ \kappa = 1, \ldots, n)$$

was of order n. Prof. Engel points out that this is not necessarily true.

What has been proved, however, is that a simply transitive group of order r, whose structure is assigned, can be either constructed immediately or when one of order n $(< r)$ has been constructed. Proceeding similarly with the assigned structure constants of order n, a simply transitive group of order n can either be constructed immediately, or when one of order less than n has been constructed. This necessary modification in the proof (to which Prof. Engel himself appears to point) can hardly be said to take away from any simplicity to which the proof may have laid claim.

Mr. Macdonald sends the following corrections of his paper, "Some Applications of Fourier's Theorem," in the *Proceedings*, Vol. xxxv., pp. 428–443 :—

P. 435, line 16, *for* "$\cos\left(xy+\frac{n-m}{2}\pi\right)$" *read* "$\cos\left(xy-\frac{n-m}{2}\pi\right)$."

P. 435, line 18, *for* "$\cos\left(bx+\frac{n-m}{2}\pi\right)$" *read* "$\cos\left(bx-\frac{n-m}{2}\pi\right)$."

P. 436, equations (13), *for* "$\cos bx$" *read* "$\cos(bx-\frac{1}{2}n\pi)$."

P. 440, line 10, *for* "$J_n(bx^{\frac{1}{2}})$" *read* "$J_m(bx^{\frac{1}{2}})$."

P. 442, line 10, *for* "(y)" *read* "$f(y)$."

P. 442, line 15, *for* "$J_n\left\{c\sqrt{(a^2-\xi^2)}\right\}$" *read* "$J_{n-m-1}\left\{c\sqrt{(a^2-\xi^2)}\right\}$."

The following *corrigenda* have been noticed in this volume :—

P. 12, line 3, *for* "face" *read* "force."

P. 39, line 5 of the small print, *the last term should be* $-\frac{2}{\sigma r}\left[\frac{\partial\phi}{\partial t}\right]$.

P. 60, line 8, *after* "(24)" *insert* "with the statical term omitted."

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE MATHEMATICAL EXPRESSION OF THE PRINCIPLE OF HUYGENS

By J. Larmor.

[Received December 26th, 1902.—Read January 8th, 1903.]

1. In Green's *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (Nottingham, 1828), which laid the foundations of modern analysis as applied to mathematical physics, potentials were first considered from a functional point of view, and the conditions requisite to specify a potential function were laid down; as a particular case of "Green's Theorem," the value of such a function throughout a region of free space was expressed in terms solely of its own values and those of its gradient normal to the boundary of the region. The mathematical development of this general theory was extended by Helmholtz,[*] with important and novel applications, to the velocity-potential in the theory of the propagation of sound-waves in air, which satisfies a differential equation for which an analogous procedure applies. Subsequently the theorem which corresponds to the one quoted above for the gravitational or electric potential was developed in detail by Kirchhoff, who thus derived a general expression for the vibratory disturbance propagated into a medium in terms of the distribution of the disturbance and its spatial gradient, supposed given throughout the time, over any arbitrary geometrical boundary across which it must have travelled into the region under consideration. Two centuries ago Huygens explained the general features of the propagation of disturbances of elastic type, by considering

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[*] "Theorie der Luftschwingungen in Röhren mit offenen Enden," *Crelle*, Vol. lvii., 1859.

each differential element of the front of an advancing wave-train to constitute a source of disturbance from which it travels onward into the medium in advance; and in the hands of Young and Fresnel this principle of Huygens furnished not merely an explanation of the geometrical form of the advancing waves, but served, by aid of the principle of interference of undulations, as a basis for the optical principle of rays, and also for the quantitative explanation of the phenomena of diffraction in which that concept partially fails. In Fresnel's hands the circumstances of actual optical diffraction were amenable to mathematical calculation in minute detail, without any knowledge of the law of propagation of these secondary disturbances from the various elements of the wave-front, except such as could naturally be assumed to represent them in the immediate neighbourhood of the direction of propagation. Yet the formula assumed for this purpose by Fresnel involved some oversights, whose general origin was, however, readily traceable when the dynamical theory of vibrations became more explicitly developed. The occurrence of such discrepancies gave rise to the endeavour to place, if possible, the circumstances of the specification and propagation of secondary disturbances on a precise foundation. This was first essayed, and may indeed be held to have been so fully developed that subsequent discussion has added little except in the way of adaptation to new circumstances, in Sir G. Stokes's memoir on "The Dynamical Theory of Diffraction."* The formula obtained in Kirchhoff's memoir, above referred to†, was, however, advanced by him mainly as an improved mathematical expression of Huygens' principle, and is often taken to be its exact mathematical specification. That such merely analytical specification *a priori*, independently of the mechanics of the propagation as depending on the constitution of the medium, including the relation of stress to strain, does not constitute a definite problem, has been remarked by Lord Rayleigh,‡ and illustrated by various allowable laws of propagation from the various elements of the wave-front.

More recently the same subject has been treated by Prof. A. E. H. Love,§ and further developments for uniaxal crystalline media were given by Prof. A. W. Conway recently in these *Proceedings* (Vol. xxxv.).

The usual modes of deduction of the Helmholtz-Kirchhoff formula are not free from analytical complexity,‖ and can hardly be said to throw much direct light on the character of the principle which they demonstrate, or on the degree of its determinateness, which is here under consideration; there seems, therefore, to be room for the following remarks, intended as a contribution towards physical precision, by putting forward a more intuitive view of the content of the formula. It will not here be necessary to attend to all the saving restrictions that are customary in the exact analysis of formal pure mathematics*; such limitations are, in the main, sufficiently obvious, and are satisfied by the nature of the case, in continuous physical analysis, while their detailed exposition would impede and interrupt the main argument.

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* *Camb. Phil. Trans.*, 1849; or *Mathematical and Physical Papers*, Vol. II.

† "Zur Theorie der Lichtstrahlen," *Wied. Ann.*, Vol. XVIII., 1883.

‡ *Encyc. Brit.*, "Wave Theory," 1888; *Collected Papers*, Vol. III., No. 148.

§ "The Integration of the Equations of Propagation of Electric Waves," *Phil. Trans.*, Vol. CXCVII. A, 1901.

‖ Compare, *e.g.*, the discussion in Drude's *Optik*, 1900.

2. It will be simplest to begin with the analogous propositions for the more familiar case of the ordinary potential, which is, in fact, what the vibrational potential becomes when the speed of propagation is infinite.

Consider, then, a potential specified throughout all space as follows. It is a function V, single-valued and continuous as to itself and its gradient, and satisfying Laplace's equation, in all the space outside a boundary S, and, as a consequence, diminishing towards infinity, according to the law r^{-1} or a higher inverse power of r: it is zero everywhere inside this boundary. What is the distribution of attracting mass to which this potential belongs? This distribution is, as usual, determined, in Green's manner, by the singularities and discontinuities of the potential-function. It consists, in fact, of a surface density σ over the boundary S, and a double sheet of strength s over the same boundary, expressed in terms of the potential and its gradient at the surface by the formulæ

$$\sigma = -\frac{1}{4\pi}\frac{dV}{dn}, \qquad s = \frac{1}{4\pi}V,$$

in which δn is an element of the normal drawn towards the outside of the surface. For it follows, by the usual procedure, that, if V' is the potential of this distribution, then $V-V'$ is a potential function which has no singularities or discontinuities throughout all space, and is, therefore, identically null. Expressed analytically, at any point in space the potential is thus

$$-\frac{1}{4\pi}\int\frac{1}{r}\frac{dV}{dn}dS+\frac{1}{4\pi}\int V\frac{d}{dn}\frac{1}{r}dS.$$

In other words, this formula gives the value of a potential function V in the free space outside the surface in terms of the values which it and its gradient assume on the surface; it constitutes, in fact, the analytical continuation of the function outward from the surface, while inside the surface the value of this expression is everywhere null. In the case of a closed surface, as well as that of an open sheet, either side may be called the outside for the present purpose. This continuation of the function is necessarily unique and determinate,* but the form of the integral which expresses it is far from being so. We may, in fact, generalize the formula immediately in Green's manner. Consider a function V which is the potential, throughout space, of any assigned distribution of mass. Draw any surface dividing space into two regions, A and B, each of which contains part of the mass, these parts being represented by M_A and M_B. What distribution of masses, and of surface densities and normal doublets on the surface S, is required to produce a potential equal to V in region A, and equal to zero in region B? Clearly, M_A, together with

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* A more detailed discussion, in many respects parallel, will be found in Poincaré's *Traité math. de la Lumière*, 1889, chap. iii. and iv.: *cf. infra*, p. 7, footnote.

$$\sigma = -\frac{1}{4\pi}\frac{dV}{dn}, \qquad s = \frac{1}{4\pi}V.$$

What distribution is required to make the potential zero in region A and V in region B? Clearly, M_B, together with the same distribution on the surface but with sign changed if the direction of δn remain the same. Thus, a distribution of surface densities and normal doublets is found which exactly cancels the effect of M_A on the other side of the dividing surface; moreover, an infinite number of such distributions can be found, for in determining it M_B is entirely arbitrary. On changing the signs of s and σ, we infer that a distribution of surface densities and normal doublets can be assigned, in an infinite variety of ways, over any surface surrounding an attracting system, so as to form the exact equivalent of that system as regards the effect transmitted to outside space; and that one of the modes of distribution thus obtained produces null effect in inside space. The problem of electrostatic distribution is, of course, unique in a different manner from this latter; a free electric distribution on a conductor can consist (physically) of simple sources only, and cannot involve doublets. Its specification involves the solution of a problem in which the form of the surface is involved, and cannot be derived from a general continuation-formula.

Needless to say, an argument similar to the above applies to a function specified by any characteristic partial differential equation of the second order.

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* This is, of course, the Gaussian theorem that, if two systems have the same potential throughout any region, the identity extends to all places that can be reached from that region without crossing attracting matter.

3. The same procedure may now be extended to a scalar potential, which is propagated in time, *i.e.*, which satisfies a characteristic equation involving the time as a variable.

We consider first the simplest case, that of a velocity potential ϕ, say of sound-waves in air, which is propagated with velocity c in accordance with the characteristic equation

$$c^{-2}\frac{d^2\phi}{dt^2}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}.$$

It is a necessary preliminary to ascertain what distribution of sources on a surface S will create given discontinuity in the values of ϕ, and of its normal gradient, in crossing the surface, it being clear that such discontinuities in ϕ and in $d\phi/dn$ constitute the most general type involving only first differential coefficients of ϕ that can exist.

The velocity potential propagated from a simple point-source* variable with the time, of strength $f(t)$, is, with this characteristic equation, $\frac{1}{r}f\left(t-\frac{r}{c}\right)$. The velocity potential propagated from a doublet consisting of simple sources $+f(t)$ and $-f(t)$ separated by an interval δn is consequently $\delta n\,\frac{d}{dn}\left\{\frac{1}{r}f\left(t-\frac{r}{c}\right)\right\}$; thus, for a doublet of strength $F(t)$, the equivalent of $f(t)\,\delta n$, it is $\frac{d}{dn}\left\{\frac{1}{r}F\left(t-\frac{r}{c}\right)\right\}$, in which the function F comes under the differentiation. Within a region of such small extent that the functions $f(t)$ and $F(t)$ do not sensibly change in the time required for the disturbance to pass across it, these velocity potentials are of types $\frac{f(t)}{r}$ and $F(t)\frac{d}{dn}\frac{1}{r}$, so far as they relate to sources inside the region of which $f(t)$ and $F(t)$ are the strengths at this interval of time; for this modification only neglects lower inverse powers of r than those that are retained. Thus, for such a region enclosing an element δS of the surface S, and at times for which the functions $f(t)$ and $F(t)$ do not there change abruptly, the velocity potentials, subject to exceptions to be presently encountered in the case of double sheets, take, throughout any time of the order above specified, the form of simple gravitational potentials, the circumstance of propagation not sensibly interfering.†

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* [In this section (but not in § 4) a source is the analogue of a mass, the flux from unit source being 4π.]

† The very elegant and fruitful application of this principle, conjoined with the Helmholtz-Kirchhoff continuation-formula, by Helmholtz (*loc. cit.*), to trace the vibratory connection established between large regions through connecting channels of dimensions small compared with the wave-length, including the problem of reflexion from an aperture, may be recalled; cf. the simplified analysis in the chapter on Resonators in Lord Rayleigh's *Theory of Sound*.

We are therefore invited to follow the procedure of Coulomb and Laplace for the ordinary potential, and investigate the discontinuities arising from a surface distribution of simple sources and one of doublets orientated normally to the surface. The discontinuities, on crossing the surface at any point, clearly arise solely from the distribution over a surface element δS surrounding that point: for the disturbances propagated from the more distant sources are virtually the same at points on the two sides of the surface, whose distance apart is infinitesimal compared with the linear dimensions of δS, so that, as regards their effect, no discontinuity can arise.

Taking first, then, the case of a simple surface density $\sigma(t)$ spread over δS, which we may take to be uniform all over it at each instant, its effect is to transmit towards both sides a train of plane waves with fronts parallel to δS, which remain plane until the distance n to which they have travelled becomes comparable with the linear dimensions of δS. For them the value of $d\phi/dn$ at distance n and time t is $2\pi\sigma\left(t-\frac{n}{c}\right)$ but with different sign on the two sides; such a surface distribution $\sigma(t)$ of simple sources thus accounts for a discontinuity in $d\phi/dn$ of amount $4\pi\sigma(t)$, but introduces no discontinuity in ϕ itself.

This result now assists us to analyze the circumstances of a sheet of normal doublets of strength $s(t)$ per unit area; for we can replace it by two simple parallel sheets of densities $+\sigma_1(t)$ and $-\sigma_1(t)$, at an infinitesimal distance δn apart, such that $\sigma_1(t)\delta n = s(t)$. At a point at distance n from the sheet of density $+\sigma_1(t)$ the values of $d\phi/dn$ at time t arising from these two sheets are, as above, $\pm 2\pi\sigma_1\left(t-\frac{n}{c}\right)$ and $\pm 2\pi\sigma_1\left(t-\frac{n+\delta n}{c}\right)$ in which signs are to be determined by the sides of the respective component sheets on which this point lies. If the point is not between the sheets, the signs are opposite, and the sum for both is $-2\pi\,\frac{\delta n}{c}\,\frac{d}{dt}\,\sigma_1\left(t-\frac{n}{c}\right)$, which is the value of $d\phi/dn$ due to the element $s\delta S$; but it has *the same sign* on both sides of the double sheet; so that in crossing the double sheet there is no discontinuity in the value of $d\phi/dn$, though the element $s\,dS$ of the double sheet contributes $\frac{4\pi}{c}\,\frac{ds}{dt}$ to that quantity on each side. But, between the sheets, the value of $d\phi/dn$ arising from them is of a higher order of magnitude, being a sum instead of a difference, and is $4\pi\sigma_1\left(t-\frac{n}{c}\right)$, or simply $4\pi\sigma_1(t)$ when $\sigma_1(t)$ is not discontinuous; and this value integrated across the interval δn gives a discontinuity in ϕ itself, on crossing the double sheet, of amount $4\pi\,\delta n\,\sigma_1(t)$, that is, $4\pi s(t)$.

Collecting these results, we see that a discontinuity in ϕ over a surface S of amount $\chi(x, y, z, t)$ and a discontinuity in $d\phi/dn$ equal to $\psi(x, y, z, t)$ over the same surface are accounted for respectively by a double sheet on the surface of strength s equal to $\chi/4\pi$, and a single sheet of density σ equal to $\psi/4\pi$.*

We are thus in a position to proceed exactly as in § 2. Consider any system of sources, and let ϕ, a function of (x, y, z, t), be their velocity potential in infinite free space. Assign any surface S, dividing space into two regions A and B, and let m_A and m_B stand for the sources as divided between the two regions. What distribution of sources would give rise to a velocity potential equal to ϕ in region A and equal to zero in region B? Clearly the sources m_A, together with a distribution (σ, s) over S given by $4\pi\sigma = -\dfrac{d\phi}{dn}$, $4\pi s = \phi$; for, if ϕ' is the potential arising from this distribution, and Φ is a function equal to ϕ in region A and to zero in region B, then $\Phi - \phi'$ will be a potential having no singularities or discontinuities throughout infinite space, and must therefore be null by simple physical intuition, or analytically by the usual type of theorem of determinacy based on the energy of disturbance being of necessity essentially positive. As the total effect within the region B is null, we can thus say that ϕ_A, the part of it arising from sources m_A outside the region, is given at time t by

$$-4\pi\phi_A = -\int\frac{dS}{r}\left[\frac{d\phi}{dn}\right] + \int dS\,\frac{d}{dn}\frac{[\phi]}{r},$$

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* These discontinuities are, I find, treated in elegant form by Poincaré (*loc. cit. ante*) for the more restricted case of simple harmonic vibrations of type e^{ipt}. The characteristic equation then assumes the special form

$$\nabla^2\phi + p^2\mathrm{c}^{-2}\phi = 0,$$

and the potential of a source m is $m\dfrac{e^{i\kappa r}}{r}$ where $\kappa = p/\mathrm{c}$. And, if there is a volume distribution of sources constituting a density ρ, the equation is modified, for purposes of continuous analysis, to

$$\nabla^2\phi + p^2\mathrm{c}^{-2}\phi = -4\pi\rho.$$

The application of Green's theorem is direct and simple—in fact in Helmholtz's original manner. As regards discontinuities in crossing surface distributions of sources, the key to the discussion is the remark that the potential of a source may be written in the form $\dfrac{m}{r} + m\dfrac{e^{i\kappa r}-1}{r}$, of which the latter part can never become infinite; so that the discontinuities are just the same as for the gravitational potential, corresponding to the statement in the text.

[*February* 14*th*.—So in the more general circumstances here treated $r^{-1}f\left(t - \dfrac{r}{\mathrm{c}}\right)$ can, under the restrictions specified, be expanded for small values of r by Taylor's theorem in the form $\dfrac{f(t)}{r} - \dfrac{1}{\mathrm{c}}f'\left(t - \theta\dfrac{r}{\mathrm{c}}\right)$, $(\theta < 1)$, of which only the first term contributes to interfacial discontinuities.]

in which $[\phi]$ and $[d\phi/dn]$ are the values of these quantities for the element δS at time $t-r/c$. This formula expresses the vibrational potential due to sources m_A within a surface S, throughout the region B outside that surface, as determined by the values which it and its gradient assume at that surface. It is, so to speak, an *analytical continuation* beyond that surface of a function satisfying the aforesaid characteristic differential equation. Such a continuation must be unique, and it is determined by the values assumed by ϕ alone on the surface: as therefore $d\phi/dn$ is determined by a knowledge of ϕ over the surface, the data for the formula here expressing the continuation are redundant;[*] if arbitrarily assigned, they will usually be self-contradictory, and the formula thus nugatory.[†] Moreover, the formula determines ϕ_A in terms of the surface distribution of ϕ, equal to $\phi_A+\phi_B$, where ϕ_B is due to an entirely arbitrary distribution of sources within the region B to which the formula relates. Thus the quantities integrated in it are very widely arbitrary, and the element of the integral corresponding to δS in no sense represents any influence actually propagated from that part of the surface. The formula is purely analytical, and in no degree a mathematical formulation of the physical principle of Huygens which relates to propagation of actual disturbance. In fact, if m_B vanishes, the formula represents a distribution of surface disturbances which does not radiate at all into the region A; whereas, if the doctrine of rays and of Huygens' (or rather Fresnel's) zones could really be derived from the formula by a mathematical process devoid of limitations, radiation ought to reach every internal point P along the directions of the normals drawn from P to the surface S.

4. Similar considerations are applicable to functions determined by an æolotropic characteristic equation: for example, they apply to the

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[*] With regard to the remarks in the text that, for the determination of a field of physical activity in terms of the succession of disturbances over its boundary, the double set of *data* over the boundary are redundant, an important reservation must, however, be made when the field is of limited extent. In that case the field possesses a system of free types of vibration in which either single set of boundary disturbances is null; and when the two sets are of such nature that the energy lost at a surface depends on their product, *i.e.*, when they may be characterized generally as *data* of force and of velocity, these free vibrations, when once excited, will go on for ever in the absence of viscosity: each type constitutes a spectrum of definite related periods, usually of vast number, when harmonically analyzed. Each type thus forms an undetermined addition to the disturbance that is determined by the succession of values in time of that set of boundary *data* which it does not alter, an addition which, however, is made definite by a knowledge of the state of the whole region at the time from which the analysis begins; if the beginning is remote, and slight viscosity of amount otherwise unimportant is assumed, the influence of the initial state of the system dies away and the indeterminacy is practically non-existent. The nature of the transition to the gravitational potential, where no such reservations occur, on making the velocity of propagation infinitely great, is readily recognizable.

[†] In the demonstration this is safeguarded by both being taken as due to m_A and m_B.

equation of propagation of heat in a crystalline medium, expressible in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\rho\frac{d\theta}{dt},$$

where (u, v, w) is the current of heat, determined, in terms of the gradient of the temperature θ, by the constitution of the medium in the form

$$u=\kappa_1\frac{\partial\theta}{\partial x}+\kappa_{12}\frac{\partial\theta}{\partial y}+\kappa_{13}\frac{\partial\theta}{\partial z},\qquad v=\kappa_{21}\frac{\partial\theta}{\partial x}+\kappa_2\frac{\partial\theta}{\partial y}+\kappa_{23}\frac{\partial\theta}{\partial z},$$

$$w=\kappa_{31}\frac{\partial\theta}{\partial x}+\kappa_{32}\frac{\partial\theta}{\partial y}+\kappa_3\frac{\partial\theta}{\partial z},$$

in which $\kappa_{rs}=\kappa_{sr}$, unless the medium has rotational structure or quality.

If the surface element δS has the posure (λ, μ, ν), and it carries a surface density σ of simple sources of heat, they will produce equal and opposite gradients of temperature on its two sides; but opposed tangential gradients would involve discontinuous temperature; hence the gradient on either side due to them must be wholly along the normal, say $(\lambda, \mu, \nu)\,d\theta/dn$. Also

$$\tfrac{1}{2}\sigma=\lambda u+\mu v+\nu w=\kappa_{\lambda\mu\nu}\,d\theta\,/\,dn,$$

where $\kappa_{\lambda\mu\nu}=\kappa_1\lambda^2+\kappa_2\mu^2+\kappa_3\nu^2+(\kappa_{23}+\kappa_{32})\mu\nu+(\kappa_{31}+\kappa_{13})\nu\lambda+(\kappa_{12}+\kappa_{21})\lambda\mu$. Thus a discontinuity of amount $\psi(t)$ in the value of $d\theta/dn$ involves a distribution of sources on the surface of density $\kappa_{\lambda\mu\nu}\psi(t)$. By superposition of the effects of two sheets $+\sigma$ and $-\sigma$ at an infinitesimal distance τ apart, it appears that a double sheet of intensity $s(=\sigma\tau)$ does not introduce any discontinuity in $d\theta\,/\,dn$ in crossing it, but only one of amount $\kappa_{\lambda\mu\nu}^{-1}s$ in θ itself. Thus, when once the law of propagation of temperature, around a point-source which changes with the time, is found, as can be done in finite terms by the usual Fourier analysis,[*] we can express the distribution of temperature throughout any region in terms of the distribution in space and time of the temperature and its normal gradient along the boundary of the region.

5. The case of electrodynamic or optical propagation in any medium, isotropic or otherwise, can be immediately dealt with, the nature and source of singularities in the electric vectors being well known.[†] A discontinuity at any surface is completely expressed by the amounts of

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[*] [For a homogeneous medium, see Maxwell, *Electricity*, § 301. The statements above concerning the discontinuities in θ arising from distributions of sources may be verified analytically from this expression, or more briefly by linear transformation of the space so as to reduce the characteristic equation to the Laplacean form, in the manner of Stokes, *Camb. and Dub. Math. Journal*, 1851; *Collected Papers*, Vol. III., p. 203.]

[†] Analytically, this case supplies the analysis for the vectorial characteristic equation

$$c^2\nabla^2(P, Q, R)=\frac{d^2}{dt^2}(P, Q, R)\quad\text{subject to}\quad\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0;$$

and for the æolotropic generalization thereof.

the abrupt changes in the tangential electric and magnetic forces; for the normal components of the forces at the two sides of the surface are determined from the tangential ones directly by the fundamental circuital relations of Ampère and Faraday, and therefore definite discontinuity in them also is involved, being demanded solely by the intrinsic constitution of the æther, whatever be the sources that are supplied on the interface. Now, discontinuity in the tangential magnetic force implies simply a surface stream of electricity in δS, at right angles to the direction of the change in this force, and equal in intensity to it divided by 4π. The question arises whether all these elements of flux fit together into a circuital current sheet on the interface, or whether they lead to accumulation of electricity on it through not satisfying the condition of continuity of flow. If the current sheet were circuital, so as to produce no accumulation of charge, the tangential magnetic force would be restricted by being derived from a surface potential. Such a restriction is not called for; indeed, in the present case of purely vibratory motions, the slight accumulations of electricity provide potential energy for the system. If the physical constant of the transmitting medium, now æther *plus* matter, were complex, so that conduction currents would exist, there could be more permanent electric accumulation; but in all cases the amount of charge thereby supplied is required to produce the accompanying discontinuity in the normal electric force, which is thus provided for.

Any discontinuity in the tangential electric force, provided it has a surface potential, is accounted for by a distribution of varying electric doublets normal to the surface. These may be considered to constitute a double current-sheet, not circuital, but with accompanying double sheet of electric density: of this the former part accounts for the accompanying discontinuity of the normal magnetic force. But when such a surface potential does not exist we must take the sources on δS to constitute an element of a "magnetic current-sheet," which may be considered to arise from a varying sheet of tangential magnetization, after the analogy of the electric current-sheet of the previous case.

The result then specifies a current-sheet of varying intensity, equal at each instant to the discontinuity in the tangential magnetic force divided by 4π, and at right angles to its direction; also a sheet of "magnetic current" of varying intensity, equal at each instant to the discontinuity in the tangential electric force divided by 4π, and at right angles to its direction; these together form the cause of all the discontinuity that can exist at the interface.*

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* It is possible, theoretically, to construct solvable cases of vibrating systems, in Green's manner, by drawing the lines of electric force, for standing waves due to any known system

The previous mode of argument is now applicable. Consider the field of activity throughout space of any electrodynamic system; divide space into two regions A and B, and let the parts of the sources that are situated in them be denoted by m_A and m_B. Consider another field constituted of this one in the region A, and zero field in the region B; and specify the surface distribution of electric and magnetic sources that is required in addition to m_A, in order to constitute it. This superficial system with sign changed must produce an effect in region B the same as that of the m_A sources situated outside that region; and we have thus learned how to effect the continuation of any electrodynamic field beyond a surface up to which it is known.

As in the previous cases, the data of arbitrarily assigned tangential forces, both electric and magnetic, over the boundary, are redundant, and so, if given independently, will probably be self-contradictory. A knowledge of either tangential force by itself renders the problem physically determinate, and so involves knowledge of the other also.

This analysis holds good for the most general type of æolotropic medium, because no relations involving the constitution of the material medium have been employed. When the medium is æolotropic the disturbance emanating from a source of either type will spread in two-sheeted wave-fronts of the Fresnel type, and its distribution may be obtained in finite terms when the strength of the source varies in a general manner expressed by an arbitrary function. The case of a uniaxal medium has been dealt with in elegant manner by Prof. Conway, *loc. cit.**

[*February* 14*th*.—When the medium is free æther the expressions are as follows. If in crossing an element of surface δS there is discontinuity $4\pi f(t)$ in the tangential magnetic force, an electric current-sheet is required, of intensity $f(t)$, in the direction δs at right angles to the direction of this discontinuity. This will produce a magnetic field of intensity $-\sin(r, \delta s)\frac{d}{dr}\frac{1}{r}f\left(t-\frac{r}{c}\right)\delta S$ in circles around δs; also an electric field in part derived from a potential function $-c^2\cos(r, \delta s)\frac{d}{dr}\frac{1}{r}f_1\left(t-\frac{r}{c}\right)\delta S$,

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of sources in infinite space, and transforming one or more of their orthogonal trajectories, where such exist, into perfectly conducting surfaces. This procedure is always available for cylindrical or axial systems; but there does not appear to be any simple surface, other than the obvious one of a plane, which can play the same part as a sphere does in the theory of electrostatic images.

* For the adaptation to practical optical diffraction formulæ, the memoir of Kirchhoff may be referred to, or that of Prof. Love.

where $f_1(t)$ represents $\int f(t)\,dt$, and in part of intensity $\frac{1}{r} f'\left(t-\frac{r}{\mathrm{c}}\right)\delta S$ parallel to δs. If in crossing δS there is also discontinuity $4\pi\phi(t)$ in the tangential electric face, a magnetic current-sheet is required of intensity $\phi(t)$ in direction $\delta\sigma$ at right angles to it. This will produce a magnetic field in part derived from a potential function $-\cos(r, \delta s)\frac{d}{dr}\frac{1}{r}\phi_1\left(t-\frac{r}{\mathrm{c}}\right)\delta S$ and in part of intensity $\frac{1}{\mathrm{c}^2 r}\phi'\left(t-\frac{r}{\mathrm{c}}\right)\delta S$ parallel to $\delta\sigma$; also an electric field of intensity $\sin(r, \delta s)\frac{d}{dr}\frac{1}{r}\phi\left(t-\frac{r}{\mathrm{c}}\right)\delta S$ in circles around $\delta\sigma$. These two fields, integrated over the surface S, form the continuation of an electrodynamic field which involves tangential magnetic force $4\pi f(t)$ and electric force $4\pi\phi(t)$ at this surface.]

6. This method of analytical continuation of fields of physical activity has now been sufficiently illustrated. It is to be observed that in no case may the bounding surface in these formulæ form a physical obstacle: the propagation is supposed to be free throughout all space, or, if there are obstacles, the law of propagation of disturbance from the various sources must take them into account and be modified thereby.

Having ascertained ways of distributing sources over any surface surrounding the origin of the disturbance in such manner as to produce the same effects, the way is open for explaining why it is only the parts of these sources that lie near the ray drawn to any external point that are usually effective at that point; and the law of their effect is at once expressible in its main features. But the argument must, in the nature of the case, be based on general considerations, rather than exact analysis, as we have seen how to distribute sources in an infinite variety of ways so as to produce no effect at all.

We may finally briefly advert to the more directly dynamical treatment of diffraction theory which was set forth with exhaustive generality by Sir George Stokes in 1849, and in which may be recognized, as Poincaré has remarked (*loc. cit.*), a less ambiguous foundation for Huygens' principle. The mode of operation is to consider the state of the medium as given throughout, at any time, by the strain or displacement and the velocity of each element of its volume, and to compute thence its state at any subsequent time by considering each such element of volume thus disturbed as a source from which disturbance is propagated. The law of this propagation will, of course, depend on the nature and position of any obstacles that may exist in the medium: on the hypothesis of their absence it has been worked out with full generality for elastic vibratory media, and all subsequent applications to more special problems are virtually included in Stokes's analysis.

The expression of the disturbance produced at any point at subsequent times can be reduced from a volume integral to a surface integral, which must be some one of the type discussed above ; but the process of integration by parts prevents the terms from any longer representing real components derived from corresponding surface sources. The essential element in the theory is that a disturbance in an element of volume lasting for a small time is completely exhausted in sending out a definite spherical pulse of disturbance which travels ahead without leaving any tail behind it. The principle of Poisson for air-waves, that the disturbance at any point at time t depends only on the initial disturbance of the medium (including velocity, but *not* any spatial gradient) at the surface of a sphere of radius ct with this point as centre, has therefore general application. This is at the base of the principle of Huygens, for it shows that what happens at this point in the interval δt at time t depends on the initial state of a spherical shell of the medium of thickness $c\delta t$ and radius ct. If this shell is drawn so as just to meet the region of initial disturbance, it appears that the disturbance at its centre depends only on the initial state of disturbance at this place of contact, and as the exciting disturbance travels on the place of contact travels along the ray. The explanation of Young and Fresnel as to why the point is not disturbed at subsequent times by the more distant elements of the initial disturbance that are off the line of the ray essentially depends on the feature of clean compact propagation without residual effect above adverted to.*

An interesting question is suggested by Prof. Lamb's recent discussion, in these *Proceedings*, of the propagation of elastic waves in two dimensions, *e.g.*, waves on a stretched membrane. Here each pulse leaves a trail, the type of argument above sketched requires modification, and the type of the Fresnel formula for ray-propagation by diffractional interference is altered. The nature of the difference can be seen by considering the two-dimensional case as the analogue of one of cylindrical propagation in three dimensions of space: the preceding considerations then apply, as each element of disturbance travels in a pulse, but the Fresnel zone that is effective is determined by a sphere of radius ct, instead of a cylinder ; the rays persist, but the diffraction formula cannot be constructed on the basis of the disturbance from a cylindrical source, and must, in fact, be deduced from the three-dimensional one.

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* Cf. Sir G. Stokes's Wilde Lecture, *Proc. Manchester Lit. and Phil. Soc.*, 1897 [or in Vol. IV. of *Mathematical and Physical Papers*, now in the press].

ON FUNCTIONS OF SEVERAL VARIABLES

By H. F. BAKER.

[Received and Read January 8th, 1903.]

THIS paper is mainly concerned with the problem, suggested by Weierstrass (*Ges. Werke*, Vol. II., p. 163), of showing that a function without finite essential singularities can be expressed as a quotient of two integral functions. In the *Acta Math.*, Vol. XXII., 1898, M. Poincaré had worked out in more detail, for the case of periodic functions, the suggestion considered in his earlier paper in *Acta Math.*, Vol. II., 1883, of expressing the real part of an integral function by the potential of the construct over which the function vanishes; this potential is a $(2p-2)$-fold integral, if p be the number of complex variables. In the *Trans. Camb. Phil. Soc.*, Vol. XVIII., 1899, p. 431, the present writer showed that the imaginary part of the function could be introduced concurrently with the real part, and the whole expressed as a $(2p-1)$-fold integral. So far as the form of the subject of integration only is concerned this integral is a particular case of one suggested by Kronecker in 1869 (*Werke*, Vol. I., p. 198), but it differs in that it is taken, not over a closed $(2p-1)$-fold, but over a $(2p-1)$-fold limited by a $(2p-2)$-fold defined by the vanishing of a function of the complex variables. In the recently published *Acta Math.*, Vol. XXVI., pp. 57–80, M. Poincaré has again given a solution of Weierstrass's problem, "qui tient pour aussi dire le milieu entre celle de M. Cousin et la mienne," depending upon an infinite series of $(2p-1)$-fold integrals. It would appear that this solution also can be deduced from the $(2p-1)$-fold integral used in the writer's previous paper, and the main object of the present paper is to explain this as simply as possible. Another object, however, is to attempt to put a point of view which appears to open a whole series of important questions.

CONTENTS.

§ 1. A certain $(n-1)$-fold Integral.—§ 2. Properties for $n=2$.—§ 3. In general for a Closed $(n-1)$-fold.—§ 4. New Theorem for Unclosed $(n-1)$-fold.—§ 5. Use to define an Integral Function vanishing in a given manner.—§ 6. Convergence Conditions; case of a Polynomial.—§§ 7, 8. Case of Periodic Functions.—§ 9. Application to Weierstrass's Problem.—§ 10. Deduction of M. Poincaré's last Solution.—§ 11. Of Functions and Power Series in several Variables.—§ 12. Mittag-Leffler's Theorem for Functions of several Variables.—§ 13. Example of this.

1. Independent complex variables being denoted by $(\xi_1 \dots \xi_p)$ or $(\tau_1 \dots \tau_p)$, put $\xi = x_{2s-1}+ix_{2s}$, $\tau_s = t_{2s-1}+it_{2s}$, and speak of $(x_1 \dots x_n)$ or

$(t_1 \dots t_n)$ where $n = 2p$, as the coordinates of a point in n dimensions, calling the positive square root of $(x_1-t_1)^2+\dots+(x_n-t_n)^2$ the distance between these points. Imagine an analytical $(n-1)$-fold, expressed near any point of itself by a single power series $F(x_1 \dots x_n) = 0$, the function $F(t_1 \dots t_n)$ passing from negative to positive as (t) passes from the inside, so called, to the outside; let $(d_k x_1 \dots d_k x_n)$, for $k = 1 \dots (n-1)$, be independent sets of differentials on the surface, and dS_{n-1}, the so-called element of extent, be the positive square root of the determinant formed by multiplying into itself, row into row, the array of $(n-1)$ rows and n columns $(d_k x_1 \dots d_k x_n)$; and, taking $dt_1 \dots dt_n$ towards the outside from the point x of the $(n-1)$-fold so as to satisfy the $(n-1)$ equations $dt_1 d_k x_1+\dots+dt_n d_k x_n = 0$, let $l_r = h^{-1}\partial F/\partial t_r$, $l_1^2+\dots+l_n^2 = 1$, where h is real and positive. Further put

$$P(x, t) = -\frac{1}{n-2}\left[(x_1-t_1)^2+\dots+(x_n-t_n)^2\right]^{-\frac{1}{2}(n-2)}$$

or

$$= \tfrac{1}{2}\log\left[(x_1-t_1)^2+(x_2-t_2)^2\right],$$

the latter when $n = 2$, and

$$H = P(x, t)-P(x, 0)+\left(t\frac{\partial}{\partial x}\right)P(x, 0)-\dots+\frac{(-1)^{k-1}}{k!}\left(t\frac{\partial}{\partial x}\right)^k P(x, 0)$$

where k is a definite positive integer which may be zero and

$$\left(t\frac{\partial}{\partial x}\right) = t_1\frac{\partial}{\partial x_1}+\dots+t_n\frac{\partial}{\partial x_n}.$$

Then, taking (x) for a point of the $(n-1)$-fold, and (t) for any point of space, consider the integral

$$F = \frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

where $\varpi = 2\pi^p/(p-1)!$ is the total solid angle in $2p$ dimensions, and $f(\xi)$ is a function of $\xi_1 \dots \xi_p$ which therefore satisfies the p equations $(\partial/\partial x_{2s-1}+i\,\partial/\partial x_{2s})f(\xi) = 0$.

2. Notice, first, that for $n = 2$, putting $l_1 = dx_2/ds$, $l_2 = -dx_1/ds$, $dS_{n-1} = ds$, the integral is

$$\frac{1}{2\pi i}\int f(\xi)\,d\xi\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)$$

and

$$\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2} = \frac{x_1-t_1-i(x_2-t_2)}{(x_1-t_1)^2+(x_2-t_2)^2}-\frac{x_1-ix_2}{x_1^2+x_2^2}+\dots+\frac{(-1)^{k-1}}{k!}\left(t\frac{\partial}{\partial x}\right)^k\frac{x_1-ix_2}{x_1^2+x_2^2}$$

$$= \frac{1}{\xi-\tau}-\frac{1}{\xi}-\frac{\tau}{\xi^2}-\dots-\frac{\tau^k}{\xi^{k+1}};$$

so that, taken round a closed curve within which $f(\tau)$ is not singular, it gives

$$\epsilon_1 f(\tau) - \epsilon_2 \left[f(0) + \frac{\tau}{1!} f'(0) + \frac{\tau^2}{2!} f''(0) + \ldots + \frac{\tau^k}{k!} f^{(k)}(0) \right]$$

where ϵ_1, ϵ_2 are both unity if τ and the origin be within the closed curve, but one or both of them otherwise zero; while, if we put $2\pi i$ in place of $f(\xi)$, and integrate from infinity along a curve not passing through the origin, up to ξ, the integral has the value

$$\log\left(1 - \frac{\tau}{\xi}\right) + \frac{\tau}{\xi} + \frac{\tau^2}{2\xi^2} + \ldots + \frac{\tau^k}{k\xi^k},$$

which is the logarithm of Weierstrass's prime factor for an integral function having ξ for a simple zero.

3. Consider now values of n greater than 2. The integral F is unaltered by a slight deformation of the $(n-1)$-fold of integration, neither a singularity of $f(\tau)$, nor the point τ, nor the origin being passed over; for the condition this should be so is only

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) f(\xi) \left(\frac{\partial H}{\partial x_1} - i\frac{\partial H}{\partial x_2}\right) + \left(\frac{\partial}{\partial x_3} + i\frac{\partial}{\partial x_4}\right) f(\xi) \left(\frac{\partial H}{\partial x_3} - i\frac{\partial H}{\partial x_4}\right) + \ldots = 0,$$

namely,

$$f(\xi)\left[\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \ldots\right] = 0;$$

further, when taken over a small closed $(n-1)$-fold given by

$$(x_1 - t_1)^2 + \ldots + (x_n - t_n)^2 = \epsilon^2,$$

within which we suppose $f(\xi)$ is not singular and the origin is not found, the integral is equal to

$$\frac{1}{\varpi}\int f(\xi) \left\{ \frac{x_1 - t_1 + i(x_2 - t_2)}{\epsilon} \left[\frac{x_1 - t_1 - i(x_2 - t_2)}{\epsilon^n} - \frac{x_1 - ix_2}{(x_1^2 + \ldots)^{\frac{1}{2}(n)}} + \ldots \right] + \ldots \right\} \epsilon^{n-1}\, d\omega,$$

which, when $\epsilon = 0$, gives $f(\tau)$; and, in fact, if $(y_1 \ldots y_n)$ be any point, $\eta_1 = y_1 + iy_2$, &c., the integral

$$J = \frac{1}{\varpi}\int f(\xi) \left\{ (l_1 + il_2)\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) P(x, y) + \ldots \right\} dS_{n-1}$$

taken over $(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2 = \epsilon^2$ is exactly equal to $f(\eta)$, independently of ϵ, provided $f(\xi)$ is throughout non-singular; and hence, as may be proved directly,

$$\left(\tau_1\frac{\partial}{\partial \eta_1} + \ldots + \tau_p\frac{\partial}{\partial \eta_p}\right) f(\eta)$$

$$= \left(t\frac{\partial}{\partial y}\right) J = \frac{1}{\varpi}\int f(\xi) \left\{ (l_1 + il_2)\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\left(t\frac{\partial}{\partial y}\right) P(x, y) + \ldots \right\} dS_{n-1},$$

or, say, as
$$\frac{\partial P(x, y)}{\partial y_i} = -\frac{\partial P(x, y)}{\partial x_i},$$

$$\left(\tau\frac{\partial}{\partial\eta}\right)f(\eta) = -\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\left(t\frac{\partial}{\partial x}\right)P(x, y)+\dots\right\}dS_{n-1},$$

and in general

$$\left(\tau\frac{\partial}{\partial\eta}\right)^m f(\eta) = \frac{(-1)^m}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\left(t\frac{\partial}{\partial x}\right)^m P(x,y)+\dots\right\}dS_{\ -1}.$$

Thus, our integral, denoted above (§ 1) by F, when taken over a closed $(n-1)$-fold within which $f(\tau)$ is not singular, has, as for $n=2$, the value $\epsilon_1 f(\tau)-\epsilon_2(1, \tau)_k$ where ϵ_1, ϵ_2 are both unity when (τ) and the origin are included but either or both otherwise zero, and $(1, \tau)_k$, arising as the value when $y=0$ of

$$f(\eta)+\frac{1}{1!}\left(\tau\frac{\partial}{\partial\eta}\right)f(\eta)+\frac{1}{2!}\left(\tau\frac{\partial}{\partial\eta}\right)^2 f(\eta)+\dots+\frac{1}{k!}\left(\tau\frac{\partial}{\partial\eta}\right)^k f(\eta),$$

is the integral polynomial in $\tau_1 \dots \tau_p$ of order k constituting the initial terms of the Taylor expansion of $f(\tau)$ about the origin.

4. It appears that the integral F also represents a function of $\tau_1 \dots \tau_p$ when the $(n-1)$-fold over which it is taken is not closed, provided its boundary consists of an $(n-2)$-fold given by the vanishing of functions of the complex variables. We have $\partial H/\partial t_m = -\partial K/\partial x_m$ where K is obtained from H by changing k into $k-1$; hence

$$\frac{\partial F}{\partial t_{2r-1}}+i\frac{\partial F}{\partial t_{2r}}$$
$$= -\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_{2r-1}}+i\frac{\partial}{\partial x_{2r}}\right)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1}.$$

Now let
$$\zeta_m = \frac{1}{\varpi}\int\left\{(l_{m,1}+il_{m,2})f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-2}$$

integrated over the $(n-2)$-fold which bounds the $(n-1)$-fold, where $l_{r,s}$ denote the $\frac{1}{2}n(n-1)$ direction cosines of the $(n-2)$-fold; this contour integral can, by the generalized Green-Stokes theorem, be expressed as an integral over the $(n-1)$-fold, given by

$$\zeta_m = \frac{1}{\varpi}\int l_m\left\{\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$
$$-\frac{1}{\varpi}\int\left\{(l_1+il_2)\frac{\partial}{\partial x_m}f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1},$$

of which the first integral has an identically vanishing integrand; thus

$$\begin{aligned}\zeta_{2r-1}+i\zeta_{2r} &= -\frac{1}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_{2r-1}}+i\frac{\partial}{\partial x_{2r}}\right)f(\xi)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1}\\ &= -\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial}{\partial x_{2r-1}}+i\frac{\partial}{\partial x_{2r}}\right)\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1}\\ &= \frac{\partial F}{\partial t_{2r-1}}+i\frac{\partial F}{\partial t_{2r}}.\end{aligned}$$

This is true whatever be the character of the bounding $(n-2)$-fold, provided $f(\xi)$ remain finite on the $(n-1)$-fold and $(n-2)$-fold, neither the origin or the point (t) be upon these, and provided any singularities which either construct may possess do not affect the validity of the transformation theorem; in case the bounding $(n-2)$-fold be given by the vanishing of functions of the complex variables (τ), we have, at any point in the neighbourhood of which one such function gives the $(n-2)$-fold $l_{2r-1,\,2s-1}+il_{2r-1,\,2s}+i(l_{2r,\,2s-1}+il_{2r,\,2s}) = 0$, and therefore $\zeta_{2r-1}+i\zeta_{2r} = 0$, giving $\partial F/\partial t_{2r-1}+i\partial F/\partial t_{2r} = 0$, which proves the theorem in question. The theorem therefore holds for an $(n-2)$-fold extending to infinity, for instance one given by the vanishing of a single integral function, provided the integrals are convergent. For more details we refer to the writer's paper, *Camb. Phil. Trans.*, Vol. XVIII., where the theorem was also used for the case of $f(\xi) = 1$.

5. Suppose now that $\Theta(\tau_1 \dots \tau_p)$ is an integral function of $\tau_1 \dots \tau_p$, capable, therefore, of expression about any finite point as a power series converging for all finite values of $\tau_1 \dots \tau_p$. The equation $\Theta(\tau) = 0$ defines an $(n-2)$-fold which we may denote by I; we suppose that, when (τ^0) is a point subject to no conditions but $\Theta(\tau^0) = 0$, the lowest terms when $\Theta(\tau)$ is arranged in ascending powers of the differences $\tau_1-\tau_1^0 \dots \tau_p-\tau_p^0$ are linear; then the intersection in the neighbourhood of (τ^0) of the $(n-2)$-fold I with any 2-fold given by such equations as

$$\tau_2-\tau_2^0 = m_2(\tau_1-\tau_1^0), \quad \dots, \quad \tau_p-\tau_p^0 = m_p(\tau_1-\tau_1^0)$$

consists only of the point (τ^0) counted once; and the increment of $\log\Theta(\tau)$, for a closed 1-fold path lying in this 2-fold, is $2\pi i$. Imagine now a very large closed $(n-1)$-fold S, afterwards to pass off entirely to infinity—it will enclose part of I, intersecting this in $(n-3)$-folds; and imagine that the interior of S is rendered simply connected by an $(n-1)$-fold diaphragm P bounded on one part by the $(n-2)$-fold I and on the other by the $(n-2)$-fold in which P intersects S; this diaphragm may be regarded as arising by the coincidence of the two $(n-1)$-fold sheets of an $(n-1)$-fold Π which encloses I and shuts it off from the interior of S, save only that, as

Π degenerates into the two sides of P, it gives rise also to a cylindrical $(n-1)$-fold Σ surrounding, and having every point of itself very near to, some point of the $(n-2)$-fold I. The portion of Π interior to S, and the whole of S other than the ultimately vanishing portion cut out from it by the $(n-2)$-fold in which Π intersects S, form together a closed $(n-1)$-fold in which $\log\Theta(\tau)$ is single valued and everywhere developable. Thus the integral $\frac{1}{\varpi}\int \log\Theta(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$ taken over this has a value $\log\Theta(\tau)-(1,\tau)_k$. Now let Π degenerate into the two sides of the diaphragm P, together with the cylindrical $(n-1)$-fold Σ surrounding I, and this itself degenerate into I. As the values of $\log\Theta(\xi)$ on the two sheets of Π differ by $2\pi i$, the integral contains a portion

$$\Phi=\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

taken over one side of the diaphragm P. As regards the portion taken over Σ, it can be shown that, if the points of Σ be defined by equations

$$\tau_m-\tau_m^0=\frac{1}{h}\left(\frac{\partial\Theta}{\partial\tau_m^0}\right)'\epsilon(\cos\theta+i\sin\theta)$$

where (τ^0) is a point of I, and $(\partial\Theta/\partial\tau_m^0)'$ is the conjugate complex of $\partial\Theta/\partial\tau_m^0$, and h the positive square root of the sum of the squares of the moduli of such quantities, then the element of extent of Σ has a form $dS_{n-1}=\epsilon\, d\theta\, dS_{n-2}$ where dS_{n-2} is a corresponding element of extent for I; it is clear that $\epsilon\log\Theta$ vanishes when ϵ vanishes; we shall assume, therefore, that when Σ degenerates into I the $(n-1)$-fold integral over Σ vanishes. We have then

$$\log\Theta(\tau)=(1,\tau)_k+\Phi+W$$

where $$W=\frac{1}{\varpi}\int\log\Theta(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

taken over the $(n-1)$-fold S with the exclusion of a vanishing portion bounded by the $(n-2)$-fold which is the limit of the intersection of Π with S. We notice that Φ is not continuous across P.

Now suppose the $(n-1)$-fold S to be taken at greater and greater distance; assume that the integer k can be chosen so that the $(n-1)$-fold integral Φ over one side of the diaphragm P remains convergent when taken over what ultimately becomes an infinite diaphragm P bounded only by the zero $(n-2)$-fold I; it is by no means asserted that this is so for any integral function Θ, but it appears that it is so for a very extensive class of functions, including those for which I is periodic. It follows then from the equation above that W does not become infinite or indeterminate for general positions of (τ); it is part of the purpose of the introduction of the $(n-1)$-fold Π to ensure that (τ) is not infinitely near to the $(n-2)$-fold I. But, in fact, the form of W shows that it cannot change from definite and finite to indefinite or infinite in consequence of any variation in the position of (τ) so long as this point remains in the interior of S; the indefinite approach of (τ) to the $(n-2)$-fold I introduces no infinite elements into W. We infer therefore, since by § 4 the integral Φ is a function of $\tau_1 \dots \tau_p$, that the limit of W when S passes off to infinity is an integral function of $\tau_1 \dots \tau_p$, possibly a constant or zero. This indeterminateness, too, in the expression for $\log \Theta(\tau)$ was to be anticipated: since the other integral Φ, over P, depends only on the position of the zero $(n-2)$-fold I, which is the same for any function Θe^{Ψ}, where Ψ is an integral function, as for Θ.

We have then the result following:—

> An integral function having the zero $(n-2)$-fold I, supposed of multiplicity 1, is given by the exponential of the integral
>
> $$\Phi = \frac{2\pi i}{\varpi} \int \left\{ (l_1 + i l_2) \left(\frac{\partial H}{\partial x_1} - i \frac{\partial H}{\partial x_2} \right) + \dots \right\} dS_{n-1},$$
>
> taken over an infinite $(n-1)$-fold diaphragm P bounded by I, provided k can be chosen so that this is convergent.

We shall, however, find it desirable to bear in mind the more complete form

$$\log \Theta(\tau) = (1, \tau)_k + \frac{2\pi i}{\varpi} \int \left\{ (l_1 + i l_2) \left(\frac{\partial H}{\partial x_1} - i \frac{\partial H}{\partial x_2} \right) + \dots \right\} dS_{n-1} + W,$$

where W is the integral function arising as the limit of the $(n-1)$-fold integral previously put down, involving $\log \Theta(\xi)$ under the integral sign.

It has already been shown that this result is valid for the simple integral function of *one* variable $1-\tau/\xi$, the diaphragm P being then a curve coming from infinity and terminated at ξ; it is therefore also valid for any integral polynomial in the one variable τ, the diaphragm consisting then of several such curves, each terminated in one of the zeros of the polynomial; and therefore also valid for any integral function of τ of finite *genre*, the diaphragm consisting then of an infinite number of such curves. And, similarly, in the general investigation above we have not intended to assume that I consists of only one piece nor that P is simply connected; it may, moreover, be divided into unconnected portions by branches of I, in case this is, as would appear necessary for the case of periodic functions, a self-intersecting structure: it is believed, however, that the language employed can be understood in a sense suitable for all cases.

6. As a simple case where the assumption as to the convergence of the integral Φ over the $(n-1)$-fold diaphragm P is justified, we may take $p=2$, and $\Theta(\xi)=\xi_1-(a+ib)$; then the $(n-2)$-fold I is $x_1=a$, $x_2=b$, and the diaphragm P is given by $x_1=a$, $x_2<b$; also $l_1=1$, $l_2=l_3=l_4=0$; taking $k=0$, we find

$$\frac{2\pi i}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{b}\left\{\frac{a-t_1-i(x_2-t_2)}{[(a-t_1)^2+(x_2-t_2)^2+\dots]^2}-\frac{a-ix_2}{[a^2+x_2^2+\dots]^2}\right\}dx_2\,dx_3\,dx_4$$

$$=\log\left(1-\frac{\tau}{a+ib}\right).$$

In general, if the diaphragm P be such that for distant parts of space, at distance R from the origin, the extent of the diaphragm contained in n-fold extent V is at most of the order of magnitude of $R^{k-1}V$, the integral in question is convergent. For then, as the quantities $\partial H/\partial x_{2s-1}-i\,\partial H/\partial x_{2s}$ are ultimately of order $R^{-(n+k)}$, the portion of the integral over the part of the diaphragm for which $r<R<r_1$ is, in absolute value,

$$<\frac{\mu(r_1^n-r^n)r_1^{k-1}}{r^{n+k}}, \qquad <\mu(1+\epsilon)^k\,\frac{(1+\epsilon)^n-1}{\epsilon}\left(\frac{1}{r}-\frac{1}{r_1}\right)$$

where μ is a fixed quantity and $r_1=r(1+\epsilon)$.

It is, however, to be remarked that, just as we have been able to infer the convergence of the integral W from the assumption of the convergence of the integral Φ, so the converse process may be possible. In absolute value W is of the order of

$$\frac{1}{\varpi}\int|\log\Theta(\xi)|\,\{R^{-(n+k)}\}\,R^{n-1}\,d\omega,$$

and vanishes if the limit of $R^{-(k+1)}\log|\Theta(\xi)|$ is zero as (x) passes to infinity in any direction not asymptotic to the zero $(n-2)$-fold I. For instance, if $\Theta(\xi)$ be an integral polynomial, this is so for $k=0$. Thus

Any integral polynomial $\Theta(\tau)$ can be represented in the form

$$\frac{\Theta(\tau)}{\Theta(0)}=\exp\left[\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}\right]$$

where $H=P(x,t)-P(x,0)$ and the integral is over the infinite $(n-1)$-fold diaphragm limited by the $(n-2)$-fold on which $\Theta(\tau)=0$. It is supposed, as usual, that this $(n-2)$-fold is of multiplicity unity and does not contain the origin.

7. For the case of periodic functions consider first $p=1$. The diaphragm consists then, say, of straight lines directed from the origin,

one passing from each of the, say m, zeros in every parallelogram of periods, to infinity; the number of parallelograms in a circle of radius r, when r is large, is of the order r^2; the whole length of diaphragm in the annulus $r < R < r_1$ is then of order $m(r_1-r)r^2$; the ratio of this to the area of the annulus is

$$\frac{m}{\pi}\frac{(r_1-r)r^2}{r_1^2-r^2} = \frac{m}{\pi}r\frac{r}{r_1+r},$$

and is ultimately of order r; thus the condition of convergence is satisfied by taking $k = 2$.

For larger values of p, the $(n-2)$-fold I may be periodic in the sense that it is possible to divide n-fold space into period cells, the interior of any one of these being given by p equations

$$\tau_i = \lambda+2\omega_{i,1}\lambda_1+\ldots+2\omega_{i,2p}\lambda_{2p} \quad (i = 1, \ldots, p),$$

where λ is a constant and $\lambda_1 \ldots \lambda_{2p}$ are real variables each between 0 and 1, in such a way that the portion of I in every cell is a repetition of that in any one cell. In that case the $(n-1)$-fold diaphragm P is presumably not periodic, there being in a distant cell, in addition to the portion bounded by the part of I actually contained therein, also portions continued from less distant cells and bounded by parts of I contained in these. It appears sufficient in order to show that the condition for the convergence of the integral is satisfied, also in this case by taking $k = 2$, to remark that the $(n-1)$-fold integral over the diaphragm P can be reduced to an $(n-2)$-fold integral over the bounding $(n-2)$-fold I; or, similarly, that a part of the $(n-1)$-fold integral over a distant portion of P can be reduced to a boundary $(n-2)$-fold integral; and that the condition for the convergence of the $(n-2)$-fold integral is that for an integral of the form $\int dS_{n-2}/R^{n+k-1}$; assuming that the extent of I included in any period cell is finite, the convergence of this last is reducible to that of the Eisenstein series

$$\sum_{m_1=-\infty}^{\infty} \ldots \sum_{m_n=-\infty}^{\infty} [\phi(m_1 \ldots m_n)]^{-\frac{1}{2}(n+k-1)},$$

where $\phi(m_1 \ldots m_n)$ is a definite quadratic form in the n integers $m_1 \ldots m_n$; this series converges if $n+k-1 > n$ or $k = 2$.

8. When $k = 2$ the second partial differential coefficient

$$-\frac{2\pi i}{\varpi}\frac{\partial^2}{\partial\tau_i\partial\tau_j}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\}dS_{n-1}$$

is equal to $$-\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial H_0}{\partial x_1}-i\frac{\partial H_0}{\partial x_2}\right)+\dots\right\}dS_{n-1}$$

where $H_0=P(x,\ t)-P(x,\ 0)$; if 2ω, a set of p complex quantities $(a_1+ia_2,\ a_3+ia_4,\ \dots)$, be a period, it may be shown, by associating with any point (x) of the diaphragm of integration the equally arising point $x-a$, and with this the point $x-2a$, and so on, that the increment of the second partial differential coefficient, say $\wp_{ij}(\tau)$, which is given by

$$\wp_{ij}(\tau+2\omega)-\wp_{ij}(\tau)=-\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial K}{\partial x_1}-i\frac{\partial K}{\partial x_2}\right)+\dots\right\}dS_{n-1},$$

where $K=P(x-a,\ t)-P(x,\ t)$, is zero. Thus the quotient, for an integral function Θ whose logarithm is represented by the $(n-1)$-fold integral over the diaphragm P, expressed by $\Theta(\tau+2\omega)/\Theta(\tau)$, is the exponential of a linear function of $\tau_1\dots\tau_p$. From this it follows, if (ξ) be a distant point obtained by addition of a general period from a finite point (τ), that the quotient $\Theta(\xi)/\Theta(\tau)$ is the exponential of a quadratic function of n integers $m_1\dots m_n$; and the condition previously remarked for the evanescence of the integral W, that $R^{-(k+1)}\log|\Theta(\xi)|$ should ultimately vanish, is clearly satisfied by $k=2$; for R is the square root of a definite quadratic function of the n integers $m_1\dots m_n$.

The integral function Θ cannot be itself periodic; for, if we write $\Theta=U+iV$, the $(n-1)$-fold integral, taken over the perimeter of a period cell, $\int U\left(l_1\frac{\partial V}{\partial x_2}-l_2\frac{\partial V}{\partial x_1}+\dots\right)dS_{n-1}$ would then be zero; it is, however, equal to $\int\left\{\frac{\partial}{\partial x_1}\left(U\frac{\partial V}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(U\frac{\partial V}{\partial x_1}\right)+\dots\right\}dS_n$ taken through the cell, namely, to $\int\left\{\left(\frac{\partial U}{\partial x_1}\right)^2+\left(\frac{\partial U}{\partial x_2}\right)^2+\dots\right\}dS_n$. On the other hand, it is not the case that every integral function whose zero $(n-2)$-fold is periodic is such that its second partial differential coefficients are periodic. We have shown that an integral function with such periodic differential coefficients can be found whose zero $(n-2)$-fold is an arbitrary given periodic $(n-2)$-fold; any other integral function having the same zero $(n-2)$-fold, with the same multiplicity, is obtainable from this by multiplication with a factor which is the exponential of an integral function. Integral functions whose second partial differential coefficients have, as here, sets of simultaneous periods are those considered by Frobenius under the name of "Jacobian functions"; it follows from his investigations, sketched in the present writer's *Abelian Functions*, pp. 579 *et seq.*, that the periods cannot be taken arbitrarily, and that the functions can be expressed by theta functions. This property then, as follows from the remarks to be next made, attaches to all single-valued analytic functions of p variables without finite essential singularities which have $2p$ sets of simultaneous periods—a theorem stated (on the authority of Hermite) to have been known to Riemann.

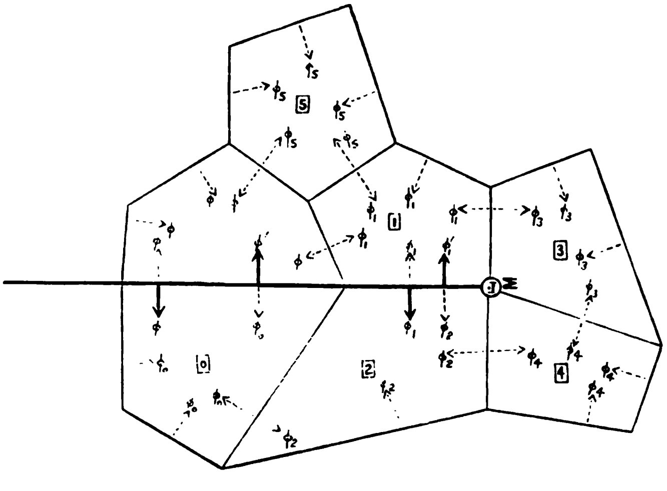
9. Suppose an analytic function is known to exist and to be single-valued and to be capable of expression about every finite point (τ^0) as the quotient of two power series converging in sufficiently near neighbourhood of (τ^0). If these series are both divisible by another power series vanishing at (τ^0), this factor may be supposed divided out (Weierstrass, *Ges. Werke*, t. II., p. 151); but further there exists a finite region about (τ^0), in the common region of convergence of the two series, such that, if the two series be developed about any point of the region, the resulting series have no common factor vanishing at this point (Weierstrass, *loc. cit.*, p. 154). Let a part of this region bounded by points all at the same distance from (τ^0) be called the proper region of (τ^0), and denoted by K_0, the expression of the function valid therein being ψ_0/ϕ_0, where ψ_0, ϕ_0 are power series in $\tau_1-\tau_1^0, \ldots, \tau_p-\tau_p^0$, having no common factor vanishing at any point of K_0. There may quite well be points of K_0 at which ψ_0 and ϕ_0 both vanish, these lying on the $(n-4)$-fold where the original function is not definite. Now let any finite region of space be for the moment called a suitable or unsuitable region according as it lies entirely within the proper region of some point within or upon the boundary of itself, or does not. Take any finite portion of space, however great, bounded by a closed $(n-1)$-fold: for definiteness we take the portion bounded by the $2n$ plane $(n-1)$-folds expressed by $x_s = -a_s$, $x_s = a_s$; let it be divided by planes into 2^{2n} similar rectangular cells each of extent 2^{-2n}-th of the original; these again divided into 2^{-2n} equal cells, and so on continually. After a finite number of steps, the original region must consist of sub-divisions each of which is a suitable region according to the definition above. For consider any indefinitely continued series of sub-divisions each of which is a sub-division of the preceding one of the series; the series will have a definite limiting point, say (τ^0), lying within or upon the boundary of every sub-division of the series. As the series is indefinitely continued, a stage can be assigned beyond which every sub-division is of less than an assigned extent, and therefore a stage can be assigned beyond which all the sub-divisions of the series lie entirely in the proper region of (τ^0), which by hypothesis is a spherical region about (τ^0) of assignable radius. Thus it is clear that in this series of sub-divisions we reach a suitable region after a finite number of steps; so that there exists no indefinitely continued series of wholly unsuitable regions, each contained in the preceding one of

the series. Thus the original cubical region, and similarly any finite portion of space, can be divided into a finite number of not-overlapping regions, each having the property of being entirely contained in the proper region of some point within or upon the boundary of itself.

Denote any one of these regions by R_0; considering then the interface between two regions R_0, R_1 and the equality $\psi_0/\phi_0 = \psi_1/\phi_1$ between the two possible expressions of the original function existing on this interface as belonging to the proper regions of both the points (τ^0) and (τ^1), which we may, as above, associate respectively with the regions R_0 and R_1; and assuming that a power series in $(p-1)$ complex variables which vanishes at the origin and for all the points of a (limited) $2p-2$-fold continuous about the origin, with the possible exception of a $(2p-4)$-fold part of this for which nothing is known, necessarily has zero coefficients; we can infer that, in the region common to the proper regions K_0, K_1, the $(n-2)$-folds $\psi_0 = 0$, $\psi_1 = 0$ are the same, as also those expressed by $\phi_0 = 0$, $\phi_1 = 0$. We thus build up the idea of a zero $(n-2)$-fold I_1, and an infinity $(n-2)$-fold I_2, for the original function, whose common $(n-4)$-fold intersection consists of the points where the function is unassigned. Expressing the equations of these $(n-2)$-folds, as in the previous part of this paper, by integral functions Θ_1, Θ_2, the original function has a form $e^{\Theta_3}\Theta_1/\Theta_2$, where Θ_3 is an integral function, and Θ_1, Θ_2 have no common zero save where the function is unassigned.

10. Suppose now that, as in what precedes, the whole of any finite portion of n-fold space is divided into regions R, R_0, R_1, …, separated by $(n-1)$-fold interfaces, the diaphragm P, limited by the zero $(n-2)$-fold I of an integral function Θ, forming part of the system of interfaces; and that each region R_s is within the domain of one of the component series ϕ_s by which Θ is expressed, while on the interface S_{rs} between R_r and R_s the ratio ϕ_r/ϕ_s is not zero or infinite.

The integral $\dfrac{2\pi i}{\varpi}\displaystyle\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$ over the diaphragm P may be decomposed into a sum of parts, one part for each of the portions of P which forms a face of a region R; if such a portion of P be denoted by H, the corresponding part of the integral may be regarded as arising from the sum of two integrals each of the form $\dfrac{1}{\varpi}\displaystyle\int\log\phi\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$, and taken over H, but with opposite directions and signs for l_1, l_2, … and values of $\log\phi$ differing by $2\pi i$; these two elementary integrals we shall represent diagrammatically by oppositely drawn arrows named ϕ and ϕ', the dash associated with the latter indicating that the corresponding values of $\log\phi$ is greater by $2\pi i$ than for the other integral; to these parts must be added, as in § 5, an ultimately vanishing integral taken over a cylindrical $(n-1)$-fold Σ enclosing and everywhere very near to I; we suppose, in addition to what has been said, that the division of space into regions R is so taken that I is very nearly an intersection of interfaces, but is shut off from the regions by portions of Σ which form part of the perimeter of these; in the diagram below the $(n-2)$-fold I is denoted by a dot and the cylindrical $(n-1)$-fold Σ by a small closed curve. We have seen that an integral $\frac{1}{\varpi}\int f(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$ taken over a closed $(n-1)$-fold within which $f(\tau)$ is not singular is zero, unless (τ) or the origin is enclosed; in the former case it gives $f(\tau)$, in the latter an integral polynomial in $\tau_1 \dots \tau_p$ of order k. With the exception of these possibilities, to be afterwards referred to, we may now make a further decomposition of the $(n-1)$-fold integral over P in a way perhaps best explained with the help of the diagram.



Here the arrows in dark line represent the original portions of our integral over the parts H of the diaphragm P which are contained in the system of interfaces. For each region R_s we supply an inwardly directed integral $\frac{1}{\varpi}\int \log\phi_s\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\}dS_{n-1}$ over its whole surface; this is zero, save that we must supply a correction $\log\phi(\tau)$ for the single region containing (τ), besides a correction, a polynomial in $\tau_1 \dots \tau_p$, for the single region containing the origin. An inspection of the figure shows that the result is an integral over every interface R_{rs}, of the form $\frac{1}{\varpi}\int \log\left(\frac{\phi_r}{\phi_s}\right)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\} dS_{n-1}$, wherein, as has been said, $\log(\phi_r/\phi_s)$ is finite and developable over this interface; together with integrals over the outside faces of the most external of the regions; these latter together form an integral of the character of that denoted in § 5 by W; we shall here denote this sum by $-W'$, so that W' is directed outwards over the whole outside $(n-1)$-fold S bounding the portion of space considered. We have thus the result—

> The integral over the $(n-1)$-fold diaphragm P,
>
> $$\frac{2\pi i}{\varpi}\int\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\} dS_{n-1},$$
>
> is equal to $\log\phi(\tau)+\Sigma J_{rs}-W'+(1,\tau)_k$, where $\phi(\tau)$ is the series for the region R containing (τ), the symbol $(1,\tau)_k$ denotes an integral polynomial of order k, and J_{rs} denotes the integral
>
> $$\frac{1}{\varpi}\int \log\left(\frac{\phi_r}{\phi_s}\right)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\} dS_{n-1}$$
>
> taken over the interface separating the regions R_r and R_s, and the sum of these integrals is to be taken for all the interfaces.

To the two sides of this equation we may add the integral W of § 5,

$$\frac{1}{\varpi}\int \log\Theta(\xi)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\ldots\right\} dS_{n-1},$$

taken over the whole outside $(n-1)$-fold S. It is manifest that the difference $W-W'$ is ultimately an integral function, the function $\log\Theta/\phi$ being finite and not zero over S; thus, if our diaphragm integral is convergent, the sum ΣJ_{rs} is ultimately convergent, and the sum W' is ultimately an integral function. Thus in this case we have the further result—

> An integral function having the given zero construct I is represented by $\log\phi(\tau)+\Sigma J_{rs}$.

Now putting, for abbreviation,

$$l_1\frac{\partial}{\partial x_1}+\ldots+l_n\frac{\partial}{\partial x_n}=\frac{\partial}{\partial\nu},\qquad l_1\frac{\partial}{\partial x_2}-l_2\frac{\partial}{\partial x_1}+\ldots+l_{n-1}\frac{\partial}{\partial x_n}-l_n\frac{\partial}{\partial x_{n-1}}=\frac{\partial}{\partial\sigma},$$

so that a function $U+iV$ of $x_1+ix_2,\ \ldots,\ x_{n-1}+ix_n$ satisfies

$$\left(\frac{\partial}{\partial\nu}+i\frac{\partial}{\partial\sigma}\right)(U+iV)=0,$$

we have

$$\int (U+iV)\left\{(l_1+il_2)\left(\frac{\partial H}{\partial x_1}-i\frac{\partial H}{\partial x_2}\right)+\dots\right\} dS_{n-1}$$

$$=\int (U+iV)\left(\frac{\partial H}{\partial \nu}-i\frac{\partial H}{\partial \sigma}\right) dS_{n-1}$$

$$=\int \left[(U+iV)\frac{\partial H}{\partial \nu}-H\frac{\partial (U+iV)}{\partial \nu}\right] dS_{n-1}-i\int \frac{\partial}{\partial \sigma}[(U+iV)H]\, dS_{n-1},$$

and any one of the integrals J_{rs} can be written in the form

$$J_{rs}=K_{rs}-\frac{i}{\varpi}\int \frac{\partial}{\partial \sigma}\left(H \log \frac{\phi_r}{\phi_s}\right) dS_{n-1},$$

where $$K_{rs}=\frac{1}{\varpi}\int \left[\frac{\partial H}{\partial \nu}\log \frac{\phi_r}{\phi_s}-H\frac{\partial}{\partial \nu}\left(\log \frac{\phi_r}{\phi_s}\right)\right] dS_{n-1},$$

taken over the interface R_{rs}. Consider the sum of the integrals

$$-\frac{i}{\varpi}\int \frac{\partial}{\partial \sigma}\left(H \log \frac{\phi_r}{\phi_s}\right) dS_{n-1}$$

or $$-\frac{i}{\varpi}\int \left(l_1\frac{\partial}{\partial x_2}-l_2\frac{\partial}{\partial x_1}+\dots\right)(H \log \phi_r)\, dS_{n-1}$$

$$-\frac{i}{\varpi}\int \left(l_1\frac{\partial}{\partial x_2}-l_2\frac{\partial}{\partial x_1}+\dots\right)(H \log \phi_s)\, dS_{n-1},$$

where in the former $(l_1, l_2, \dots)$ are directed inwards to the region R_r and in the latter inwards to the region R_s; an integral $\int \partial/\partial\sigma\,[Hf(\xi)]\, dS_{n-1}$ is easily seen to be unaltered by a deformation of the $(n-1)$-fold of integration provided no singularity of $f(\xi)$ is passed over, and to give zero when taken over a vanishing closed $(n-1)$-fold containing (τ) or the origin. Hence it is easily seen that the sum of these integrals is ultimately zero. Compounding them together by a process the reverse of that followed in obtaining the sum ΣJ_{rs}, there result three parts.

(1) a sum of integrals $-\dfrac{i}{\varpi}\displaystyle\int \frac{\partial}{\partial \sigma}(H \log \phi)\, dS_{n-1}$ over the outside $(n-1)$-fold S bounding the space considered, ultimately zero because H is ultimately of order $R^{-(n+k)}$, and the part of S where $\phi = 0$ and $\log \phi = \infty$ is excluded from integration by the cylindrical $(n-1)$-fold Σ;

(2) an integral $\dfrac{2\pi}{\varpi}\displaystyle\int \frac{\partial H}{\partial \sigma}\, dS_{n-1}$ over the diaphragm P;

(3) an outward integral $\dfrac{i}{\varpi}\displaystyle\int \frac{\partial}{\partial \sigma}(H \log \phi)\, dS_{n-1}$ over the cylindrical

$(n-1)$-fold Σ; this, in virtue of $i\partial\phi/\partial\sigma = -\partial\phi/\partial\nu$, is equal to

$$\frac{1}{\varpi}\int\left(i\log\phi\frac{\partial H}{\partial\sigma} - \frac{H}{\phi}\frac{\partial\phi}{\partial\nu}\right)dS_{n-1},$$

which, if we put $dS_{n-1} = \epsilon\, d\theta\, dS_{n-2}$ as in § 5, and allow ϵ to vanish, so that Σ degenerates into I, becomes, since $\phi = 0$ on I,

$$-\frac{1}{\varpi}\int d\theta\int H\frac{\phi}{\phi}dS_{n-2} = -\frac{2\pi}{\varpi}\int H\,dS_{n-2};$$

and it is the fact that the sum of this last integral, over the boundary I of the diaphragm P, and the integral in (2), over this diaphragm, is zero. We have thus shown, for cases when our original integral is convergent, that

> An integral function having the given zero $(n-2)$-fold I may be obtained by adding, to the sum of integrals
>
> $$K_{rs} = \frac{1}{\varpi}\int\left[\log\frac{\phi_r}{\phi_s}\frac{\partial H}{\partial\nu} - H\frac{\partial}{\partial\nu}\left(\log\frac{\phi_r}{\phi_s}\right)\right]dS_{n-1}$$
>
> over the interfaces, the quantity $\log\phi(\tau)$ associated with the region containing (τ).

This is the result obtained, for the less general case of periodic functions, by M. Poincaré in his last paper, *Acta Math.*, Vol. XXVI., pp. 67, 73, 78; while the identity just remarked,

$$\frac{2\pi}{\varpi}\int\frac{\partial H}{\partial\sigma}dS_{n-1} = \frac{2\pi}{\varpi}\int H\,dS_{n-2},$$

occurring *Trans. Camb. Phil. Soc.*, Vol. XVIII., p. 431, establishes the identity of the real part of the function considered by M. Poincaré, *Acta Math.*, Vol. XXII., p. 168, with the real part of the function

$$\frac{2\pi i}{\varpi}\int\left(\frac{\partial H}{\partial\nu} - i\frac{\partial H}{\partial\sigma}\right)dS_{n-1},$$

which is fundamental in the present paper.

11. To some readers the constant use of the language of hyperspace, and in particular the frequent mention of $(n-2)$-folds, in this paper, may seem to augur badly for the theory of the functions involved; on the contrary, the writer believes that this point of view, already found with some explicitness in Kronecker's *Berlin Monatsber.* paper of 1869, is of the greatest importance for the development of the theory of functions of several variables. It is desired to add here some general remarks, including a view of Mittag-Leffler's theorem for functions of more than one variable, which may add a little to the elucidation of the ideas.

In the plane which we employ in describing the properties of functions of one complex variable τ, the *element* is the point; it is, in general, at a point or a set of discrete points that a function of the complex variable τ has an assigned value, and we speak of infinite values of τ as lying in the neighbourhood of a point $1/\tau = 0$; and the region of existence of a power series in the difference $\tau - \tau^0$ is necessarily a circle described about and entirely enclosing the point τ^0, there being no point outside the circle at which the series converges. Suppose, however, we have, for instance, a power series in the two differences $\tau_1 - \tau_1^0$, $\tau_2 - \tau_2^0$, say $\phi(\tau_1, \tau_2)$; we have the theorem that, if the series converges for values τ_1', τ_2' such that $|\tau_1' - \tau_1^0| = R_1$, $|\tau_2' - \tau_2^0| = R_2$, a region of convergence is given by the equations $|\tau_1 - \tau_1^0| < R_1$, $|\tau_2 - \tau_2^0| < R_2$; but in a region of convergence of this form the radii R_1, R_2 are not in general to be regarded as fixed or independent; it may happen, indeed, that by taking one of them sufficiently small the other may be taken arbitrarily great; in that case, though, putting $\tau_1 = t_1 + it_2$, $\tau_2 = t_3 + it_4$, and using the language of hyperspace, we may take a sphere

$$(t_1 - t_1^0)^2 + (t_2 - t_2^0)^2 + (t_3 - t_3^0)^2 + (t_4 - t_4^0)^2 = R^2,$$

and choose R less than the less of R_1 and R_2, to have its greatest value so that the power series converges at every interior point, yet this sphere will not in general contain all the points at which the series converges; the points at which the series $\phi(\tau_1, \tau_2)$ has any value which it takes within this sphere lie, in fact, upon a continuum of two dimensions satisfying $\phi_1 d\tau_1 + \phi_2 d\tau_2 = 0$, which generally intersects the sphere, in a locus of one dimension, and passes to indefinite distance.

As an example of a power series whose region of existence is easily seen to pass to infinity, we may take the series obtained from the series

$$\text{(B)} \quad 1 + \tau_1 e^{-\tau_2} + \tau_1^2 e^{-2\tau_2} + \ldots$$

by arranging according to terms of increasing dimension, that is, the series

$$\text{(A)} \quad 1 + \tau_1 + (-\tau_1\tau_2 + \tau_1^2) + \left(\frac{\tau_1\tau_2^2}{2} - 2\tau_1^2\tau_2 + \tau_1^3\right) + \ldots;$$

if $\tau_1 = r_1 e^{i\theta_1}$, $\tau_2 = r_2 e^{i\theta_2}$, the sum of the moduli of the terms of (B) is $1 + r_1 e^{r_2} + r_1^2 e^{2r_2} + \ldots$, which converges, only when $r_1 < 1$, for $r_1 < e^{-r_2}$; for any values of τ_1, τ_2 satisfying this condition the series (B) can be arranged as a convergent power series (A). Conversely, if the series (A) converges for any pair of values $\tau_1 = \tau_1'$, $\tau_2 = \tau_2'$, it will converge absolutely for values $|\tau_1| < |\tau_1'|$, $|\tau_2| < |\tau_2'|$ and converge therefore to the same value when arranged in the form (B); but the series (B) converges only when $|\tau_1| < |e^{\tau_2}|$, or, if $\tau_2 = t_3 + it_4$, only when $r_1 < e^{t_3}$;

its region of convergence, represented on the two planes of τ_1 and τ_2, consists therefore of the interior of a circle, centre the origin, of radius r_1, in the first plane, taken with the region to the right of the straight line parallel to the imaginary axis in the second plane at distance $\log r_1$ to the right of this axis; there can, therefore, be absolute convergence only when $\log r_1$ is negative or $r_1 < 1$, and then only when $r_2 < \log 1/r_1$ or $r_1 < e^{-r_2}$. Thus the region of convergence of the series (A) consists of the interior of the three-fold $r_1 e^{r_2} = 1$; within this r_1 cannot be as great as unity, but can be as near as may be desired by taking r_2 small enough; on the other hand, r_2 can be as great as may be desired by taking r_1 small enough. We may then describe the region as *spindle-shaped.* It contains in its interior an infinite number of bi-cylindrical regions of convergence, each bounded by portions of two three-folds $r_1 = R_1$, $r_2 = R_2$, for which $R_2 = \log(1/R_1)$; this pair of three-folds intersects in a two-fold lying on the boundary of the spindle-shaped region. The spherical region of convergence of greatest radius is given by

$$t_1^2+t_2^2+t_3^2+t_4^2 = R^2$$

where R is the real positive quantity between $\frac{1}{2}$ and 1 for which $R = e^{-R}$. The two-fold on which the function represented by the series has the value, unity, which it has for $\tau_1 = 0$, $\tau_2 = 0$, has for equation $\tau_1 = 0$; it clearly passes to infinity; it intersects the boundary of the spherical region of convergence in the locus of one dimension $t_1 = 0$, $t_2 = 0$, $t_3^2+t_4^2 = R^2$; it intersects the perimeter of a bi-cylindrical region of convergence $|\tau_1| < R_1$, $|\tau_2| < R_2$ in the one-fold $t_1 = 0$, $t_2 = 0$, $t_3^2+t_4^2 = R_2^2$; but it does not intersect the perimeter of the spindle-shaped region of convergence in any finite point. Finally, it has appeared that this latter region is by no means co-extensive with the region of existence of the function represented by the power series, which is $(1-\tau_1 e^{-\tau_2})^{-1}$.

Thus the hyperspace of n, $= 2p$, dimensions which we speak of to describe the properties of a function of p complex variables is one of which the elements are not the points, but $(n-2)$-folds (cf. §§ 2, 4); the closed perimeters separating off regions where the function has an assigned character or value are $(n-1)$-folds which are not most naturally spheres, but multi-cylindrical surfaces often passing to an indefinite distance, and the infinity of the space consists not of one point, but of points lying on one or more of the $(n-2)$-folds, p in number, expressed by $\tau_1^{-1} = 0, \ldots \tau_p^{-1} = 0$; in general a point of the space is a derived element obtained by the co-intersection of p $(n-2)$-folds; whereas in the case of one variable the whole space is expressed by two equations of the form $|\tau| < R$, $|\tau^{-1}| < R$, in the case of p variables there are 2^p regions necessary to include the whole space, first the finite region $|\tau_1| < R_1 \dots |\tau_p| < R_p$, and then all the regions of the form

$$|\tau_1^{-1}| < R_1 \dots |\tau_m^{-1}| < R_m, \quad |\tau_{m+1}| < R_{m+1} \dots \quad |\tau_p| < R_p,$$

where $m = 1, 2, \dots, p$.

12. If now we have a single-valued function $f(\tau)$, either rational in the variables $\tau_1 \dots \tau_p$, or without essential singularity for finite values of these, whose infinity $(n-2)$-fold I does not pass through the origin, and a multi-cylindrical region $|\tau_1| < R_1 \dots |\tau_p| < R_p$ be constructed whose $(n-1)$-fold boundary, given by the aggregate of p sets of equations

$$|\tau_1| < R_1 \dots |\tau_i| = R_i \dots |\tau_p| < R_p,$$

excludes the $(n-2)$-fold I, the function $f(\tau)$ can be expanded in this region as a power series in $\tau_1 \dots \tau_p$; and this expansion may be valid in a region of greater extent including in its interior all such possible multi-cylindrical regions; let C denote an $(n-1)$-fold excluding I, and including the origin, within which the expansion of $f(\tau)$

$$f(\tau) = \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} A_{m_1 \dots m_p} \tau_1^{m_1} \dots \tau_p^{m_p}$$

is uniformly convergent, in the sense that, for any assigned small positive quantity ϵ, values of $\mu_1 \dots \mu_p$ can be assigned such that

$$F(\tau) = f(\tau) - \sum_{m_1=0}^{\mu_1-1} \dots \sum_{m_p=0}^{\mu_p-1} A_{m_1 \dots m_p} \tau_1^{m_1} \dots \tau_p^{m_p}$$

$$= \sum_{m_1=\mu_1}^{\infty} \dots \sum_{m_p=\mu_p}^{\infty} A_{m_1 \dots m_p} \tau_1^{m_1} \dots \tau_p^{m_p}$$

is in absolute value less than ϵ for all points interior to C; so that the region may be a multi-cylindrical region, or an interior spherical region, or be more extended.

Now suppose we have an enumerable series of such functions, infinity $(n-2)$-folds and $(n-1)$-folds, $f_1(\tau)$, $f_2(\tau)$, ..., I_1, I_2, ..., C_1, C_2, ... such that any one $(n-1)$-fold C_s excludes I_s, I_{s+1}, ..., but includes C_{s-1}, C_{s-2}, ... and the origin, while any finite point of space is interior to only a finite number of the $(n-1)$-folds C_1, C_2, ... ; and, taking a convergent series of real positive terms ϵ_1, ϵ_2, ..., subtract from the expansion of $f_s(\tau)$ in C_s the sufficient polynomial that the remainder

$$F_s(\tau) = f_s(\tau) - \sum_{m_1=0}^{\mu_1-1} \dots \sum_{m_p=0}^{\mu_p-1} A_{s\, m_1 \dots m_p} \tau_1^{m_1} \dots \tau_p^{m_p}$$

may within C_s be less than ϵ_s in absolute value, the integers $\mu_1 \dots \mu_p$ presumably depending on s. Taking, then, any point (τ^0) not upon any one of the $(n-2)$-folds $I_1, I_2, \ldots$, but exterior, say, to $C_1 \ldots C_{s-1}$ and interior to $C_s C_{s+1} \ldots$, the infinite series $F_s(\tau)+F_{s+1}(\tau)+\ldots$ whose terms are power series in $\tau_1 \ldots \tau_p$ absolutely less respectively than $\epsilon_s, \epsilon_{s+1}, \ldots$, is uniformly convergent about this point and its sum can be arranged as a power series in the differences $\tau_1-\tau_1^0 \ldots \tau_p-\tau_p^0$; about this point also each of the finite number of functions $F_1(\tau) \ldots F_{s-1}(\tau)$ is non-singular and capable of expression as a power series in these differences, though possibly incapable of expression as a power series in $\tau_1 \ldots \tau_p$. Thus

> The series $F(\tau) = F_1(\tau)+F_2(\tau)+F_3(\tau)+\ldots$ represents a single-valued function developable about every finite point not upon any one of the $(n-2)$-folds $I_1, I_2, \ldots$; while, as precisely the same reasoning applies if one of the $(n-2)$-folds and the corresponding function be omitted from consideration, and the difference $F_s(\tau)-f_s(\tau)$ is an integral polynomial, it follows that the difference $F(\tau)-f_s(\tau)$ is a function whose region of existence excludes only the $(n-2)$-folds other than I_s.

It is, of course, in the specification of the behaviour of the function $F(\tau)$ in the neighbourhood of the $(n-2)$-folds I_s that the subsidiary functions $f_s(\tau)$ have their chief utility; yet we have assumed a knowledge of the expansion of $f_s(\tau)$ about the origin in order to form the functions $F_s(\tau)$; it is worth remarking that when the functions $f_s(\tau)$ are known only in the immediate neighbourhood of the $(n-2)$-folds I_s, essentially a similar final theorem can be obtained. For one variable, if $f_s(\tau)$ be given only in an annulus surrounding the point I_s, consider the function

$$\phi_s(\tau) = \frac{1}{2\pi i} f_s(\tau) \int \frac{d\xi}{\xi-\tau} - \frac{1}{2\pi i}\int \frac{f_s(\xi)\,d\xi}{\xi-\tau}$$

taken round a closed curve in the annulus; it exists, is single-valued and developable about every finite point outside the inner boundary of the annulus, requires a knowledge of $f_s(\tau)$ only in the part of the annulus interior to the closed curve of integration, and is such that everywhere within this closed curve the difference $\phi_s(\tau)-f_s(\tau)$, being equal to $-\frac{1}{2\pi i}\int \frac{f_s(\xi)\,d\xi}{\xi-\tau}$, is non-singular (not excluding I_s). If then we use the expansion of $\phi_s(\tau)$ about the origin, just as before we used the expansion of $f_s(\tau)$, we shall obtain essentially the same character for the function $F(\tau)$. So, for any number of variables, imagine a cylindrical $(n-1)$-fold surface Γ_s passing to infinity which encloses the $(n-2)$-fold I_s, and suppose that the function $f_s(\tau)$ is known only in an annular cylindrical space which encloses Γ_s; then by § 4 we can use the expansion of the function

$$\phi_s(\tau) = \frac{f_s(\tau)}{\varpi}\int\left(\frac{\partial H}{\partial \nu} - i\frac{\partial H}{\partial \sigma}\right)dS_{n-1} - \frac{1}{\varpi}\int f_s(\xi)\left(\frac{\partial H}{\partial \nu} - i\frac{\partial H}{\partial \sigma}\right)dS_{n-1},$$

which is easily seen to be continuous as (τ) passes over Γ_s, in place of the expansion of $f_s(\tau)$, about the origin. This gives another example of the utility of the theorem of § 4.

Further, the reason for the use of the functions $F_s(\tau)$ in the place of the functions $f_s(\tau)$ is the need of a series converging uniformly about every finite point (τ^0) not upon the $(n-2)$-folds I_1, I_2, ... and of definite construction; it may be possible, in place of choosing the integers $\mu_1 \dots \mu_p$ so that $F_s(\tau)$ is absolutely less than ϵ_s for the whole interior of C_s, to choose them smaller than in that case in such a way that for the whole interior of any assigned C_r the series $F_r(\tau)+F_{r+1}(\tau)+\dots$, and therefore the series $F(\tau)$, is uniformly convergent, which is sufficient for the theorem. An example is the case where $F(\tau) = \cot \pi\tau$; another example occurs below.

The series obtained permit of integration and differentiation term by term, as may be proved in a manner quite analogous to that given by Weierstrass for one variable, or by consideration of the integrals

$$F_r(\tau) = \int F_r(\xi)\left(\frac{\partial}{\partial \nu} - i\frac{\partial}{\partial \sigma}\right) P(x, t)\, dS_{n-1}$$

taken over a closed $(n-1)$-fold within which no one of the functions $F_1(\tau)$, $F_2(\tau)$, ... is singular.

The theorem and proof are capable of extension to functions not altogether single-valued or of pseudo-rational character; in particular, analogous to the way in which Weierstrass's factor theorem for integral functions of one variable is derivable from Mittag-Leffler's theorem, there is a derivable factor theorem for functions of several variables; the expression in § 6 for the logarithm of a rational function gives rise to the necessary expansion of this logarithm about the origin.

13. To form an example of the theorem it seems natural in the first instance to consider the case when the $(n-2)$-folds I_1, I_2, ... are given by linear equations. For two complex variables $\xi = x+iy$, $\eta = z+it$, the square of the distance from the origin to the nearest point of the two-fold $(a+ib)\xi+(c+id)\eta = 1$ is $(a^2+b^2+c^2+d^2)^{-1}$: and, in fact, the square of the modulus of the left side of this equation, that is, of

$$(a+ib)(x+iy)+(c+id)(z+it),$$

is $(ax-by+cz-dt)^2+(ay+bx+ct+dz)^2$, which is equal to

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2)-(az+bt-cx-dy)^2-(at-bz-cy+dx)^2$$

and is less than unity when $x^2+y^2+z^2+t^2 < (a^2+b^2+c^2+d^2)^{-1}$; so that under this condition we have, if $u = a+ib$, $v = c+id$, the expansion

$$(1-u\xi-v\eta)^{-1} = 1+(u\xi+v\eta)+(u\xi+v\eta)^2+\dots.$$

Now take an infinite series of functions

$$w_1(1-u_1\xi-v_1\eta)^{-1}, \qquad w_2(1-u_2\xi-v_2\eta)^{-1}, \qquad \dots$$

where $w_1, w_2, \dots$ are constants, so that the numbers $R_1 = [|u_1|^2+|v_1|^2]^{-\frac{1}{2}}$, $R_2 = [|u_2|^2+|v_2|]^{-\frac{1}{2}}, \dots$ constantly increase to infinity, but in such a way that only a finite number of them are less than any assigned real positive number; then, θ being a fixed real quantity just less than unity, the $(n-1)$-folds $|\xi|^2+|\eta|^2 = (\theta R_s)^2$ are such a series as those denoted in § 12 by $C_1, C_2, \dots$; and, if ϵ be a real quantity between 0 and θ, $W_s = |w_s|$ and $\rho_s = |u_s\xi+v_s\eta|$, the sum of the moduli of the terms which follow the μ_s-th term of the expansion within C_s of $w_s(1-u_s\xi-v_s\eta)^{-1}$ in powers of $u_s\xi+v_s\eta$, namely $W_s\rho_s^{\mu_s}/(1-\rho_s)$, is, for $|\xi|^2+|\eta|^2 \leqslant (\epsilon R_s)^2$, and therefore $\rho_s \leqslant \epsilon$, less than or equal to $W_s\epsilon^{\mu_s}/(1-\epsilon)$, and the functions $F_s(\xi, \eta)$ may be defined by choosing the numbers μ_s so that $W_s\epsilon^{\mu_s}/(1-\epsilon) < \epsilon_s$ and taking

$$F_s(\xi, \eta) = \frac{w_s}{1-u_s\xi-v_s\eta} - \{1+u_s\xi+v_s\eta+\dots+(u_s\xi+v_s\eta)^{\mu_s-1}\};$$

but we have remarked that it is sufficient if the numbers μ_s be such that the series $\sum\limits_s W_s\rho_s^{\mu_s}/(1-\rho_s)$ be uniformly convergent within any assigned C_r; which is satisfied, since $\rho_s < \epsilon$, if the series $\sum\limits_s W_s\rho_s^{\mu_s}$ be so uniformly convergent; and this again, since $\rho_s \leqslant (|u_s|^2+|v_s|^2)^{\frac{1}{2}}(x^2+y^2+z^2+t^2)^{\frac{1}{2}}$, provided the series of constants $\sum\limits_s W_s[|u_s|^2+|v_s|^2]^{\frac{1}{2}\mu_s}$ be convergent.

The most obvious case is when the series $\sum\limits_s W_s$ is convergent; then we have

$$F(\xi, \eta) = \sum_{s=1}^{\infty} \frac{w_s}{1-u_s\xi-v_s\eta},$$

the aggregate of numbers $[|u_s|^2+|v_s|^2]^{-\frac{1}{2}}$ satisfying the condition of having infinity as its sole point of condensation.

For another case we may take $w_s = 1$ and $u_s = m_s^{-1}$, $v_s = n_s^{-1}$, wherein $m_1, m_2, \dots$ and $n_1, n_2, \dots$ are both series of constantly increasing integers; then clearly $\sum\limits_s \left(\frac{1}{m_s^2}+\frac{1}{n_s^2}\right)^{\frac{1}{2}\mu_s}$ converges for $\mu_s = 2$, and we have

$$F(\xi, \eta) = \sum_s \left\{\frac{1}{1-\frac{\xi}{m_s}-\frac{\eta}{n_s}} - \left(1+\frac{\xi}{m_s}+\frac{\eta}{n_s}\right)\right\} = \sum_s \frac{\left(\frac{\xi}{m_s}+\frac{\eta}{n_s}\right)^2}{1-\frac{\xi}{m_s}-\frac{\eta}{n_s}}.$$

14. Such considerations appear to the writer to have great interest as throwing light on the question, "How far do there exist Mittag-Leffler series* of simpler functions for multiperiodic functions without finite essential singularities analogous to the well known series for the elliptic function $\wp(u)$?" When we consider the periodicity of the infinity construct of such a function, it appears unlikely that such a series can be built under the hypothesis that the $(n-1)$-folds C_1, C_2, ... used in the demonstration of § 12 are spheres or multicylindrical surfaces of all finite radii; for that case, moreover. already considered by M. Appell, *Acta Math.*, Vol. II., 1883, p. 71, the proof is only a very obvious generalization of the case of one variable; we believe it to be of importance to plead for standing ground for the more general formulation.

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* [*September* 1*st*, 1903.—For triply periodic functions of two variables, such series are derivable at once from the known unsymmetrical forms given, for instance, in Part III. of the writer's note on hyperelliptic functions in *Proc. Camb. Phil. Soc.*, Vol. XII., Part III. (Easter Term, 1903). See also Painlevé, *Compt. Rend.*, April 14th, 1902.]

WAVE-MOTIONS WITH DISCONTINUITIES AT WAVE-FRONTS

By A. E. H. LOVE.

[Read January 8th, 1903.—Received January 16th, 1903.]

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1. Many of the current explanations of optical phenomena are based upon an abstract wave-theory, in which the nature of the medium transmitting the waves, and the nature of the disturbances propagated with the waves, are almost entirely disregarded.[*] This abstract theory has two distinct branches, which may be described as kinematical and analytical. To the kinematical branch belong Huygens' notion of wave-envelopes, and Fresnel's doctrine concerning the approximately rectilinear character of the propagation of light. To the analytical branch belong all investigations which proceed from the characteristic equation of wave-propagation, $\ddot{\phi} = c^2\nabla^2\phi$. In this order of ideas, apart from investigations relating to indefinitely prolonged sequences of simple harmonic waves, the two main results so far achieved have been the expression by Poisson[†] of the value of ϕ at a point, as an integral taken over a sphere surrounding the point (the integral involving initial values of ϕ and $\dot{\phi}$), and the more general expression by Kirchhoff[‡] of the value of ϕ at a point, as an integral taken over an arbitrary surface (the integral involving the values of ϕ and its differential coefficients at points on the surface and at definite times). Especial importance has been attached to Kirchhoff's formula on the

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[*] Cf. the introduction to Lord Rayleigh's article, "Wave Theory of Light," *Ency. Brit.*, 9th edition; *Scientific Papers*, Vol. III., p. 47.

[†] *Mém. de l'Institut*, Paris, t. 3, 1820. Cf. Lord Rayleigh, *Theory of Sound*, Vol. II., chap. xiv.

[‡] *Ann. Phys. Chem.* (Wiedemann), Bd. XVIII., 1883; or *Vorlesungen ü. math. Optik* (Leipzig, 1891).

ground that it harmonizes the kinematical and analytical branches of the theory.*

The possibility of fitting results, arrived at in the abstract wave-theory, into a theory of light based on some hypothesis concerning the nature of the medium, or the nature of waves of light, may not be assumed without investigation, For example, some functions, which satisfy the characteristic equation, cannot represent displacements in an elastic solid transmitting transverse waves, for they are not components of a circuital vector. It is not my purpose here to criticize the abstract wave-theory from this point of view, but rather to add something to it in one respect in which it is incomplete. In the analytical branch of the theory, the methods employed presuppose a high degree of continuity of all the functions involved. For example, Poisson's integral formula was first obtained on the supposition that the function ϕ is regular throughout all the region of space in which the formula is used. Yet any actual radiation must move onwards from previously disturbed, into previously undisturbed, parts of the medium; and there must be some degree of discontinuity between these parts. The admissible degrees of discontinuity and the effects of them are to be examined; and it is not possible to dispense with a consideration of the peculiarities which may arise from the constitution of the medium. The better to bring out the necessary conditions for the validity of Kirchhoff's and Poisson's integral formulæ, proofs of them are prefixed to the examination of the effects of discontinuity; and a proof of the theorem of determinacy of the solution is added, because the validity of this theorem when there are wave-boundaries might be called in question. A summary of the results obtained is given at the end of the paper.

Abstract Wave-Theory.

2. *Kirchhoff's Formula.*—Let ϕ be a function of x, y, z, t representing some kind of disturbance, transmitted through an elastic medium according to a partial differential equation of the form

$$\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi, \tag{1}$$

where c is a constant, which may depend upon the nature of the medium and the nature of the disturbance.

Suppose in the first place that, throughout a specified region of space and a specified interval of time, ϕ and its differential coefficients of the first order are continuous functions of x, y, z and of t; and suppose also that the differential coefficients of the second order, such as $\partial^2\phi/\partial t^2$, $\partial^2\phi/\partial x^2$ are finite.

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* See, for example, Drude, *Lehrbuch d. Optik* (Leipzig, 1900), p. 158.

Let O be any point (x_0, y_0, z_0), which need not be in the specified region of space; and consider the function ψ obtained from ϕ by substituting $t-r/\mathrm{c}$ for t, r denoting the distance of any point in the specified region from the point O. The function ψ satisfies a partial differential equation of the second order, which may be obtained by expressing the differential coefficients of ψ in terms of the differential coefficients of ϕ, when $t-r/\mathrm{c}$ is substituted for t in the latter after performing the differentiations. We denote by $[\phi]$, $\left[\frac{\partial\phi}{\partial t}\right]$, $\left[\frac{\partial\phi}{\partial x}\right]$, ... the result of substituting $t-r/\mathrm{c}$ for t in ϕ, $\frac{\partial\phi}{\partial t}$, $\frac{\partial\phi}{\partial x}$, ...

The transformation is effected by means of equations of the forms

$$\psi=[\phi],\quad \frac{\partial\psi}{\partial t}=\left[\frac{\partial\phi}{\partial t}\right],\quad \frac{\partial\psi}{\partial x}=\left[\frac{\partial\phi}{\partial x}\right]-\frac{x-x_0}{r\mathrm{c}}\left[\frac{\partial\phi}{\partial t}\right],$$

and we find

$$\frac{\partial^2\psi}{\partial x^2}=\left[\frac{\partial^2\phi}{\partial x^2}\right]-\frac{2(x-x_0)}{r\mathrm{c}}\left[\frac{\partial^2\phi}{\partial x\,\partial t}\right]+\frac{(x-x_0)^2}{r^2\mathrm{c}^2}\left[\frac{\partial^2\phi}{\partial t^2}\right]-\left(\frac{1}{r}-\frac{(x-x_0)^2}{r^3}\right)\frac{1}{\mathrm{c}}\left[\frac{\partial\phi}{\partial t}\right],$$

so that $$\nabla^2\psi=[\nabla^2\phi]+\frac{1}{\mathrm{c}^2}\left[\frac{\partial^2\phi}{\partial t^2}\right]-\frac{2}{\mathrm{c}}\left[\left(\frac{x-x_0}{r}\frac{\partial}{\partial x}+\frac{y-y_0}{r}\frac{\partial}{\partial y}+\frac{z-z_0}{r}\frac{\partial}{\partial z}\right)\frac{\partial\phi}{\partial t}\right]-\frac{2}{\mathrm{c}r}\left[\frac{\partial\phi}{\partial t}\right].$$

Now $$\left(\frac{x-x_0}{r}\frac{\partial}{\partial x}+\ldots+\ldots\right)\frac{\partial\psi}{\partial t}=\left[\left(\frac{x-x_0}{r}\frac{\partial}{\partial x}+\ldots+\ldots\right)\frac{\partial\phi}{\partial t}\right]-\frac{1}{\mathrm{c}}\left[\frac{\partial^2\phi}{\partial t^2}\right].$$

Hence we have $$\nabla^2\psi=[\nabla^2\phi]-\frac{1}{\mathrm{c}^2}\left[\frac{\partial^2\phi}{\partial t^2}\right]-\frac{2}{\mathrm{c}}\left(\frac{x-x_0}{r}\frac{\partial}{\partial x}+\ldots+\ldots\right)\frac{\partial\psi}{\partial t}-\frac{2}{\mathrm{c}r}\frac{\partial\psi}{\partial t}.$$

If the interval of time, during which ϕ satisfies equation (1) at all points of the specified region, includes all the times represented by $t-r/\mathrm{c}$, then ψ satisfies the equation

$$\nabla^2\psi+\frac{2}{\mathrm{c}}\left(\frac{x-x_0}{r}\frac{\partial}{\partial x}+\ldots+\ldots\right)\frac{\partial\psi}{\partial t}+\frac{2}{\mathrm{c}r}\frac{\partial\psi}{\partial t}=0$$

at all points of this region. This equation may be written

$$\nabla^2\psi+\frac{2r}{\mathrm{c}}\left\{\frac{\partial}{\partial x}\left(\frac{x-x_0}{r^2}\frac{\partial\psi}{\partial t}\right)+\frac{\partial}{\partial y}\left(\frac{y-y_0}{r^2}\frac{\partial\psi}{\partial t}\right)+\frac{\partial}{\partial z}\left(\frac{z-z_0}{r^2}\frac{\partial\psi}{\partial t}\right)\right\}=0.$$

We multiply the left-hand member of this equation by $1/r$, and integrate it through a volume lying entirely within the specified region of space. The volume integral can be transformed immediately into an integral taken over the boundary of the volume. Let S denote this boundary, and let ν denote the direction of the normal to S drawn into the volume; then we have

$$\iint\left(\psi\frac{\partial r^{-1}}{\partial\nu}-r^{-1}\frac{\partial\psi}{\partial\nu}-\frac{2}{\mathrm{c}r}\frac{\partial r}{\partial\nu}\frac{\partial\psi}{\partial t}\right)dS=0, \tag{2}$$

provided that the point O does not lie within the volume of integration. This volume may be bounded externally by a closed surface S_1 and internally by a closed surface S_2, surrounding the point O; and we may pass to a limit by contracting S_2 indefinitely. The value of the integral taken over S_2 is then $-4\pi\psi(0)$, where $\psi(0)$ is the value of ψ at the point O, which is the same as the value of ϕ at the point O. We have therefore

$$\phi_0 = \frac{1}{4\pi} \iint \left(\psi \frac{\partial r^{-1}}{\partial \nu} - r^{-1} \frac{\partial \psi}{\partial \nu} - \frac{2}{cr} \frac{\partial r}{\partial \nu} \frac{\partial \psi}{\partial t} \right) dS_1,$$

where ϕ_0 is the value of ϕ at time t at the point O. Also we have

$$\frac{\partial \psi}{\partial \nu} = \left[\frac{\partial \phi}{\partial \nu} \right] - \frac{1}{c} \frac{\partial r}{\partial \nu} \left[\frac{\partial \phi}{\partial t} \right],$$

and thus the integral may be expressed in terms of $[\phi]$. In the result the value of ϕ at time t and at any point O within a surface S is expressed in terms of the values of ϕ, $\partial\phi/\partial\nu$, $\partial\phi/\partial t$ at points on the surface S and at times represented by $t - r/c$, r being the distance of a point on the surface from O.

The result of the transformation is expressed by the equation

$$\phi = \frac{1}{4\pi} \iint \left\{ [\phi] \frac{\partial r^{-1}}{\partial \nu} - r^{-1} \left[\frac{\partial \phi}{\partial \nu} \right] - \frac{1}{cr} \frac{\partial r}{\partial \nu} \left[\frac{\partial \phi}{\partial t} \right] \right\} dS. \tag{3}$$

This may also be written

$$\phi = \frac{1}{4\pi} \iint \left\{ - r^{-1} \left[\frac{\partial \phi}{\partial \nu} \right] + \frac{\partial}{\partial \nu} \frac{[\phi]}{r} \right\} dS, \tag{4}$$

where $\partial[\phi]/\partial\nu$ means, not $\partial\psi/\partial\nu$, but the result of differentiating the r which occurs explicitly in $[\phi]$. In the operation of forming $\partial\psi/\partial\nu$, the function ψ is differentiated at a point within the surface, and the point is then moved on to the surface; in the operation of forming $\partial[\phi]/\partial\nu$, ϕ is first formed at a point (x', y', z') of the surface, then $t - r''/c$ is substituted for t, where r'' is the distance from O of a point (x'', y'', z'') within the surface, the function of (x'', y'', z'') which thus arises is differentiated, and the point (x'', y'', z'') is then moved up to (x', y', z'). The formula (3) will be referred to as Kirchhoff's formula.*

3. The theorem expressed by equation (3) may be applied to determine the values of ϕ in space outside a closed surface S, provided ϕ and its differential coefficients of the first order are continuous functions of x, y, z, and continuous functions of t, in the region of space outside the surface, and in an interval of time which is unlimited in the negative sense; and provided also that ϕ tends to zero, at infinite distances, in the order r^{-2} at least. If ϕ tends to zero at infinite distances in the order r^{-1}, then $\frac{\partial \phi}{\partial r} + \frac{1}{c} \frac{\partial \phi}{\partial t}$ must tend to zero at infinite distances in the order r^{-2} at least. This condition would be satisfied if ϕ is of the form $f(R - ct)/R$, where R is distance from a point, which may be other than O.

The nature of the discontinuity at the surface S, of the right-hand member of (3), considered as a function of the position of O, may be investigated directly by the ordinary methods employed in the theory of potential. Denoting the function by Φ, and supposing that the normal ν is drawn to the interior of S, the results are expressed by the equations*

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* The proof given here is essentially due to Beltrami, *Rom. Acc. Linc. Rend.* (Ser. 5), t. IV., 1895.

$$\Phi \text{ inside } - \Phi \text{ outside} = \phi,$$

$$\frac{\partial \Phi}{\partial \nu} \text{ inside } - \frac{\partial \Phi}{\partial \nu} \text{ outside} = \frac{\partial \phi}{\partial \nu}.$$

The results may be utilized to show that any disturbance, which can be propagated in accordance with equation (1), must be due to sources of disturbance, at which the equation is, at some time, not satisfied.

4. *Poisson's Formula.*—When the surface S is a sphere having O as centre, r is the radius of the sphere, and the result may be written

$$\phi = \frac{1}{4\pi} \iint \left\{ \frac{[\phi]}{r^2} + \frac{1}{r} \left[\frac{\partial \phi}{\partial r} \right] + \frac{1}{cr} \left[\frac{\partial \phi}{\partial t} \right] \right\} dS. \tag{5}$$

If the sphere is of radius ct, the quantities $[\phi]$, ... are the initial values of the quantities ϕ, ... at the points on the surface. Let ϕ_0 be the initial value of ϕ, and $\dot{\phi}_0$ that of $\partial \phi / \partial t$ at any point within or on the surface, and let $\bar{\phi}_0$, $\bar{\dot{\phi}}_0$ denote the average values of these quantities over the surface of the sphere of radius ct. We have

$$\bar{\phi}_0 = \frac{1}{4\pi c^2 t^2} \iint \phi_0 \, dS, \qquad \bar{\dot{\phi}}_0 = \frac{1}{4\pi c^2 t^2} \iint \dot{\phi}_0 \, dS,$$

and it is clear that $\bar{\phi}_0$ is a function of t, which we may denote by $f(t)$. We wish to evaluate the differential coefficient $f'(t)$. We have

$$f(t') - f(t) = \frac{1}{4\pi} \iint (\phi_0)_{r = ct'} \, d\omega - \frac{1}{4\pi} \iint (\phi_0)_{r = ct} \, d\omega,$$

where $d\omega$ is the element of solid angle subtended at O by an element dS of the sphere. Further, we have

$$(\phi_0)_{r = ct'} - (\phi_0)_{r = ct} = c(t' - t) \left(\frac{\partial \phi_0}{\partial r} \right)_{r = r'},$$

where r' is a value of r intermediate between ct' and ct. Hence we have

$$f'(t) = \mathrm{Lt}_{t' = t} \frac{f(t') - f(t)}{t' - t} = \frac{c}{4\pi} \iint \left(\frac{\partial \phi_0}{\partial r} \right)_{r = ct} d\omega = \frac{c}{4\pi c^2 t^2} \iint \left[\frac{\partial \phi}{\partial r} \right] dS$$

or

$$\frac{\partial \bar{\phi}_0}{\partial t} = \frac{1}{4\pi t} \iint \frac{1}{r} \left[\frac{\partial \phi}{\partial r} \right] dS.$$

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* Cf. Larmor's proof of the formula (4) in these *Proceedings*, *supra*, p. 1.

Hence we find for ϕ Poisson's formula

$$\phi = \frac{d}{dt}(t\bar{\phi}_0) + t\bar{\dot{\phi}}_0. \tag{6}$$

5. *Theorem of Determinacy of the Solution.*—The theorems already obtained enable us to express the value of ϕ at a point on one side of a closed surface S, and at time t, in terms of the surface values of ϕ and its differential coefficients at previous times, the difference of times being in every case just so long as is required to travel from the surface element to the point with the velocity c. In the case of Kirchhoff's integral (3) the data are redundant, inasmuch as a knowledge of the surface values of ϕ alone is sufficient, to determine those of $\partial\phi/\partial t$, and is, in fact, sufficient, with a knowledge of the initial values of ϕ and $\partial\phi/\partial t$, to determine ϕ throughout the region of space in which it satisfies the differential equation (1) and the assigned conditions of continuity. This statement constitutes a part of the theorem of determinacy of the solution for the equation in question; and it follows from it that, if ϕ and $\partial\phi/\partial t$ are given initially and ϕ is given on the surface, $\partial\phi/\partial\nu$ can have only one definite value at any point on the surface. The surface values of $\partial\phi/\partial\nu$ also would suffice in the same way for the determination of ϕ, and this statement constitutes the other part of the theorem of determinacy.

To prove the theorem of determinacy, we denote by W the integral

$$\iiint \tfrac{1}{2}\left\{c^{-2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} dx\,dy\,dz, \tag{7}$$

taken through the volume within the surface S. Then

$$\begin{aligned}\frac{\partial W}{\partial t} &= \iiint \left\{c^{-2}\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\,\partial t} + \ldots + \ldots\right\} dx\,dy\,dz \\ &= \iiint \left\{c^{-2}\frac{\partial\phi}{\partial t}\left(\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi\right)\right. \\ &\qquad \left. + \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial t}\right)\right\} dx\,dy\,dz \\ &= -\iint \left(\frac{\partial\phi}{\partial\nu}\frac{\partial\phi}{\partial t}\right) dS,\end{aligned}$$

since ϕ satisfies the equation (1) throughout the volume of integration.

On integrating with respect to t between extreme values t_0 and t, we have

$$W - (W)_{t_0} = -\int_{t_0}^{t} dt \iint \frac{\partial\phi}{\partial\nu}\frac{\partial\phi}{\partial t}\,dS. \tag{8}$$

Now, if there could be two functions satisfying the initial conditions and the boundary condition, as well as the conditions of continuity and the differential equation (1), their difference would be a function ϕ for which $(W)_{t_0}$ vanishes, and either $\partial\phi/\partial\nu$ or $\partial\phi/\partial t$ would vanish at every point of S; this function would also satisfy the conditions of continuity and the differential equation (1). Equation (8) shows that, for such a function, W vanishes, and therefore the function is constant throughout the volume of integration, and throughout the interval of time beginning at t_0. Being zero at the time t_0, it is zero always.

In order that the theorem may be applicable to space outside a closed surface, it is necessary, either that at infinite distances ϕ should tend to zero in an order higher than r^{-1}, or that the limiting form of $r\phi$ at infinite distances should be specified. For example, if $r\phi$ tends to the form $A\cos\kappa(r-ct+\epsilon)$, the constants A, κ, ϵ must be specified.

6. *Wave-Boundaries without essential Discontinuity.* — Kirchhoff's integral is often used to determine the effects of disturbances which at some time are confined within a definite region of space. The particular form (6), which is here referred to as Poisson's integral, has been used to show that the disturbance spreads in such a way that the boundary of the disturbed portion of the medium moves normally to itself with velocity c. Stokes* has, in fact, founded upon Poisson's integral a well-known description of the propagation through the medium of a disturbance which is initially local. Duhem† has observed that the validity of the transformations employed in the proof of Poisson's integral depends on the continuity of ϕ and its differential coefficients of the first order across the boundary of the disturbed portion of the medium; and he has observed further that the functions which must be employed to represent the disturbance are of a non-analytical character, inasmuch as they vanish identically throughout certain regions of space and certain intervals of time, and have finite values in other regions of space and the same intervals of time. It is clear that, when the conditions of continuity above stated are satisfied, Stokes's description of the propagation of the disturbance does not stand in need of any modification on account of the non-analytical character of the functions.

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* "Dynamical Theory of Diffraction," *Camb. Phil. Soc. Trans.*, Vol. IX. (1849); *Math. and Phys. Papers*, Vol. II., p. 259.

† *Hydrodynamique, Elasticité, Acoustique*, Paris, 1901, t. I., Livre 1, chap. vii. Duhem does not consider any cases of wave-motion involving essential discontinuity. In his work the functions representing the disturbance and their differential coefficients of the first order are continuous at the wave-boundaries.

An example of a disturbance initially confined within boundaries, and expressed by non-analytical functions, is afforded by spherical waves. We may take

$$\phi_0 = f(R), \qquad \dot{\phi}_0 = -\mathrm{c}\left\{f'(R) + R^{-1}f(R)\right\},$$

where $f(\zeta)$ is a function of its argument ζ which vanishes unless $R_1 > \zeta > R_2$, and both $f(\zeta)$ and $f'(\zeta)$ vanish* when $\zeta = R_1$ or $\zeta = R_2$. For example, $f(\zeta)$ might be $(R_1-\zeta)^2(\zeta-R_2)^2$, when ζ lies between R_1 and R_2, and zero elsewhere, or again $f(\zeta)$ might be $\sin^2 \pi (R_1-\zeta)/(R_1-R_2)$ in the interval between R_1 and R_2 and zero elsewhere. The value of ϕ at any point and at any subsequent time may be determined by means of Poisson's integral, and it will be found† that

$$\phi = \frac{R-\mathrm{c}t}{R} f(R-\mathrm{c}t).$$

The assumed value of $\dot{\phi}_0$ is dictated by the condition that the waves are to be propagated wholly outwards. If any other form for $\dot{\phi}_0$ is assumed, the result is different. If, for instance, we take $\dot{\phi}_0 = 0$, we find three forms for ϕ by the application of Poisson's integral, viz.,

when $R > R_1$, $$\phi = \frac{R-\mathrm{c}t}{2R} f(R-\mathrm{c}t),$$

when $R_1 > R > R_2$, $$\phi = \frac{R-\mathrm{c}t}{2R} f(R-\mathrm{c}t) + \frac{R+\mathrm{c}t}{2R} f(R+\mathrm{c}t),$$

when $R_2 > R$, $$\phi = \frac{R+\mathrm{c}t}{2R} f(R+\mathrm{c}t).$$

The solution thus arrived at cannot hold at the origin after $t = R_2/\mathrm{c}$, and it cannot hold at any point after $t = (R+R_2)/\mathrm{c}$. If, for example, $R_2 = 0$, it cannot hold at any point at any time t, which is such that Poisson's sphere, of radius $\mathrm{c}t$ with its centre at the point, includes the origin of R. A result of this kind may indicate one of two things: either the initial state is one which could not be arrived at by natural means, or the state of the medium is not adequately represented by the solution of the differential equation (1) when ϕ is large. In the case of sound waves, for example, the ordinary approximations, by which the equations of motion are reduced to linear equations, would cease to be valid.

7. *Wave-Boundaries with Discontinuity.*—It would be an undue limitation of the theory to restrict it by those conditions of continuity in which Kirchhoff's integral and Poisson's integral have been proved to hold good. If the medium were an elastic solid, these conditions would require not merely the physically necessary continuity of the displacement at the boundary of an advancing wave, but also the continuity of the components of strain, rotation, and velocity, and there is no physical necessity for the continuity of these quantities. It is important to determine the modification (if any) which should be made in Kirchhoff's and Poisson's integrals if the conditions of continuity are not satisfied. One method of investigating this question is to assume that the abrupt variation may be treated as the limit of a continuous, though rapid, variation.‡ This assumption is liable to the objection that, if ϕ undergoes a continuous rapid change in

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* $f'(\zeta)$ is, of course, a regressive differential coefficient at $\zeta = R_1$, and a progressive differential coefficient at $\zeta = R_2$.

† The integrations are exactly similar to those in Article 13, *infra*.

‡ This method is suggested, but is not worked out in detail, by Stokes, *loc. cit.* (*Papers*, Vol. II., p. 275).

a very small distance, its differential coefficients might tend to become infinite, whereas they also ought to undergo continuous rapid changes from zero to finite values. This observation suggests that the effect of abrupt variation of ϕ, and the effect of abrupt variation of its differential coefficients of the first order, require to be treated by different methods. As we have already seen that abrupt variation of the differential coefficients of higher orders produces no effect, it appears to be legitimate to regard the abrupt variations of the differential coefficients of the first order as limits of continuous though rapid variation, provided that there is no abrupt variation of ϕ.

8. Let Σ denote the boundary of the disturbed portion of the medium at any time. This boundary cannot be fixed, and we therefore take it to be in motion. The position at a neighbouring subsequent time can be derived from the position at a previous time by subjecting the points of Σ to a displacement, directed at each point along the normal to Σ at the point. We take ω to be the normal velocity with which Σ advances; then ω may be a function of position on Σ. We denote by N the direction of the normal to Σ in the sense in which Σ advances. If the equation of Σ at time t has the form $F(x, y, z, t) = 0$, we must have

$$\frac{\partial F}{\partial t} + \omega \left\{ \cos(x, N)\frac{\partial F}{\partial x} + \cos(y, N)\frac{\partial F}{\partial y} + \cos(z, N)\frac{\partial F}{\partial z} \right\} = 0.$$

We proposed to assume that ϕ does not change discontinuously in crossing Σ. Since ϕ is zero in the undisturbed part of the medium, it has the limiting value zero when Σ is approached from the side where there is disturbance at the time t. We may therefore take $\phi = 0$ to be the equation of Σ. Hence we must have, at every point of Σ,

$$\frac{\partial \phi}{\partial t} + \omega \left\{ \cos(x, N)\frac{\partial \phi}{\partial x} + \cos(y, N)\frac{\partial \phi}{\partial y} + \cos(z, N)\frac{\partial \phi}{\partial z} \right\} = 0. \tag{9}$$

Again, since $\phi = 0$ on Σ, the gradient of ϕ in any direction on the tangent plane of Σ at any point must vanish; so that, if s is any direction at right angles to N, we must have

$$\cos(x, s)\frac{\partial \phi}{\partial x} + \cos(y, s)\frac{\partial \phi}{\partial y} + \cos(z, s)\frac{\partial \phi}{\partial z} = 0. \tag{10}$$

From this equation and (9) it follows that we must have at every point of Σ

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t}\cos(x, N) + \omega\frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial \phi}{\partial t}\cos(y, N) + \omega\frac{\partial \phi}{\partial y} = 0, \\ \frac{\partial \phi}{\partial t}\cos(z, N) + \omega\frac{\partial \phi}{\partial z} = 0. \end{aligned}\right\} \tag{11}$$

Herein $\partial\phi/\partial t$, $\partial\phi/\partial x$, ... are supposed to be calculated from the values of ϕ in the disturbed part of the medium close to Σ.

9. We consider now the case in which Σ advances into the undisturbed part of the medium, so that parts of the medium which previously were undisturbed become disturbed immediately after the passage of Σ over them. Let S_0 be a surface as close as we please to the position of Σ at time t_0, but in the portion of the medium which is then undisturbed; and let S_1 be a surface as close as we please to the position of Σ at a later time t_1, but in the portion of the medium which is then disturbed. We integrate the two members of equation (1) through a small volume standing on an element ΔS of S_0 and enclosed by S_0, S_1 and normals to S_0 at the boundary of ΔS; and we regard the differential coefficients $\partial\phi/\partial t$, $\partial\phi/\partial x$, ... as undergoing continuous but rapid variations in this volume. The integral $\iiint c^2\nabla^2\phi\,dx\,dy\,dz$ can be transformed into a surface integral taken over the boundary of the small volume, say $\iint -c^2\frac{\partial\phi}{\partial n}\,dS$. If t_1-t_0 is a very short interval of time, the walls of the volume which are formed by normals to S_0 cannot contribute anything sensible to this integral, when Σ is anywhere between S_0 and S_1; for $\partial\phi/\partial n$ vanishes at the points where these walls cut Σ, by equation (10). Further, at any such instant, $\partial\phi/\partial n$ vanishes on S_1; and thus this integral is sensibly equal to $-c^2\Delta S(\partial\phi/\partial N)$ at Σ. Herein $\partial\phi/\partial N$ is supposed to be calculated from the values of ϕ in the disturbed part of the medium close to Σ.

Let a be a coordinate measured along the normal to ΔS, and ϵ the value of a for a point on Σ at any time t' between t_0 and t_1. At time t_0, $\partial\phi/\partial t$ vanishes everywhere between S_0 and S_1; at time t', it vanishes between Σ and S_1. Hence the integral $\iiint\frac{\partial^2\phi}{\partial t^2}\,dx\,dy\,dz$ taken through the specified small volume is equal to

$$\Delta S\int_0^{\epsilon}\frac{\partial^2\phi}{\partial t^2}\,da,$$

at any time t' between t_0 and t_1. Now we may take a to be the distance described by Σ in the direction N in the interval between t_0 and t; then

$$da = \omega\,dt, \qquad \epsilon = \omega(t'-t_0),$$

and the integral in question is equal to

$$\Delta S\,\omega^2\int_0^{\omega(t'-t_0)}\frac{\partial^2\phi}{\partial a^2}\,da.$$

Now, when $a = 0$, $\partial\phi/\partial a$ is the value of $\omega^{-1}\ \partial\phi/\partial t$ on the surface S_0 at time t_0, and this is zero; also, when $a = \omega(t' - t_0)$, $\partial\phi/\partial a$ is the value of $\omega^{-1}\ \partial\phi/\partial t$ on the surface Σ at time t', *i.e.*, the value taken by $\partial\phi/\partial t$ at a point as Σ passes the point. Hence the integral in question is $\Delta S\,\omega(\partial\phi/\partial t)$.

It follows that the results of integrating the left-hand and right-hand members of (1) through the specified volume are $\Delta S\,\omega(\partial\phi/\partial t)$ and $-c^2\Delta S(\partial\phi/\partial N)$, and these must be equal. We deduce an equation which holds at every point of Σ, viz.,

$$\omega\frac{\partial\phi}{\partial t} + c^2\frac{\partial\phi}{\partial N} = 0.$$

From this equation and (11) we find that $\omega = c$, or the surface Σ advances with the same velocity as a wave-boundary which does not involve discontinuity. The result, $\omega = c$, is equivalent to Huygens' principle of wave-envelopes.

On introducing this result into equations (11), we obtain the equations

$$\frac{\partial\phi}{\partial t}\cos(x, N) + c\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial t}\cos(y, N) + c\frac{\partial\phi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial t}\cos(z, N) + c\frac{\partial\phi}{\partial z} = 0. \tag{12}$$

These equations must be satisfied at the surface separating the undisturbed from the disturbed portions of the medium. The result holds for a receding wave as well as an advancing wave, provided that the sense of N is that of the motion of the boundary, and provided that ϕ vanishes at the boundary.

The rather intricate analysis of this article constitutes an abstract proof of the proposition that the velocity with which the wave-boundary advances is the velocity c; that is to say, it is the same as the velocity of waves which do not present any essential discontinuity. From a physical point of view, this conclusion might perhaps be assumed. In the case of any mechanical medium a less abstract proof can be given. Such a proof would involve in each case the formation of an equation to determine the change of momentum of a thin slice of the medium, over which a small part of the wave-boundary passes during a very short interval of time.

10. We may apply the results (12) to show that the theorem of determinacy (Article 5) holds good without modification when ϕ vanishes at the boundary of the disturbed portion of the medium. We consider the case where the disturbance is due to sources within a closed surface S, and the disturbed portion of the medium outside S is within a moving boundary Σ at time t. We suppose that the integration in the expression (7), denoted by W, is taken through the volume between S and Σ. In

addition to the terms by which $\partial W/\partial t$ would be expressed if Σ were fixed, we have the term

$$c\iint \tfrac{1}{2}\left\{c^{-2}\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}d\Sigma$$

on account of the motion of the boundary (through $c\Delta t$ along the normal to Σ in the interval Δt); and this term is, by equations (12), equal to $c^{-1}\iint\left(\frac{\partial\phi}{\partial t}\right)^2 d\Sigma$. The contribution of the surface Σ to the terms by which $\partial W/\partial t$ would be expressed if Σ were fixed is $\iint\frac{\partial\phi}{\partial N}\frac{\partial\phi}{\partial t}d\Sigma$, because N is directed away from the volume between S and Σ; and this term is, by equations (12), equal to $-c^{-1}\iint\left(\frac{\partial\phi}{\partial t}\right)^2 d\Sigma$. Hence these terms cancel one another, and equation (8) holds good. The proof of the theorem can be completed as before.

11. If we wish to judge the validity of an attempt to determine, by means of Kirchhoff's integral or Poisson's integral, the state of the medium at a point O outside a surface S, within which there are sources of disturbance, we must revert to the process by which those integrals were obtained. In the performance of that process we assumed that the function ψ (which is derived from ϕ by substituting $t-r/c$ for t) is continuous, and its differential coefficients of the first order also are continuous, within the volume of integration. If the disturbed portion of the medium outside S has a boundary Σ, in crossing which the differential coefficients of ϕ are discontinuous at time t, the surfaces Σ_1, Σ_2, ... which are the positions of Σ at different times t_1, t_2, ... are a family of parallel surfaces. The locus of points at which differential coefficients of ψ are discontinuous at time t is a surface Σ', which is the locus of the curve of intersection of any surface such as Σ_1 and a sphere of radius $c(t-t_1)$ with its centre at O. It follows that, until the surface Σ passes the point O, the surface Σ' does not begin to be formed. Until this instant, ϕ vanishes at O. Immediately after Σ has passed O, Σ' is formed; and then the volume of integration must not contain any points which lie on the side of Σ' remote from O, and thus it is bounded partly by S and partly by Σ'. When we transform the volume integral, we shall have to take account of the contribution of the surface Σ' to the right-hand member of (3). This contribution is

$$\frac{1}{4\pi}\iint\left\{[\phi]\frac{\partial r^{-1}}{\partial \nu}-r^{-1}\left[\frac{\partial\phi}{\partial\nu}\right]-\frac{1}{cr}\frac{\partial r}{\partial\nu}\left[\frac{\partial\phi}{\partial t}\right]\right\}d\Sigma', \tag{13}$$

in which ν denotes the direction of the normal to Σ', drawn towards that side of Σ' on which O lies, and $[\phi]$, ... are the values of ϕ, ... at the points of Σ', at the instants when the boundaries such as Σ_1 are passing these points. Thus $[\phi]$ is equal to zero. Also, by equations (12), we have

$$\left[\frac{\partial\phi}{\partial\nu}\right] = -\frac{1}{c}\cos(\nu, N)\left[\frac{\partial\phi}{\partial t}\right],$$

N denoting the direction of the normal, at any point of Σ', to that surface of the family Σ which passes through the point at time $t-r/c$, and the sense of N being that in which Σ moves. Hence the expression (13) may be written

$$\frac{1}{4\pi c}\iint\frac{1}{r}\{\cos(\nu, N)-\cos(\nu, r)\}\left[\frac{\partial\phi}{\partial t}\right]d\Sigma'.$$

We may show that $\cos(\nu, N) = \cos(\nu, r)$ at all points of Σ', and therefore that this expression vanishes identically. This may be proved by means of the geometrical properties of the surface Σ'.* Let Σ_1, Σ_2 be neighbouring positions of Σ at times t_1, t_2. The points in which Σ' cuts Σ_1 lie on a sphere, centre O and radius $c(t-t_1)$; a similar statement holds for the surface Σ_2. Let P_1 (Fig. 1) be a point common to Σ' and Σ_1; and let P_2 be that point, common to Σ' and Σ_2, which is in the plane through OP_1 and the direction N. The normal distance between the parallel surfaces Σ_1 and Σ_2 is equal to $c(t_2-t_1)$, and this is equal to OP_1-OP_2. It follows that that tangent line at P_1 to the surface Σ', which is in the plane OP_1N, bisects the angle OP_1N. Now the plane OP_1N is at right angles to that tangent line of Σ' at P_1, which also touches Σ_1; for this line is at right angles to N, being in the tangent plane of Σ_1 at P_1, and it is at right angles to OP_1, being in the tangent plane at P_1 to a sphere with centre

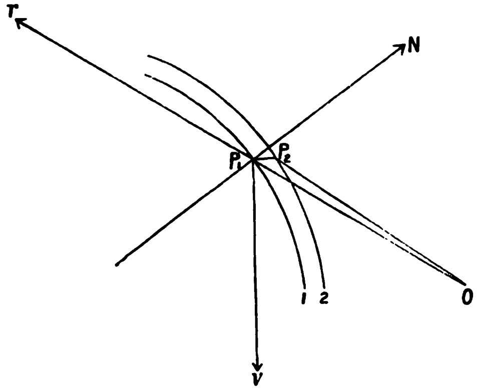

FIG. 1.

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* An analytical proof can be devised quite easily.

at O and radius OP_1. Hence the plane OP_1N contains the normal ν to the surface Σ' at P_1. It follows that the normal ν is also a bisector of the angle OP_1N, and therefore $\cos(\nu, N) = \cos(\nu, r)$, as stated.

An example which shows well the genesis of the surface Σ' is afforded by waves diverging from a point source. If R denotes distance from the source, and the boundary Σ at time t is given by the equation $R = R_0 + ct$, the surface Σ' is a prolate spheroid given by the equation $R + r = R_0 + ct$. If h denotes the distance of O from the source, the spheroid does not exist when $t < (h - R_0)/c$. When $t = (h - R_0)/c$, the spheroid is the straight line joining O to the source. As t increases the spheroid expands, and O and the source are its foci. The property expressed by the equation $\cos(\nu, N) = \cos(\nu, r)$ is a known property of the spheroid, viz., that the normal bisects the angle between the focal radii.

Since the contribution of Σ' to the expression (3) vanishes, we may apply Kirchhoff's integral to determine the value of ϕ at any point O outside a surface S, which contains all the sources of disturbance, in terms of the values of $[\phi]$, $[\partial\phi/\partial\nu]$, $[\partial\phi/\partial t]$ on S, exactly as in the case where the differential coefficients of ϕ of the first order are continuous outside S; and we may also apply Poisson's integral to determine the value of ϕ at a point in terms of the initial values of ϕ and $\partial\phi/\partial t$ in all space.

[*Note added February 14th*, 1903.—It may be remarked that the property of contributing zero to Kirchhoff's integral is possessed by other surfaces as well as Σ'. The surface Σ' is derived from the surface Σ by replacing t in the equation of Σ by $t - r/c$; and it is therefore determined by the surface Σ and by the choice of a point, the origin of r. The surface Σ is one at which ϕ vanishes, but its differential coefficients are discontinuous. If ϕ vanishes at a surface, which moves normally to itself with velocity c, and the differential coefficients of ϕ are continuous at the surface, a surface analogous to Σ' may be derived by changing t into $t - r/c$ in the equation $\phi(x, y, z, t) = 0$. The equation $\partial\phi/\partial t + c\,\partial\phi/\partial N = 0$ will then be satisfied at the surface $\phi = 0$, and the equation $\partial\phi/\partial s = 0$ will hold for any tangential direction ds at any point on this surface. Equations (12) are therefore satisfied at all points of this surface $\phi = 0$. When a new surface is derived by changing t into $t - r/c$ as above, this new surface will possess the properties expressed by the equations

$$[\phi] = 0, \qquad \left[\frac{\partial\phi}{\partial\nu}\right] = -\frac{1}{c}\cos(\nu, N)\left[\frac{\partial\phi}{\partial t}\right], \qquad \cos(\nu, N) = \cos(\nu, r);$$

and from these equations it follows that the surface in question contributes nothing to Kirchhoff's integral.]

12. The method of the last article cannot be applied in case ϕ does not vanish at the boundary Σ; for in this case $[\phi]$ does not vanish at the surface Σ', and the equations (12) do not hold good. In any particular case, there may be equations corresponding with (12), which must hold at the boundary of the disturbed portion of the medium; but they do not enable us to discard the contribution of Σ' to the right-hand member of

(3). For example, in the case of waves diverging symmetrically from a source, the condition

$$\frac{\partial\phi}{\partial R}+\frac{\phi}{R}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0$$

must hold at the spherical boundary Σ, and the property expressed by the equation $\cos(\nu, R)=\cos(\nu, r)$ must hold at Σ'; and thus the contribution of Σ' to the right-hand member of (3) is

$$\frac{1}{4\pi}\iint[\phi]\frac{1}{r}\left(\frac{1}{R}-\frac{1}{r}\right)\cos(r, \nu)\,d\Sigma',$$

but we have no means of evaluating this integral. If the motion has gone on for a long time, and the boundary Σ is at a great distance from O, this correction of Kirchhoff's formula is unimportant.

The result that ϕ is not given by Kirchhoff's formula, when it is discontinuous at a moving boundary, is not in conflict with Larmor's intuitional proof of the formula;* for in that proof it is assumed that ϕ has no singularities other than the actual sources of disturbance.

13. As an example we may take spherical waves expressed by the equation

$$\phi=R^{-1}f(R-ct),$$

where $f(\zeta)$ is zero when ζ is either R_1 or R_2, and also when ζ is not between these values.

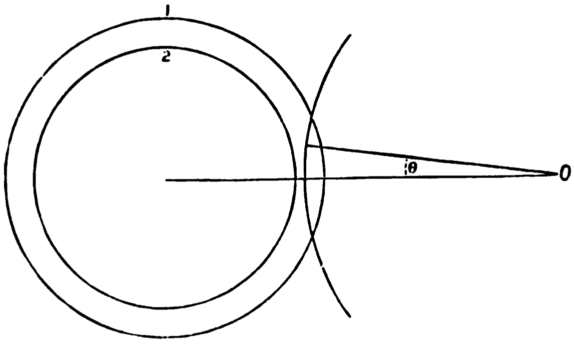

FIG. 2.

The initial values of ϕ and $\partial\phi/\partial t$ are given by the equations

$$\phi_0=R^{-1}f(R), \qquad \dot{\phi}_0=-cR^{-1}f'(R)$$

in the interval for R, $R_1\geqslant R\geqslant R_2$. Outside this interval ϕ_0 and $\dot{\phi}_0$ vanish.

Let R_0 be the distance of any point O from the origin of R; we shall form the expression for ϕ at O by means of Poisson's integral. We have

$$\phi=\frac{1}{4\pi}\iint\left(\frac{\phi_0}{r^2}+\frac{\dot{\phi}_0}{cr}+\frac{1}{r}\frac{\partial\phi_0}{\partial r}\right)dS,$$

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* *Supra*, p. 1.

where the integration is taken over a sphere with centre O and radius $r\ (= \mathrm{c}t)$. We suppose this sphere to cut into the region to which the initial disturbance is confined.

Let θ be the angle between any radius vector r and the line joining O to the origin of R; we have

$$R^2 = R_0^2 + r^2 - 2R_0 r\cos\theta, \qquad R_0 r\sin\theta\, d\theta = R\,dR;$$

also we have

$$\frac{\partial\phi_0}{\partial r} = \frac{\partial\phi_0}{\partial R}\cos(R, r) = \frac{\partial\phi_0}{\partial R}\,\frac{R^2+r^2-R_0^2}{2Rr}.$$

Hence

$$\phi = \tfrac{1}{2}\int\left\{\frac{R^2+r^2-R_0^2}{2rR_0}\,\frac{d}{dR}\,\frac{f(R)}{R} + \frac{f(R)}{rR_0} - \frac{f'(R)}{R_0}\right\}dR,$$

where the limits of integration are certain values of R, say R_1' and R_2'. The integral is

$$\Big|_{R'_2}^{R'_1}\left\{\frac{R^2+r^2-R_0^2}{2rR_0}\,\frac{f(R)}{R} - \frac{f(R)}{R_0}\right\}.$$

When $R_0 > R_1$, and r lies between R_0-R_1 and R_0-R_2, we have the arrangement in the figure, and $R_1' = R_1$, $R_2' = R_0-r$, so that $\phi = R_0^{-1}f(R_0-\mathrm{c}t)$.

When $R_0 > R_1$ and $r > R_0-R_2$, the limits are $R_1' = R_1$, $R_2' = R_2$, and the integral vanishes.

When R_0 lies between R_1 and R_2, and r is not too great, the limits are R_0+r and R_0-r; the terms at R_0+r vanish, and the terms at R_0-r give us $\phi = R_0^{-1}f(R_0-\mathrm{c}t)$. When the upper limit becomes R_1, instead of R_0+r, the terms at the upper limit again vanish. When $r > R_0-R_2$ the lower limit becomes R_2, and then ϕ vanishes.

If $R < R_2$, we have $R_2' = R_2$, and at first $R_1' = R_0+r$, and then ϕ vanishes. As r increases we come to have $R_1' = R_1$, and ϕ still vanishes.

This example may be utilized to illustrate the result that the value of ϕ is not given by Poisson's integral, unless ϕ vanishes at the boundary of the disturbed portion of the medium. The integration is effected by the same process whatever function $f(R)$ may be, and it does not depend at all upon the vanishing of $f(R)$ at R_1 and R_2. The process might be applied to the function $\dot{\phi}$, derived from ϕ by differentiating with respect to t, and Poisson's integral would give

$$\tfrac{1}{2}\Big|_{R'_2}^{R'_1} - \mathrm{c}\left\{\frac{R^2+r^2-R_0^2}{2rR_0}\,\frac{f'(R)}{R} - \frac{f'(R)}{R_0}\right\}.$$

It will be sufficient to take the case $R_0 > R_1$, $R_0-R_2 > r > R_0-R_1$, for which $R_1' = R_1$, $R_2' = R_0-r$; the result would be

$$-\mathrm{c}\left\{\frac{f'(R_0-\mathrm{c}t)}{R_0} + \frac{(R_1-\mathrm{c}t)^2-R_0^2}{4R_1R_0\mathrm{c}t}f'(R_1)\right\}.$$

The true value of $\dot{\phi}$ might be calculated from ϕ, and it is $-\mathrm{c}R_0^{-1}f'(R_0-\mathrm{c}t)$.

The analysis of this article would be applicable to the propagation of a sound wave, which, at some instant, forms a *pulse*, and is propagated wholly outwards.[*] The condensation $(-\mathrm{c}^{-2}\dot{\phi})$ is discontinuous at the wave-boundary, and cannot be calculated from Poisson's integral formula, although it satisfies the characteristic equation (1).

Adaptation to particular Media.

14. So far we have been concerned with abstract wave-theory; and we have found necessary conditions which must be satisfied at the boundary of a wave, when the function representing the disturbance is continuous

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[*] In the paper as originally written the analysis of trains of waves into a succession of such pulses was considered, and an application of this analysis to the theory of the rectilinear propagation of light was given. This part is now withdrawn, as it requires to be completed in some important respects, and would in any case be more appropriate as a separate communication.—*January 30th*, 1903.

in crossing the boundary. In addition to these conditions, the function may have to satisfy others, imposed by the constitution of the medium or the nature of the disturbance, if it is to represent waves of a specified type transmitted through a specified medium.* We shall consider, in order, waves in an isotropic elastic solid medium, sound waves in air, and electric waves in free æther.

15. *Waves in an Isotropic Elastic Solid Medium.*—There are two types of waves, dilatational and distortional; and, when waves of either type are being propagated, the components u, v, w of the displacement satisfy an equation of the form (1), in which c must be the velocity appropriate to the type of waves. The continuity of the medium requires u, v, and w to be continuous at the boundaries of a wave; and the conclusions of the abstract theory are therefore applicable to u, v, and w separately. In particular the displacement due to a limited initial disturbance may be calculated by means of Poisson's integral formula. The same formula may not in general be applied to calculate the dilatation, or the components of strain, or of rotation; but the values of these quantities can be calculated from the displacement when the latter is known. The condition that (u, v, w) vanishes at a wave-boundary, combined with the condition that the wave-boundary advances with the appropriate velocity c, shows that some sets of functions, which can represent displacements in an elastic solid executing small motions, cannot represent wave-motions arising from a limited initial disturbance. For example, the system of dilatational waves expressed by the equations

$$(u, v, w) = A\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\sin\kappa(R-\mathrm{c}t)}{R}$$

cannot arise from a limited initial disturbance; for the displacement vanishes at the spheres which are given by the equation

$$\sin\kappa(R-\mathrm{c}t)-\kappa R\cos\kappa(R-\mathrm{c}t) = 0,$$

but these surfaces do not move forward with the velocity c.

The proof that the velocity of advance of the wave-boundary is the velocity c, equal to $\sqrt{\{(\lambda+2\mu)/\rho\}}$ or to $\sqrt{(\mu/\rho)}$, as the case may be, can be obtained by forming an equation to determine the change of momentum of a thin slice of the medium over which the wave-boundary advances in a very short time. The method is equivalent to that adopted by Christoffel † in the case of an æolotropic medium.

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* The importance of these conditions was emphasized by Dr. Larmor at the meeting at which the paper was communicated.

† *Ann. di Mat.* (Ser. 2), t. VIII., 1877.

16. *Application to Theory of Sound.*—In the case of sound waves of small amplitude given by a velocity potential ϕ, we may not assume that ϕ is continuous at the boundary of an advancing disturbance. We might in this case proceed, as in the case of an elastic solid, by making use of a displacement of the elements of the aerial medium from a state in which it is at one uniform pressure and density. For the purposes of the theory of sound this procedure would involve unncessary complication.

With the ordinary approximations the velocity, pressure and density at a point are determined by means of ϕ in the forms

$$\text{velocity} = \text{gradient of } \phi = \left(\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z}\right),$$

$$\text{excess of pressure in disturbed part of medium} = -\rho_0\frac{\partial\phi}{\partial t},$$

$$\text{condensation} = -\mathrm{c}^{-2}\frac{\partial\phi}{\partial t},$$

ρ_0 being the density in the undisturbed state. The equation of conservation of mass of an elementary portion of a thin stratum, over which the boundary Σ moves in time Δt, is

$$\rho_0\left(1-\mathrm{c}^{-2}\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial N}-\omega\right) = -\rho_0\omega,$$

and the equation of momentum* for the same portion is

$$\rho_0\omega\Delta t\frac{\partial\phi}{\partial N} = -\rho_0\frac{\partial\phi}{\partial t}\Delta t.$$

Hence we have the two equations

$$\omega\frac{\partial\phi}{\partial t}+\mathrm{c}^2\frac{\partial\phi}{\partial N} = 0, \qquad \frac{\partial\phi}{\partial t}+\omega\frac{\partial\phi}{\partial N} = 0; \tag{14}$$

so that $\omega^2 = \mathrm{c}^2$.

The equation $\omega = \mathrm{c}$, which is thus proved to hold at the boundary of an advancing sound wave of small amplitude, shows that the boundary of the wave moves normally to itself with a velocity equal to the constant c which occurs in the characteristic equation, a result usually deduced from Poisson's integral formula.† The equation

$$\frac{\partial\phi}{\partial N} = -\mathrm{c}^{-1}\frac{\partial\phi}{\partial t}$$

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* The difficulty that arises from the consideration of the changes of the energy when the amplitude is not small are avoided in the case of small amplitude. Cf. Rayleigh, *Theory of Sound*, Vol. II., § 250.

† Rayleigh, *Theory of Sound*, Vol. II., § 274.

is expressible in the form

$$\text{velocity} = c\ (\text{condensation}),$$

a result which is known to hold in the case of plane waves; it is here shown to hold at the boundary of an advancing wave of any form.

The equation of impulsive motion which we have used is obtained by resolving momenta along the normal to Σ. By resolving them in any tangential direction we could show that the tangential component of velocity must vanish at all points on Σ. It follows that Σ is an equipotential surface, and that the equations (12) hold at every point of it. From this result we may deduce the theorem of determinacy. Since ϕ is not a physically determinate quantity, but values of ϕ which differ by a constant are physically equivalent, it follows that, when there is a single boundary, we may take ϕ to be zero at that boundary. But when there is more than one boundary it may be impossible to take ϕ to be zero at all the boundaries. In a spherical sound wave ϕ is of the form $R^{-1}f(R-ct)$, so that equations (14) cannot be satisfied at a spherical surface unless $\phi = 0$ at that surface. Whenever we know that $\phi = 0$ at the wave-boundary, we may determine the motion by means of Poisson's integral formula. There are few problems in the theory of sound, as developed hitherto, which require the application of any such formula.

17. *Electric Waves. Conditions at a Wave-Boundary.*—In the case of electric waves in free æther we may not assume that any of the vector quantities, by which the disturbance is determined, is continuous at a wave-boundary. The conditions which must hold at such a boundary may be investigated by means of the equations which govern the changes of state of the medium. We shall write these equations in the form*

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}(X, Y, Z) &= \operatorname{curl}(\alpha, \beta, \gamma), \qquad \operatorname{div.}(X, Y, Z) = 0, \\ -\frac{1}{c}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \operatorname{curl}(X, Y, Z), \qquad \operatorname{div.}(\alpha, \beta, \gamma) = 0. \end{aligned}\right\} \tag{15}$$

in which (X, Y, Z) denotes electric force at a point, measured electrostatically, (α, β, γ) denotes magnetic force at the point, measured in electromagnetic units, and c is the velocity of light in free æther. We may integrate the left-hand and right-hand members of the equations of (15) which involve differential coefficients with respect to t through a small volume in the neighbourhood of a surface Σ, which bounds the disturbed portion of the æther at any instant, and moves with velocity ω in the direction of the normal N at any point of it. The process is precisely similar to that of Article 9. The result will be expressed by six equations: viz., three of the form

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* Cf. Hertz, *Electric Waves*, p. 138. There are some differences between these equations and those used by Hertz which arise from his employment of a left-handed system of axes of (x, y, z). The system adopted in the text is right-handed.

$$\omega c^{-1} X = \cos(z, N)\beta - \cos(y, N)\gamma, \tag{16}$$

and three of the form

$$-\omega c^{-1} \alpha = \cos(z, N) Y - \cos(y, N) Z. \tag{17}$$

These equations may be interpreted in the following statements* :—

(i.) At a wave-boundary the electric and magnetic forces on the side where there is disturbance are tangential to the boundary.

(ii.) These forces are at right angles to each other.

(iii.) The velocity of propagation of the boundary is the velocity c.

(iv.) The electric and magnetic forces are equal in magnitude at any point of the boundary, and the directions of the electric force, the magnetic force, and the direction of propagation (N), in this order, are parallel to the positive senses of a right-handed system of axes of (x', y', z').

The result (iii.) is proved by observing that the equations are incompatible unless $\omega^2 = c^2$. In obtaining the result (iv.), ω has been put equal to c.

18. We may utilize these results to prove that the theorem of determinacy is unaffected by the existence of a moving boundary to the disturbed portion of the medium. The theorem is that the state of the medium outside a fixed boundary S is determined by the initial state and the values either of the tangential components of the electric force, or of the tangential components of the magnetic force, at all points on S. If Σ is the moving boundary at time t, and W denotes the value of the integral

$$\iiint (X^2 + Y^2 + Z^2 + \alpha^2 + \beta^2 + \gamma^2)\, dx\, dy\, dz$$

taken through the space between S and Σ, the terms of $\partial W/\partial t$ contributed by Σ are

$$c \iint (X^2 + Y^2 + Z^2 + \alpha^2 + \beta^2 + \gamma^2)\, d\Sigma$$
$$+ c \iint \{\cos(x, N)(\beta Z - \gamma Y) + \cos(y, N)(\gamma X - \alpha Z) + \cos(z, N)(\alpha Y - \beta X)\}\, d\Sigma,$$

of which the first depends on the motion of Σ, and the second arises from the transformation of the volume integral in the expression for $\partial W/\partial t$. The sum of these two terms vanishes identically, and the proof proceeds exactly as in the case where there is no moving boundary.*

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* Equivalent results to those here obtained appear to have been employed by Heaviside, *Electrical Papers*, Vol. II., pp. 405 *et seq.*

19. Unless we know some quantity, which satisfies equation (1), and serves to determine the electric and magnetic forces at any point, in such a way that these forces obey the conditions (16) and (17), with $\omega = \mathrm{c}$, at surfaces at which this quantity vanishes, we cannot utilize Kirchhoff's integral or Poisson's integral to determine the state of the medium at one place and time in terms of the states at other places and at previous times. We can only say that the use of these integral formulæ yields approximately correct results in cases of long continued radiation, and at places which are very distant from the wave-boundary.

In the dynamical theory of the æther propounded by Larmor,† in which magnetic force represents velocity of some kind in the æther, such a quantity as is required may be found in the displacement corresponding with this velocity, for it seems reasonable to suppose that this displacement is continuous. If we introduce this displacement as a circuital vector (u, v, w), of which the components satisfy the equation (1), we have the relations

$$(\alpha, \beta, \gamma) = \frac{\partial}{\partial t}(u, v, w), \qquad (X, Y, Z) = \mathrm{c}\,\mathrm{curl}\,(u, v, w). \tag{18}$$

If we assume that (u, v, w) vanishes at a wave-boundary, we may use the integral formulæ in question to determine these components separately, and we may deduce the values of (α, β, γ) and (X, Y, Z). It can be proved without difficulty that the conditions (16) and (17), with $\omega = \mathrm{c}$, hold at any surface at which (u, v, w) vanishes.

Whether we adopt this theory or not, we may construct formulæ to represent electric waves with boundaries at which conditions (16) and (17) are satisfied. Such formulæ are most conveniently expressed in terms of the vector (u, v, w); and, if (u, v, w) does not vanish at the wave-boundaries, the conditions (16) and (17) must be introduced. The vanishing of (u, v, w) at the wave-boundaries is, as we have seen already, sufficient to secure the satisfaction of (16), (17); so that any displacement, corresponding with distortional waves in an elastic solid, may be interpreted as a "magnetic displacement," corresponding with electric waves in free æther.

In the case of spherical waves diverging from a source, such formulæ may be expressed in terms of a scalar function ϕ, which satisfies equation (1) and is of the form $r^{-1}f(r-\mathrm{c}t)$, and a spherical solid harmonic (ω_n) of degree n. There are two types of such formulæ.* One type is obtained by putting

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* Cf. A. E. H. Love, *Phil. Trans. R. S.* (Ser. A), Vol. CXCVII., 1901.

† *Phil. Trans. R. S.* (Ser. A.), Vol. CLXXXV., 1894.

$$(u,\ v,\ w) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left(\frac{f}{r}\right) . \left(y\frac{\partial \omega_n}{\partial z} - z\frac{\partial \omega_n}{\partial y},\ z\frac{\partial \omega_n}{\partial x} - x\frac{\partial \omega_n}{\partial z},\ x\frac{\partial \omega_n}{\partial y} - y\frac{\partial \omega_n}{\partial x}\right), \tag{19}$$

and the other is obtained from the right-hand member of this equation by the operation *curl*. If the conditions (16) and (17) can be satisfied at any spherical surface, at which the function f vanishes, the formulæ can represent electric waves having that surface as a wave-boundary.

20. A particular example, of some importance, is arrived at by supposing that $f = \sin\kappa(r-\mathrm{c}t)$, and that $\omega_n = -z$, so that $n = 1$. The vector (u, v, w) is then given by the equations

$$(u,\ v,\ w) = \left(-\frac{\partial}{\partial y},\ \frac{\partial}{\partial x},\ 0\right)\frac{\sin\kappa(r-\mathrm{c}t)}{r}, \tag{20}$$

and the magnetic and electric forces are given by the equations

$$\left.\begin{aligned} (\alpha,\beta,\gamma) &= \left(-\frac{\partial^2}{\partial y\,\partial t},\ \frac{\partial^2}{\partial x\,\partial t},\ 0\right)\frac{\sin\kappa(r-\mathrm{c}t)}{r}, \\ (X,\ Y,\ Z) &= \mathrm{c}\left(-\frac{\partial^2}{\partial x\,\partial z},\ -\frac{\partial^2}{\partial y\,\partial z},\ \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\sin\kappa(r-\mathrm{c}t)}{r}. \end{aligned}\right\} \tag{21}$$

The boundaries at which f vanishes are spheres moving radially outwards with velocity c. The surfaces to which the electric and magnetic forces are both tangential are surfaces of revolution given by an equation of the form

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\frac{\sin\kappa(r-\mathrm{c}t)}{r} = \text{const.} \tag{22}$$

These two families of surfaces have no surface in common, and the second system contains no member which moves normally to itself with velocity c. The equations (21) cannot therefore represent electric waves which have a beginning in time.

The formulæ (21) are equivalent to those given by Hertz † to represent the waves emitted by his oscillator. It appears that they stand in need of some modification in consequence of the necessary existence of a wave-boundary. We have seen that such a boundary must contain the directions of the electric and magnetic forces at every point on it, and it must move normally to itself with velocity c.

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* A. E. H. Love, *loc. cit.*

† *Electric Waves*, pp. 141, 142. I cannot find that Hertz paid any attention to the effects of a boundary of his waves. The figure (Fig. 27), which he draws to represent the state of the field at time $t = 0$, is the figure corresponding with the beginning of a period in an indefinite sequence of oscillations.

21. A particular example, which may possibly be useful for the representation of waves sent out by a Hertzian oscillator, will be found by taking $n = 1$, $\omega_n = -z$, and assuming for $f(r-ct)$ the following rules of calculation:—(i.) Outside the interval $2p\pi/\kappa < \zeta < 2q\pi/\kappa$, $f(\zeta) = 0$; (ii.) throughout this interval $f(\zeta) = \sin^2 \frac{1}{2}\kappa\zeta$. In this statement p and q are any positive integers ($p < q$).

With this assumption for f, the vector (u, v, w) is given by the rules:—(i.) Outside the interval $2p\pi/\kappa < r-ct < 2q\pi/\kappa$, $(u, v, w) = 0$; (ii.) throughout this interval

$$(u, v, w) = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 0\right)\frac{\sin^2 \frac{1}{2}\kappa(r-ct)}{r}. \tag{23}$$

Throughout the same interval the magnetic and electric forces are given by

$$\left.\begin{aligned} (\alpha, \beta, \gamma) &= \tfrac{1}{2}\kappa c\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}, 0\right)\frac{\sin \kappa(r-ct)}{r}, \\ (X, Y, Z) &= -\tfrac{1}{2}c\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\partial r^{-1}}{\partial z} \\ &\quad + \tfrac{1}{2}c\left(\frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}, -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\frac{\cos \kappa(r-ct)}{r}. \end{aligned}\right\} \tag{24}$$

Both these vectors vanish outside the stated interval. Since (u, v, w) vanishes at the surfaces $\sin \frac{1}{2}\kappa(r-ct) = 0$, the conditions (16) and (17) are satisfied at these surfaces; and, since these surfaces move with the velocity c, they can be boundaries of a wave. It may be observed that neither the electric force nor the magnetic force vanishes at the boundary of the wave.

* Equations (24) represent a system of electric waves which have a definite front and rear; the waves are generated at a source and advance into a region of space in which there is no field of magnetic or electric force. Behind the rear of the wave also, there is neither magnetic nor electric force. Within the wave there are two superposed fields of force, viz., (i.) a steady electrostatic field, the same as would be due to an electric

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[* The two final paragraphs of this article have been re-written at the suggestion of the referees, to both of whom the author is much indebted for the care and trouble they have bestowed upon the paper.—*February 14th*, 1903.]

doublet at the same source; and (ii.) an alternating electromagnetic field of the same kind as that represented by Hertz's formulæ (21). The adjustments of amplitude and phase which are required for the identification of the alternating field with that represented by Hertz's formulæ can always be made. Both the steady field and the alternating field are carried along with the wave, and neither can exist without the other.

In the application to waves emitted by a Hertzian oscillator, we may regard the disturbance expressed by (24) as propagated into a region within which the electrostatic field expressed by the equation

$$(X, Y, Z) = \tfrac{1}{2}c\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\partial r^{-1}}{\partial z}$$

is already established. During the passage of the waves the electrostatic field is obliterated, and after the passage of the waves it is restored.* The forces, which could be detected at a point during the passage of the waves, are equivalent to those expressed by Hertz's formulæ.

Summary.

22. In the abstract wave-theory discussed in the first part of this paper it has been shown that there are three types of waves with boundaries, viz., (i.) waves in which neither the function representing the disturbance, nor any of its differential coefficients of the first order, is discontinuous at the boundaries; (ii.) waves in which the function is continuous at the boundaries, but its differential coefficients are not; (iii.) waves in which the function is discontinuous at the boundaries.

In case (i.) the conditions which are necessary to secure the validity of known proofs of Kirchhoff's and Poisson's integral formulæ, and of the theorem of determinacy, hold good.

In case (ii.) it is shown that the values, at the wave-boundary, of the temporal and spatial gradients of the function are connected by definite relations, which may be regarded as generalizations of a relation known to hold for progressive plane waves of sound, viz., the condensation is equal to the ratio of the velocity of the aerial medium to the velocity of sound. It is shown also that the velocity of advance of a wave-boundary, involving discontinuity of this type, is the same as that of a wave which does not involve discontinuity. The theorem of determinacy holds without modification. When a wave advances into the previously undisturbed part of the medium, no disturbance takes place at a point until the wave-boundary passes the point. At any instant t after the wave has passed a point O, the function which expresses the disturbance at O can be calculated by Kirchhoff's formula, provided that the surface, to which the formula is applied, lies entirely within the region of continuity of the function and its first differential coefficients, and provided further that these conditions of continuity hold at times previous to t by intervals not exceeding those required to travel from the points of the surface to the point O with the velocity c. This region is therefore bounded, partly by a surface S which surrounds the sources of disturbance, and partly by a surface Σ' which is the locus of certain points on the various positions of the wave-boundary at times previous to t; and these points on this family of surfaces are just so distant from the point O that waves setting out from them, at the instants when the actual wave-boundary passes them, would reach O at the time t. It is then proved that the contributions of the elements of the surface Σ' to the disturbance at O vanish identically; and the validity of Kirchhoff's and Poisson's integral formulæ is thus shown to be unimpaired by the existence of a wave-boundary involving discontinuity in the temporal and spatial gradients of the disturbance.

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* It ought perhaps to be made clear that this discussion is not intended primarily as a contribution to the theory of a Hertzian oscillator, but as an illustration of possible types of electric waves with boundaries. A theory of the action of the oscillator ought to take account of the dissipation. Cf. J. J. Thomson, *Recent Researches*, pp. 396 *et seq.*

In the class (iii.) of waves with boundaries, in which the function representing the disturbance is discontinuous at a wave-boundary, Kirchhoff's and Poisson's integral formulæ cannot be applied to the calculation of the function. This conclusion is demonstrated directly, and it is illustrated by calculations concerning known types of waves.

The abstract theory is applicable directly to dilatational and distortional waves in an isotropic elastic solid medium, because the components of the displacement satisfy the characteristic equation of wave motion, and are necessarily continuous. The application to sound waves in air is less direct. It is shown that the equations of motion and of continuity suffice to determine the conditions which must be satisfied at a wave-boundary. It appears to be possible that the application of Poisson's formula to determine the velocity potential of sound waves due to a limited initial disturbance may not always be legitimate; but it is shown that the principal results, which, in the theory of sound, are deduced from this formula, are not invalidated. These results relate to the principle of wave-envelopes, and to the velocity of advance of a wave-boundary.

In the application of the abstract theory to electric waves in free æther, difficulties arise from our ignorance of the constitution of the medium. The electric and magnetic forces are known to obey the two circuital relations of Faraday in regions in which they are continuous; but nothing is known about displacements, or necessary physical conditions of continuity. It has been shown here that a wave-boundary, at which electric and magnetic forces are discontinuous, must advance normally to itself with the velocity of light; and that these forces are tangential to the boundary, at right angles to each other, and numerically equal, when measured in terms of certain units. A method is described for constructing formulæ to represent electric waves with spherical boundaries. The formulæ given by Hertz, to represent the field around his oscillator, are considered as an example, and they are shown to be incapable of representing waves with boundaries, outside which there is neither magnetic force nor electric force; and a modification of these formulæ is proposed. The modified formulæ differ from Hertz's by the addition of an electrostatic field due to an invariable electric doublet occupying the position of the oscillator. Hertz's formulæ represent the disturbance within a wave which advances through the field of this doublet.

SERIES CONNECTED WITH THE ENUMERATION OF PARTITIONS

By the REV. F. H. JACKSON.

[Received in revised form March 9th, 1903.*]

1. The principal object of this paper is to investigate a theorem of which the following are interesting particular cases :—

$$\frac{\Gamma(p^{\gamma-a-\beta}z)\,\Gamma(p^{\gamma}z)}{\Gamma(p^{\gamma-a}z)\,\Gamma(p^{\gamma-\beta}z)}$$

$$= 1+p^{l}\frac{p^{a}-1\,.\,p^{\beta}-1}{p^{l}-1\,.\,p^{\gamma}+z}\,z+p^{2l}\frac{p^{a}-1\,.\,p^{a+l}-1\,.\,p^{\beta}-1\,.\,p^{\beta+l}-1}{p^{l}-1\,.\,p^{2l}-1\,.\,p^{\gamma}+z\,.\,p^{\gamma+l}+z}\,z^{2}+\ldots, \tag{1}$$

$$\frac{\Gamma(p^{\gamma-a-\beta})\,\Gamma(p^{\gamma}z)}{\Gamma(p^{\gamma-\beta})\,\Gamma(p^{\gamma-a}z)}$$

$$= 1+p^{\gamma-a-\beta}\frac{p^{a}-1\,.\,p^{\beta}+z}{p^{l}-1\,.\,p^{\gamma}+z}+p^{2(\gamma-a-\beta)}\frac{p^{a}-1\,.\,p^{a+l}-1\,.\,p^{\beta}+z\,.\,p^{\beta+l}+z}{p^{l}-1\,.\,p^{2l}-1\,.\,p^{\gamma}+z\,.\,p^{\gamma+l}+z}+\ldots. \tag{2}$$

Γ does not denote Euler's Gamma Function, but a certain convergent infinite product. Each of the above theorems can be reduced to

$$\frac{\Gamma(\gamma-a-\beta)\,\Gamma(\gamma)}{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)} = F(a\beta\gamma)$$

by giving particular values to p and z. Γ will then denote Euler's gamma function. The general theorems referred to are

$$[P^{x}z]^{n} = 1+\Sigma P^{rx+\frac{1}{2}(r\,.\,r-1)l}\,\frac{P^{nl}-1\,.\,P^{(n-1)l}-1\,\ldots\,P^{(n-r+1)l}-1}{P^{l}-1\,.\,P^{2l}-1\,\ldots\,P^{rl}-1}\,z^{r}, \tag{3}$$

$$[P^{x+y}z]^{n} = [P^{x}z]^{n}+\Sigma P^{rx}\frac{P^{nl}-1\,.\,P^{(n-1)l}-1\,\ldots\,P^{(n-r+1)l}-1}{P^{l}-1\,.\,P^{2l}-1\,\ldots\,P^{rl}-1}\,[P^{x}z]^{n-r}(P^{y}1)^{r}z^{r}, \tag{4}$$

$$\frac{\Gamma(P^{\gamma-a-\beta}z)\,\Gamma(P^{\gamma}z)}{\Gamma(P^{\gamma-a}z)\,\Gamma(P^{\gamma-\beta}z)} = 1-P^{l}\frac{P^{a}-1\,.\,P^{\beta}-1}{P^{l}-1\,.\,P^{\gamma}+z}\,z+\ldots. \tag{5}$$

P^{a} denotes $p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_m^{a_m}$, the product of m independent elements of the type p^{a}. Similar meanings are attached to P^{β}, P^{γ}, P^{l} respectively.

[* The two papers communicated at the meetings of December, 1902, and January, 1903, have been rearranged by the author to form a single paper.]

The differential equation of the series

$$y = 1+\frac{[a_1][a_2][a_3]\ldots[a_n]}{[1][\beta_1][\beta_2]\ldots[\beta_{n_1}]}x^{[1]}$$
$$+\frac{[a_1][a_1+1][a_2][a_2+1]\ldots[a_n][a_n+1]}{[1][2][\beta_1][\beta_1+1]\ldots[\beta_{n_1}][\beta_{n_1}+1]}x^{[2]}+\ldots \quad \left([a] \text{ denotes } \frac{p^a-1}{p-1}\right) \tag{6}$$

will be investigated: this series reduces when $p = 1$ to the hypergeometric series with n elements in the numerator and n_1 elements in the denominator of each term.

The following and many similar theorems due to Cauchy, Euler, Gauss are particular cases of (3):—

$$(1+xz)(1+x^2z)\ldots(1+x^mz)$$
$$= 1+\sum_{n=1}^{n=m}\frac{x^m-1\,.\,x^{m-1}-1\,.\,x^{m-2}-1\ldots x^{m-n+1}-1}{x-1\,.\,x^2-1\,.\,x^3-1\ldots x^n-1}x^{\frac{1}{2}(n\,.\,n+1)}z^n$$
$$= 1+\sum_{1}^{\infty}\frac{x^{\frac{1}{2}(n\,.\,n+1)}}{x-1\,.\,x^2-1\,.\,x^3-1\ldots x^n-1}z^n \qquad (m=\infty),$$

$$\frac{1}{(1-xz)(1-x^2z)\ldots(1-x^mz)}$$
$$= 1+\sum_{1}^{\infty}\frac{x^m-1\,.\,x^{m+1}-1\,.\,x^{m+2}-1\ldots x^{m+n-1}-1}{x-1\,.\,x^2-1\,.\,x^3-1\ldots x^n-1}x^nz^n,$$

$$\frac{1}{(1-xz)(1-x^3z)\ldots(1-x^{2m-1}z)}$$
$$= 1+\sum_{1}^{\infty}\frac{x^{2m}-1\,.\,x^{2m+2}-1\,.\,x^{2m+4}-1\ldots x^{2m+2n-2}-1}{x^2-1\,.\,x^4-1\,.\,x^6-1\ldots x^{2n}-1}x^nz^n,$$

$$(1+xz)(1+x^3z)\ldots(1+x^{2m-1}z)$$
$$= 1+\sum_{n=1}^{n=m}\frac{x^{2m}-1\,.\,x^{2m-2}-1\ldots x^{2m-2n+2}-1}{x^2-1\,.\,x^4-1\ldots x^{2n}-1}x^{n^2}z^n.$$

2. The following notation will, I think, be useful and suggestive:—

$$P^l \equiv p_1^{l_1}p_2^{l_2}p_3^{l_3}\ldots p_m^{l_m}, \qquad P^a \equiv p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_m^{a_m},$$
$$P^x \equiv p_1^{x_1}p_2^{x_2}p_3^{x_3}\ldots p_m^{x_m}, \qquad P^{rx} \equiv p_1^{rx_1}p_2^{rx_2}p_3^{rx_3}\ldots p_m^{rx_m},$$
$$P^{x+sa} \equiv p_1^{x_1+sa_1}p_2^{x_2+sa_2}\ldots p_m^{x_m+sa_m},$$
$$[P^xz]^n = \mathop{\mathrm{L}}_{\kappa=\infty}\frac{(P^{x+(n-1)l}z+1)(P^{x+(n-2)l}z+1)\ldots(P^{x+(n-\kappa)l}z+1)}{(P^{x+l}z+1)(P^{x+2l}z+1)\ldots(P^{x+\kappa l}z+1)} \quad (P^l>1),$$
$$(P^y1)^r \equiv (P^y-1)(P^{y+l}-1)(P^{y+2l}-1)\ldots(P^{y(r-1)l}-1),$$
$$\begin{Bmatrix} n \\ r \end{Bmatrix} \equiv \frac{P^{nl}-1\,.\,P^{(n-1)l}-1\ldots P^{(n-r+1)l}-1}{P^l-1\,.\,P^{2l}-1\ldots P^{rl}-1}.$$

Consider the product of n factors $(P^{x}z+1)(P^{x+l}z+1)\ldots(P^{x+(n-1)l}z+1)$; denote this by $[P^{x}z]^{n}$. How must $[P^{x}z]^{n}$ be interpreted when n is not positive and integral? So long as a^{n} was regarded as the product of n factors each equal to a, a^{n} was meaningless except for positive and integral values of n; but, as soon as a^{n} was defined as a function satisfying the index law $a^{m}a^{n}=a^{m+n}=a^{n}a^{m}$, expressions such as a^{-m}, $a^{p/q}$, a^{0} were interpreted. We begin then by considering $[P^{x}z]^{n}$ as the product of n related factors; then from an inspection of the products

$$\{(P^{x}z+1)(P^{x+l}z+1)\ldots(P^{x+(n-1)l}z+1)\}$$
$$\times\{(P^{x+nl}z+1)(P^{x+(n+1)l}z+1)\ldots(P^{x+(n+m-1)l}z+1)\}$$

we observe that

$$[P^{x}z]^{n}[P^{x+nl}z]^{m}=[P^{x}z]^{m+n}=[P^{x}z]^{m}[P^{x+ml}z]^{n}. \tag{7}$$

This is the index law for the function $[P^{x}z]^{n}$.

The function is not completely defined by this law; for the law would be satisfied if $[P^{x}z]^{n}$ were of the form $f[P^{x}z]/f[P^{x+nl}z]$, f denoting any function we please.

$[P^{x}z]^{n}$ must be such a function as will reduce for integral values of n to a product of n factors $(P^{x}z+1)(P^{x+l}z+1)\ldots(P^{x+(n-1)l}z+1)$. The index law is, however, sufficient to give the function for negative integral values of the index.

In the relation $[P^{x}z]^{n}[P^{x+nl}z]^{m}=[P^{x}z]^{m+n}$ put $n=0$. We have

$$[P^{x}z]^{0}[P^{x}z]^{m}=[P^{x}z]^{m} \quad\text{and}\quad [P^{x}z]^{0}=1.$$

In $[P^{x}z]^{m}[P^{x+ml}z]^{n}=[P^{x}z]^{m+n}$ we put $m=-n$;

$$[P^{x}z]^{-n}[P^{x-nl}z]^{n}=[P^{x}z]^{0}=1;$$

therefore

$$[P^{x}z]^{-n}=\frac{1}{[P^{x-nl}z]^{n}}=\frac{1}{(P^{x-l}z+1)(P^{x-2l}z+1)\ldots(P^{x-nl}z+1)}.$$

With these two forms to guide us we choose in general

$$[P^{x}z]^{n}=\underset{\kappa=\infty}{\mathrm{L}}\frac{(1+P^{x+(n-1)l}z)(1+P^{x+(n-2)l}z)\ldots(1+P^{x+(n-\kappa)l}z)}{(1+P^{x-l}z)(1+P^{x-2l}z)\ldots(1+P^{x-\kappa l}z)}. \tag{8}$$

This function satisfies the index law; and, moreover, when n is positive and integral it reduces by cancelling common factors of numerator and denominator to

$$(1+P^{x+(n-1)l}z)(1+P^{x+(n-2)l}z)\ldots(1+P^{x}z)\frac{1}{(1+P^{x+(n-\kappa-1)l}z)\ldots(1+P^{x-\kappa l}z)}.$$

Now $P^{x-\kappa l}$ vanishes if $\dfrac{p_1^{x_1}p_2^{x_2}\ldots p_m^{x_m}}{p_1^{\kappa l_1}p_2^{\kappa l_2}\ldots p_m^{\kappa l_m}}=0$;

that is, if $$p_1^{l_1}p_2^{l_2}\dots p_m^{l_m} > 1 \qquad (P^l > 1).$$

And we have $[P^x z]^n = (1+P^x z)(1+P^{x+l}z)\dots(1+P^{x+(n-1)l}z)$

(n a positive integer).

Similarly, if n be a negative integer ($= -m$), the infinite product reduces to

$$\frac{1}{(1+P^{x-l}z)(1+P^{x-2l}z)\dots(1+P^{x-ml}z)}(1+P^{x-(\kappa+1)l}z)\dots(1+P^{x-(\kappa+m)l}z).$$

The m factors which involve P^κ reduce to unity if $P^l > 1$; so that

$$[P^x z]^{-m} = \frac{1}{(1+P^{x-l}z)(1+P^{x-2l}z)\dots(1+P^{x-ml}z)}.$$

3. To establish theorem (3),

$$[P^x z]^n = 1+P^x\frac{P^{nl}-1}{P^l-1}z+\dots+P^{rx+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix}n\\r\end{matrix}\right\}z^r+\dots. \tag{9}$$

There will be three steps in the proof—(1) the case of n positive and integral, (2) the case of n negative and integral, (3) the case of n unrestricted. Each step will follow from the preceding step, case (2) being a direct deduction from (1) and case (3) a direct deduction from (2). When n is a positive integer we have plainly to establish an identity; for the series contains $n+1$ terms only, the last term being $P^{nx+\frac{1}{2}(n.n-1)l}z^n$. The form for $n = 2$ or 3 is easily established; so assume that

$$[P^x z]^n = 1+P^x\frac{P^{nl}-1}{P^l-1}z+\dots+P^{rx+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix}n\\r\end{matrix}\right\}z^r+\dots+P^{nx+\frac{1}{2}(n.n-1)l}z^n.$$

Multiply both sides of the assumed identity by $1+P^{x+nl}z$: we obtain

$$[P^x z]^{n+1} = 1+P^x\frac{P^{(n+1)l}-1}{P^l-1}z+\dots+P^{(n+1)x+\frac{1}{2}(n+1.n)l}z^n.$$

The coefficient of z^r being the sum of

$$P^{rx+\frac{1}{2}(r.r-1)l}\frac{P^{nl}-1.P^{(n-1)l}-1\dots P^{(n-r+1)l}-1}{P^l-1.P^{2l}-1\dots P^{rl}-1}$$

and $$P^{x+nl}P^{(r-1)x+\frac{1}{2}(r-1.r-2)l}\frac{P^{nl}-1\dots P^{(n-r+2)l}-1}{P^l-1\dots P^{(r-1)l}-1},$$

this sum is easily seen to be

$$P^{rx+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix}n\\r-1\end{matrix}\right\}\frac{p^{(n+1)l}-1}{p^{rl}-1} = P^{rx+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix}n+1\\r\end{matrix}\right\}.$$

This is the coefficient of z^r and is of the same form for $n+1$ as for n in the assumed identity. The theorem is established for all positive integral values of the index.

4. Since $[P^xz]^n[P^{x+nl}z]^m = [P^xz]^{m+n} = [P^xz]^m[P^{x+ml}z]^n$,

then, if we expand the functions and equate the coefficients of z^r, we obtain the following theorem :—

$$\begin{Bmatrix}m+n\\ r\end{Bmatrix} = \begin{Bmatrix}m\\ r\end{Bmatrix} + P^{(m-r+1)l}\begin{Bmatrix}m\\ r-1\end{Bmatrix}\begin{Bmatrix}n\\ 1\end{Bmatrix} + \dots + P^{rml}\begin{Bmatrix}n\\ r\end{Bmatrix}$$

$$= \begin{Bmatrix}n\\ r\end{Bmatrix} + P^{(n-r+1)l}\begin{Bmatrix}n\\ r-1\end{Bmatrix}\begin{Bmatrix}m\\ 1\end{Bmatrix} + \dots + P^{rnl}\begin{Bmatrix}m\\ r\end{Bmatrix}, \qquad (10)$$

which reduces when P^l (viz., $p_1^{l_1}p_2^{l_2}\dots p_m^{l_m}$) $= 1$ to Vandermonde's theorem. This form was given by Prof. Lloyd Tanner (*Proc. London Math. Soc.*, Vol. XXVII.).

To establish the theorem for negative integral values of the index, we shall show that, if $f(-n)$ denote the series (supposed convergent)

$$1+P^x\frac{P^{-nl}-1}{P^l-1}z+\dots+P^{rx+\frac{1}{2}(r.r-1)l}\begin{Bmatrix}-n\\ r\end{Bmatrix}z^r+\dots,$$

then $$f(-n)[P^{x-nl}z]^m = [P^xz]^{m-n},$$

m being a positive integer not less than n. This, by the index law, will be sufficient to establish the equivalence of $[P^xz]^{-n}$ and $f(-n)$. The words "m being a positive integer, not less than n," do not limit in any way the generality of the proof, as might at first sight appear. Writing

$$f(-n) = 1+P^x\frac{P^{-nl}-1}{P^l-1}z+\dots+P^{rx}\begin{Bmatrix}-n\\ r\end{Bmatrix}P^{\frac{1}{2}(r.r-1)l}z^r+\dots,$$

$$[P^{x-nl}z]^m \equiv 1+P^{x-nl}\frac{P^{ml}-1}{P^l-1}z+\dots+P^{rx-rnl+\frac{1}{2}(r.r-1)l}\begin{Bmatrix}m\\ r\end{Bmatrix}z^r+\dots$$
$$+P^{mx-mnl+\frac{1}{2}(m.m-1)l}z^m,$$

the product of these two series gives

$$f(-n)[P^{r-nl}z]^m = 1+P^x\frac{P^{(m-n)l}-1}{P^l-1}z+\dots,$$

the coefficient of z^r being the sum of $r+1$ terms

$$\sum_{s=0}^{s=r} P^{(r-s)x-(r-s)nl+\frac{1}{2}(r-s.r-s-1)l}\begin{Bmatrix}m\\ r-s\end{Bmatrix}P^{sx+\frac{1}{2}(s.s-1)l}\begin{Bmatrix}-n\\ s\end{Bmatrix}$$

$$= P^{rx-rnl+\frac{1}{2}(r.r-1)l}\sum_{s=0}^{s=r}P^{s[s+(n-r)l]}\begin{Bmatrix}m\\ r-s\end{Bmatrix}\begin{Bmatrix}-n\\ s\end{Bmatrix}$$

$$= P^{rx+\frac{1}{2}(r.r-1)l}\begin{Bmatrix}m-n\\ r\end{Bmatrix}.$$

All coefficients of z^{m-n+1} and higher terms vanish owing to the presence of the factor $P^{(m-n-r+1)l}-1$ in the numerator of $\begin{Bmatrix}m-n\\ r\end{Bmatrix}$.

The infinite series reduces identically to

$$1+\sum_{r=1}^{r=m-n} P^{rx+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix} m-n \\ r \end{matrix}\right\} z^r,$$

which is $[P^x z]^{m-n}$, since $m-n$ is positive and integral; so $[P^x z]^{-n}=f(-n)$, and the theorem is established for negative integral indices.

5. Now consider the infinite product

$$[P^x z]^n = \underset{\kappa=\infty}{\mathrm{L}} \frac{(1+P^{x+(n-1)l}z)(1+P^{x+(n-2)l}z)\dots(1+P^{x+(n-\kappa)l}z)}{(1+P^{x-l}z)(1+P^{x-2l}z)\dots(1+P^{x-\kappa l}z)}.$$

The numerator of the above may be written $[P^{x+(n-\kappa)l}z]^\kappa$, κ being integral and ultimately great without limit;

$$[P^{x+(n-\kappa)l}z]^\kappa$$

$$=1+\sum_{r=1}^{r=\kappa} P^{rx+r(n-\kappa)l+\frac{1}{2}(r.r-1)l}\frac{P^{\kappa l}-1.P^{(\kappa-1)l}-1\dots P^{(\kappa-r+1)l}-1}{P^l-1.P^{2l}-1\dots P^{rl}-1}z^r$$

(κ positive and integral).

Also, since

$$\frac{1}{(1+P^{x-l}z)(1+P^{x-2l}z)\dots(1+P^{x-\kappa l}z)}=[P^x z]^{-\kappa} \quad (\kappa \text{ integral}),$$

the function $$[P^x z]^n = \underset{\kappa=\infty}{\mathrm{L}} [P^{x+(n-\kappa)l}z]^\kappa [P^x z]^{-\kappa},$$

which is

$$\left[1+\Sigma P^{rx+r(n-\kappa)l+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix} \kappa \\ r \end{matrix}\right\} z^r\right]\left[1+\Sigma P^{rx+\frac{1}{2}(r.r-1)l}\left\{\begin{matrix} -\kappa \\ r \end{matrix}\right\} z^r\right].$$

This product gives a series of which the first two terms are

$$1+P^x\frac{P^{nl}-1}{P^l-1}z+\dots.$$

The coefficient of z^r being the sum of $r+1$ terms,

$$\sum_{s=0}^{s=r} P^{(r-s)x+(r-s)(n-\kappa)l+\frac{1}{2}(r-s.r-s-1)l+sx+\frac{1}{2}(s.s-1)l}\left\{\begin{matrix} \kappa \\ r-s \end{matrix}\right\}\left\{\begin{matrix} -\kappa \\ s \end{matrix}\right\}.$$

The expression $\left\{\begin{matrix} \kappa \\ r-s \end{matrix}\right\}\left\{\begin{matrix} -\kappa \\ s \end{matrix}\right\}$ is

$$\frac{P^{\kappa l}-1.P^{(\kappa-1)l}-1\dots P^{(\kappa-r+s+1)l}-1}{P^l-1.P^{2l}-1\dots P^{(r-s)l}-1}\;\frac{P^{-\kappa l}-1.P^{(-\kappa-1)l}-1\dots P^{(-\kappa-s+1)l}-1}{P^l-1.P^{2l}-1\dots P^{sl}-1}.$$

Now $P^{-\kappa l}$ vanishes in the limit ($P^l > 1$); so that this expression may be written

$$(-1)^s\frac{P^{(r-s)\kappa l-\frac{1}{2}(r-s.r-s-1)l}}{(P^l-1)(P^{2l}-1)\dots(P^{(r-s)l}-1)(P^l-1)(P^{2l}-1)\dots(P^{sl}-1)},$$

and the coefficient of z^r becomes

$$\sum_{s=0}^{s=r} \frac{P^{(r-s)x+(r-s)\,n-\kappa\,.\,l+\frac{1}{2}(r-s\,.\,r-s-1)\,l+sx+\frac{1}{2}(s\,.\,s-1)\,l+(r-s)\kappa l-\frac{1}{2}(r-s\,.\,r-s-1)\,l}}{(P^l-1)(P^{2l}-1)\dots(P^{(r-s)\,l}-1)(P^l-1)(P^{2l}-1)\dots(P^{sl}-1)}$$

$$\equiv \frac{P^{rx}}{(P^l-1)\dots(P^{rl}-1)}\left\{P^{rnl}-P^{(r-1)\,nl}\frac{P^{rl}-1}{P^l-1}+\dots+(-1)^r P^{\frac{1}{2}(r\,.\,r-1)\,l}\right\}$$

$$\equiv P^{rx+\frac{1}{2}(r\,.\,r-1)\,l}\,\frac{P^{nl}-1\,.\,P^{(n-1)\,l}-1\,.\,P^{(n-2)\,l}-1\,\dots\,P^{(n-r+1)\,l}-1}{P^l-1\,.\,P^{2l}-1\,.\,P^{3l}-1\,\dots\,P^{rl}-1};$$

so that

$$[P^x z]^n = 1+\sum_{r=1}^{r=\infty} P^{rx+\frac{1}{2}(r\,.\,r-1)\,l}\begin{Bmatrix} n \\ r \end{Bmatrix} z^r \tag{11}$$

(for all values of n subject to convergence $P^l > 1$).

From this we can deduce

$$\mathop{\mathrm{L}}_{\kappa=\infty} \frac{(P^{x-(n-1)\,l}z+1)(P^{x-(n-2)\,l}z+1)\dots(P^{x-(n-\kappa)\,l}z+1)}{(P^{x+l}z+1)(P^{x+2l}z+1)\dots(P^{x+\kappa l}z+1)}$$

$$= 1+\sum_{r=1}^{r=\infty} P^{r\,[x-(n-r)\,l]-\frac{1}{2}(r\,.\,r-1)\,l}\begin{Bmatrix} n \\ r \end{Bmatrix} z^r \quad (P^l < 1). \tag{12}$$

6. Before proceeding to prove the theorem

$$[P^{x+y}z]^n = [P^x z]^n + \Sigma P^{rx}\begin{Bmatrix} n \\ r \end{Bmatrix}[P^x z]^{n-r}(P^y\,.\,1)^r z^r \tag{12 A}$$

it will be interesting to see how the form was arrived at.

$$P^{x+y}z+1 \equiv (P^x z+1)+P^x(P^y-1)\,z, \tag{i.}$$

also

$$P^{x+y+l}z+1 \equiv (P^{x+l}z+1)+P^{x+l}(P^y-1)\,z, \tag{ii.}$$

or

$$\equiv (P^x z+1)+P^x(P^{y+l}-1)\,z. \tag{iii.}$$

Take the product of $P^{x+y}z+1$ and $P^{x+y+l}z+1$ as follows:—Multiply the term $P^x z+1$ on the right side of (i.) by (ii.), the term $P^x(P^y-1)\,z$ on the right side of (i.) by (iii.): we obtain

$$[P^{x+y}z]^2 \equiv [P^x z]^2 + P^x\frac{P^{2l}-1}{P^l-1}[P^x z]^1(P^y\,.\,1)^1 z + P^{2x}(P^y\,.\,1)^2 z^2. \tag{A}$$

So, multiplying $[P^{x+y}z]^2$ by $P^{x+y+2l}z+1$, which may be written in three forms:—

$$(P^{x+2l}z+1)+P^{x+2l}(P^y-1)\,z, \tag{i.}$$

$$(P^{x+l}z+1)+P^{x+l}(P^{y+l}-1)\,z, \tag{ii.}$$

$$(P^x z+1)+P^x(P^{y+2l}-1)\,z; \tag{iii.}$$

multiplying the terms on the right side of (A) by (i.), (ii.), (iii.) respectively, we obtain

$$[P^{x+y}z]^3 \equiv [P^x z]^3 + P^x \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} [P^x z]^2 (P^y . 1)\, z$$
$$+ P^{2x} \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} [P^x z]^1 (P^y . 1)^2 z^2 + P^{3x} (P^y . 1)^3 z^3,$$

and, by induction for all positive integral values of the index, we arrive at form (12 A).

7. Having arrived at the form of the series, we can proceed with the general proof of

$$[P^{x+y}z]^n = [P^x z]^n + \sum_{r=1}^{r=\infty} P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} [P^x z]^{n-r} (P^y . 1)^r z^r. \tag{B}$$

Replace $\quad [P^x z]^n,\ [P^x z]^{n-1},\ \ldots,\ [P^x z]^{n-r},\ \ldots$

by the convergent infinite series which represent them, namely,

$$1 + P^x \begin{Bmatrix} n \\ 1 \end{Bmatrix} z + \ldots,\quad 1 + P^x \begin{Bmatrix} n-1 \\ 1 \end{Bmatrix} z + \ldots,\quad \ldots .$$

Then the series on the right side of (B) may be written

$$\left[1 + P^x \begin{Bmatrix} n \\ 1 \end{Bmatrix} z \quad + \ldots + P^{rx + \frac{1}{2}(r.r-1)l} \begin{Bmatrix} n \\ r \end{Bmatrix} z^r \quad + \ldots\right]$$
$$+ P^x \begin{Bmatrix} n \\ 1 \end{Bmatrix} (P^y . 1)^1 z \left[1 + P^x \begin{Bmatrix} n-1 \\ 1 \end{Bmatrix} z + \ldots + P^{rx + \frac{1}{2}(r.r-1)l} \begin{Bmatrix} n-1 \\ r \end{Bmatrix} z^r + \ldots\right]$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+ P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} (P^y . 1)^r z^r \left[1 + P^x \begin{Bmatrix} n-r \\ r \end{Bmatrix} z + \ldots + P^{rx + \frac{1}{2}(r.r-1)l} \begin{Bmatrix} n-r \\ r \end{Bmatrix} z^r + \ldots\right] + \ldots . \tag{13}$$

Collecting according to the powers of z, we obtain the series

$$1 + P^{x+y} \frac{P^{nl}-1}{P^l-1} z + \ldots + P^{r(x+y) + \frac{1}{2}(r.r-1)l} \begin{Bmatrix} n \\ r \end{Bmatrix} z^r + \ldots ;$$

for the coefficient of z^r is the sum of $r+1$ terms

$$P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} (P^y . 1)^r + P^{rx} \begin{Bmatrix} n \\ r-1 \end{Bmatrix} \begin{Bmatrix} n-r+1 \\ 1 \end{Bmatrix} (P^y . 1)^{r-1}$$
$$+ P^{rx+l} \begin{Bmatrix} n \\ r-2 \end{Bmatrix} \begin{Bmatrix} n-r+2 \\ 2 \end{Bmatrix} (P^y . 1)^{r-2} + \ldots + P^{rx + \frac{1}{2}(r.r-1)l} \begin{Bmatrix} n \\ r \end{Bmatrix},$$

which may be written

$$P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} \left[(P^y . 1)^r + \frac{P^{rl}-1}{P^l-1} (P^y . 1)^{r-1} + P^l \frac{P^{rl}-1 . P^{(r-1)l}-1}{P^l-1 . P^{2l}-1} (P^y . 1)^{r-2} + \ldots\right.$$
$$\left.\ldots + P^{\frac{1}{2}(r.r-1)l}\right] \equiv P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} P^{ry + \frac{1}{2}(r.r-1)l}. \tag{14}$$

This is the coefficient of z^r. The series

$$[P^x z]^n + \sum_{r=1}^{r=\infty} P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} [P^x z]^{n-r} (P^y . 1)^r z^r$$

has been transformed into

$$1 + \sum_{r=1}^{r=\infty} P^{r(x+y)+\frac{1}{2}(r.r-1)l} \begin{Bmatrix} n \\ r \end{Bmatrix} z^r,$$

and this, subject to convergence, is equal to $[P^{x+y} z]^n$, and Theorem (B) is completely established.

8. To transform (B) to form (5) we need the following lemma:—

$$\frac{[P^x z]^{n-r}}{[P^x z]^n} = \frac{1}{(P^{x+(n-1)l} z+1)(P^{x+(n-2)l} z+1) \ldots (P^{x+(n-\kappa)l} z+1)}. \qquad (15)$$

This follows at once from the index law

$$[P^x z]^{n-r} [P^{x+(n-r)l}]^r = [P^x z]^n.$$

Writing $\quad [P^{x+y} z]^n = [P^x z]^n + \sum_{r=1}^{r=\infty} P^{rx} \begin{Bmatrix} n \\ r \end{Bmatrix} [P^x z]^{n-r} (P^y . 1)^r z^r,$

divide both sides by $[P^x z]^n$: we obtain

$$\frac{[P^{x+y} z]^n}{[P^x z]^n} = 1 + \Sigma P^{rx} \frac{P^{nl}-1 . P^{(n-1)l}-1 \ldots P^{(n-r+1)l}-1}{P^l-1 . P^{2l}-1 \ldots P^{rl}-1}$$

$$\times \frac{P^y-1 . P^{y+l}-1 \ldots P^{y+(r-1)l}-1}{P^{x+(n-1)l} z+1 . P^{x+(n-2)l} z+1 \ldots P^{x+(n-r)l} z+1} z^r. \qquad (16)$$

Make the following substitutions:—$P^{-\alpha}$ for P^{nl}, that is, $-\alpha_1$ for nl_1, $-\alpha_2$ for nl_2, $-\alpha_3$ for nl_3, …; $P^{-\gamma}$ for $P^{x+(n-1)l}$, that is, $-\gamma_1$ for $x_1+(n-1)l_1$, $-\gamma_2$ for $x_2+(n-1)l_2$, $-\gamma_3$ for $x_3+(n-1)l_3$, …; P^{β} for P^y, that is, β_1 for y_1, β_2 for y_2, β_3 for y_3, … .

The series becomes

$$1 + \Sigma(-1)^r P^{rl} \frac{P^{\alpha}-1 . P^{\alpha+l}-1 \ldots P^{\alpha+(r-1)l}-1 . P^{\beta}-1 \ldots P^{\beta+(r-1)l}-1}{P^l-1 . P^{2l}-1 \ldots P^{rl}-1 . P^{\gamma}+z \ldots P^{\gamma+(r-1)l}+z} z^r, \qquad (17)$$

which is series (5), § 1. The function

$$\frac{[P^{x+y} z]^n}{[P^x z]^n} = \frac{(P^{x+y+(n-1)l} z+1) \ldots (P^{x+y+(n-\kappa)l} z+1)(P^{x-l} z+1) \ldots (P^{x-\kappa l} z+1)}{(P^{x+y-l} z+1) \ldots (P^{x+y-\kappa l} z+1)(P^{x+(n-1)l} z+1 \ldots (P^{x+(n-\kappa)} z+1)},$$

which becomes, on substituting α, β, γ as above,

$$\frac{(P^{\beta-\gamma}z+1)\dots(P^{\beta-\gamma-(\kappa-1)l}z+1)(P^{\alpha-\gamma}z+1)\dots(P^{\alpha-\gamma-(\kappa-1)l}z+1)}{(P^{\alpha+\beta-\gamma}z+1)\dots(P^{\alpha+\beta-\gamma-(\kappa-1)l}z+1)(P^{-\gamma}z+1)\dots(P^{-\gamma-(\kappa-1)l}z+1)},$$

$$\frac{(P^{\gamma-\beta}+z)(P^{\gamma-\beta+l}+z)\dots(P^{\gamma-\beta+(\kappa-1)l}+z)(P^{\gamma-\alpha}+z)(P^{\gamma-\alpha+l}+z)\dots(P^{\gamma-\alpha+(\kappa-1)l}+z)}{(P^{\gamma-\alpha-\beta}+z)(P^{\gamma-\alpha-\beta+l}+z)\dots(P^{\gamma-\alpha-\beta+(\kappa-1)l}+z)(P^{\gamma}+z)(P^{\gamma+l}+z)\dots(P^{\gamma+(\kappa-1)l}+z)},$$

which we write
$$\frac{\Gamma(P^{\gamma-\alpha-\beta}z)\,\Gamma(P^{\gamma}z)}{\Gamma(P^{\gamma-\alpha}z)\,\Gamma(P^{\gamma-\beta}z)}. \tag{18}$$

9. The series
$$1-p^l\frac{p^\alpha-1\,.\,p^\beta-1}{p^l-1\,.\,p^\gamma+z}z+\dots, \tag{19}$$

in which p is a simple element, can be derived from the general theorem in two ways:—(1) by making the elements $p_3, p_4, \dots, p_m$ each equal to unity, p_2 equal to $1/z$, z equal to unity, γ_2 equal to unity, $\alpha_2, \alpha_3, \dots, \alpha_m$ each equal to zero, $\beta_2, \beta_3, \dots, \beta_m$ each equal to zero; (2) by making $p_2, p_3, p_4, \dots, p_m$ each equal to unity.

The series
$$1-p^l\frac{p^\alpha-1\,.\,p^\beta-z}{p^l-1\,.\,p^\gamma+z}+\dots \tag{20}$$

is derived from the general series by making $p_3, p_4 \dots, p_m$ each $= 1$, $z = 1$, $\alpha_2 = -0$, $\beta_2 = -1$, $\gamma_2 = -1$, $p_2 = z$, all other elements $= 0$.

The series

$$1-p^l q^{\gamma_1-\alpha_1-\beta_1}\frac{p^\alpha-q^{\alpha_1}\,.\,p^\beta-q^{\beta_1}}{p^l-q^{l_1}\,.\,p^\gamma+q^{\gamma_1}z}z$$
$$+p^{2l}q^{2(\gamma_1-\alpha_1-\beta_1)}\frac{p^\alpha-q^{\alpha_1}\,.\,p^{\alpha+l}-q^{\alpha_1+l_1}\,.\,p^\beta-q^{\beta_1}\,.\,p^{\beta+l}-q^{\beta_1+l_1}}{p^l-q^{l_1}\,.\,p^{2l}-q^{2l_1}\,.\,p^\gamma+q^{\gamma_1}z\,.\,p^{\gamma+l}+q^{\gamma_1+l_1}z}z^2+\dots \tag{21}$$

can be derived by putting $p_3, p_4, \dots, p_m$ each $= 1$, $p_2 = 1/q$, $l_3, l_4, \dots, l_m$ each $= 0$; further, putting $p = 1$, $z = -1$, $l_1 = 1$, the last series reduces to

$$1+q^{\gamma-\alpha-\beta}\frac{q^\alpha-1\,.\,q^\beta-1}{q-1\,.\,q^\gamma-1}+\dots=\frac{\Gamma(q^{\gamma-\alpha-\beta})\,\Gamma(q^\gamma)}{\Gamma(q^{\gamma-\alpha})\,\Gamma(q^{\gamma-\beta})}, \tag{22}$$

given in Vol. XXVIII., *Proc. London Math. Soc.*, p. 475.

10. We now proceed to consider a theorem which may be stated as follows:—

$$\{P^{x+y}.1\}^n=[P^x z^{-1}]^n+\sum_{r=1}^{r=\infty}P^{r[x-(n-r)l]}\frac{P^{nl}-1\,.\,P^{(n-1)l}-1\dots P^{(n-r+1)l}-1}{P^l-1\,.\,P^{2l}-1\dots P^{rl}-1}$$
$$\times[P^x z^{-1}]^{n-r}[P^y z]^r z^r; \tag{23}$$

$$[P^x z]^n \text{ denotes } \underset{\kappa=\infty}{\mathrm{L}}\frac{(P^{x-(n-1)l}+z)(P^{x-(n-2)l}+z)\dots(P^{x-(n-\kappa)l}+z)}{(P^{x+l}+z)(P^{x+2l}+z)\dots(P^{x+\kappa l}+z)},$$

$$\{P^{x+y}.1\}^n \text{ denotes } \underset{\kappa=\infty}{\mathrm{L}}\frac{(1-P^{x+y-(n-1)l})\dots(1-P^{x+y-(n-\kappa)l})}{(1-P^{x+y+l})\dots(1-P^{x+y+\kappa l})},$$

$$[P^x z^{-1}]^n = 1 + \sum_{r=1}^{r=\infty} P^{rx-r.n-r.l-\frac{1}{2}(r.r-1)l} \begin{Bmatrix} n \\ r \end{Bmatrix} z^r$$

[see (11) and (12), § 5]. Consider the product of n factors

$$\frac{(P^x+z)(P^{x-l}+z)\dots(P^{x-(n-1)l}+z)}{z^n} \tag{24}$$

(denote this by $[P^x z]^n$); then we have an index law

$$[P^x z]^n [P^{x-nl} z]^m = [P^x z]^{m+n} = [P^x z]^m [P^{x-ml} z]^n. \tag{25}$$

Interpreting this law, we obtain $[P^x z]^0 = 1$,

$$[P^x z]^{-n} = \frac{1}{[P^{x+nl} z]^n} = \frac{z^n}{(P^{x+l}+z)(P^{x+2l}+z)\dots(P^{x+(n-1)l}+z)}. \tag{26}$$

From a consideration of the forms (24) and (26) we write

$$[P^x z]^n = \underset{\kappa=\infty}{\mathrm{L}} \frac{(P^{x-(n-1)l}+z)(P^{x-(n-2)l}+z)\dots(P^{x-(n-\kappa)l}+z)}{(P^{x+l}+z)(P^{x+2l}+z)\dots(P^{x+\kappa l}+z)} \quad (P^l < 1). \tag{27}$$

This function satisfies the index law; and, moreover, if n be a positive integer, reduces to $z^{-n}(P^x+z)(P^{x-l}+z)\dots$ to n factors; if n be negative $(=-m)$, the function reduces to

$$\frac{z^m}{(P^{x+l}+z)(P^{x+2l}+z)\dots(P^{x+ml}+z)}.$$

There will be three steps in the investigation:—(1) The case of n positive and integral; (2) the case of n negative and integral; (3) the case of n unrestricted.

When $n = 2$ the theorem is an identity

$$\begin{aligned}(1-P^{x+y})(1-P^{x+y-l}) &\equiv \frac{(P^x+z^{-1})(P^{x-l}+z^{-1})}{z^{-2}} - P^{x-l}\frac{P^{2l}-1}{P^l+1}\,\frac{P^x+z^{-1}}{z^{-1}}\,\frac{P^y+z}{z}\,z \\ &\qquad + P^{2x}\frac{P^y+z\,.\,P^{y-l}+z}{z^2}\,z^2 \\ &\equiv u_0+u_1 z+u_2 z^2.\end{aligned}$$

Introduce an additional factor $1-P^{x+y-2l}$ which may be written in three forms:—

$$\frac{P^{x-2l}+z^{-1}}{z^{-1}} - P^{x-2l}\frac{P^y+z}{z}\,z, \qquad \text{(i.)}$$

$$\frac{P^{x-l}+z^{-1}}{z^{-1}} - P^{x-l}\frac{P^{y-l}+z}{z}\,z, \qquad \text{(ii.)}$$

$$\frac{P^x+z^{-1}}{z^{-1}} - P^x\frac{P^{y-2l}+z}{z}\,z. \qquad \text{(iii.}$$

Multiply u_0 by (i.), $u_1 z$ by (ii.), $u_2 z^2$ by (iii.). We obtain

$$\begin{aligned}\{P^{x+y}1\}^3 \equiv [P^x z^{-1}]^3 - P^{x-2l}\begin{Bmatrix}3\\1\end{Bmatrix}[P^x z^{-1}]^2[P^y z]^1 z + P^{2(x-l)}\begin{Bmatrix}3\\2\end{Bmatrix}[P^x z^{-1}]^1[P^y z]^2 z^2 \\ - P^{3x}[P^y z]^3 z^3.\end{aligned}$$

Assume that for n a positive integer

$$\{P^{x+y}1\}^n \equiv [P^x z^{-1}]^n + \sum_{r=1}^{r=n} (-1)^r P^{r[x-(n-r)l]} \begin{Bmatrix} n \\ r \end{Bmatrix} [P^x z^{-1}]^{n-r} [P^y z]^r z^r$$
$$\equiv u_0 + u_1 z + u_2 z^2 + \ldots + u_n z^n. \tag{28}$$

Write $1-P^{x+y-nl}$ the additional factor in the $n+1$ forms

$$\frac{P^{x-nl}+z^{-1}}{z^{-1}} - P^{x-nl}\frac{P^y+z}{z}z, \tag{i.}$$

$$\frac{P^{x-(n-1)l}+z^{-1}}{z^{-1}} - P^{x-(n-1)l}\frac{P^{y-l}+z}{z}z, \tag{ii.}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{P^x+z^{-1}}{z^{-1}} - P^x\frac{P^{y-nl}+z}{z}z. \tag{$n+1$}$$

Multiply u_0 by (i.), $u_1 z$ by (ii.), …, $u_n z^n$ by $(n+1)$. We get $n+1$ pairs of terms, and, taking the second of one pair with the first of the next, we have a typical pair

$$-P^{x-(n-r+1)l}\frac{P^{y-(r-1)l}+z}{z}z \, . \, u_{r-1}z^{r-1} + \frac{P^{x-(n-r)l}+z^{-1}}{z^{-1}}u_r z^r;$$

and, since $$[P^x z^{-1}]^{n-r}\frac{P^{x-(n-r)l}+z^{-1}}{z^{-1}} = [P^x z^{-1}]^{n+1-r},$$

$$[P^y z]^{r-1}\frac{P^{y-(r-1)l}+z}{z} = [P^y z]^r.$$

The pair of terms which give rise to the $(r+1)$-th term of a new series reduce to

$$(-1)^r P^{r[x-(n-r)l]}\frac{P^{nl}-1\,.\,P^{(n-1)l}-1\,\ldots\,P^{(n-r+2)l}-1}{P^l-1\,.\,P^{2l}-1\,\ldots\,P^{rl}-1}\left\{\frac{P^{(n-r+1)l}-1}{P^{rl}-1} - P^{-rl}\right\}$$
$$\times [P^x z^{-1}]^{n+1-r}[P^y z]^r z^r$$
$$\equiv (-1)^r P^{r[x-(n-r)l]}\begin{Bmatrix} n+1 \\ r \end{Bmatrix}[P^x z^{-1}]^{n-r+1}[P^y z]^r z^r. \tag{29}$$

This is of the same form in $n+1$ as the $(r+1)$-th term of the series was in n; so we write for all positive integral values of n

$$\{P^{x+y}1\}^n = [P^x z^{-1}]^n + \Sigma P^{r[x-(n-r)l]}\begin{Bmatrix} n \\ r \end{Bmatrix}[P^x z^{-1}]^{n-r}[P^y z]^r z^r. \tag{30}$$

11. Two particular cases of this will be needed in the general proof; so we give them here, substituting ml for x, -1 for z, $-nl$ for y, r for n.

We have

$$(P^{(m-n)l}-1)(P^{(m-n-1)l}-1)\ \dots \text{ to } r \text{ factors}$$

$$\equiv (P^{ml}-1)(P^{(m-1)l}-1)\dots(P^{(m-r+1)l}-1)$$

$$+P^{(m-r)l}\frac{P^{rl}-1}{P^{l}-1}(P^{ml}-1)\dots(P^{(m-r+2)l}-1)(P^{-nl}-1)+\dots$$

$$+P^{rml}(P^{-nl}-1)\dots(P^{(-n-r+1)l}-1). \tag{31}$$

Substituting $-x-l$ for x, $x-(n-r)l$ for y, r for n, z^{-1} for z, we obtain

$$\{P^{-(n-r+1)l}1\}^{r}\equiv[P^{-x-l}z]^{r}-P^{-x-rl}\frac{P^{rl}-1}{P^{l}-1}[P^{-x-l}z]^{r-1}[P^{x-(n-r)l}z^{-1}]z^{-1}+\dots$$

$$+(-1)^{r}P^{-rx-rl}[P^{x-(n-r)l}z^{-1}]z^{-r};$$

but $\{P^{-(n-r+1)l}\}^{r}\equiv P^{-\frac{1}{2}r(2nl-rl+l)}.P^{nl-1}.P^{(n-1)l}-1\ \dots\ P^{(n-r+1)l}-1$;

therefore

$$\frac{P^{nl}-1.P^{(n-1)l}-1\ \dots\ P^{(n-r+1)l}-1}{P^{l}-1.P^{2l}-1\ \dots\ P^{rl}-1}$$

$$\equiv\frac{P^{\frac{1}{2}r(2nl-rl+l)}}{P^{l}-1\dots P^{rl}-1}\Big\{[P^{-x-l}z]^{r}-P^{-x-rl}\frac{P^{rl}-1}{P^{l}-1}[P^{-x-l}z]^{r-1}[P^{x-(n-r)l}z^{-1}]^{1}z^{-1}+\dots$$

$$+(-1)^{r}P^{-rx-rl}[P^{x-(n-r)l}z^{-1}]^{r}z^{-r}\Big\}. \tag{32}$$

12. Now denote the series (supposed convergent)

$$[P^{x}z^{-1}]^{-n}+\sum_{r=1}^{r=\infty}(-1)^{r}P^{r[x+(n+r)l}\left\{\begin{matrix}-n\\ r\end{matrix}\right\}[P^{x}z^{-1}]^{-n-r}[P^{y}z]^{r}z^{r}$$

by $f(-n)$. We shall show that

$$f(-n)\{P^{x+y+nl}1\}^{m}=\{P^{x+y}1\}^{m-n}$$

(m being positive and integral, and not $< n$). This will establish the equivalence of $f(-n)$ and $\{P^{x+y}1\}^{-n}$ by the index law. Write

$$f(-n)=u_0+u_1z+u_2z^2+u_3z^3+\dots \text{ ad inf.}$$

Now, when m is a positive integer, we have, from (30),

$$\{P^{x+nl+y}\}^{m}$$

$$\equiv[P^{x+(n+r)l}z^{-1}]^{m}-P^{x+(n+r)l-(m-1)l}\frac{P^{ml}-1}{P^{l}-1}[P^{x+(n+r)l}z^{-1}]^{m-1}[P^{y-rl}z]^{1}z$$

$$+\dots \text{ to } m+1 \text{ terms.} \tag{33}$$

Multiply $f(-n)$ by this in the following manner:—u_0 by the series (33), putting $r=0$; u_1z by the series (33), putting $r=1$; u_2z^2 by the series (33), putting $r = 2$; &c. Then we obtain

$$
\begin{aligned}
&f(-n)\{P^{x+y+nl}1\}^m \\
&= u_0\Big[[P^{x+nl}z^{-1}]^m - \dots + (-1)^r P^{r[x+nl-(m-r)l]}\left\{\begin{matrix} m \\ r \end{matrix}\right\} \\
&\qquad\qquad \times [P^{x+nl}z^{-1}]^{m-r}[P^y z]^r z^r + \dots\Big] \\
&\quad + u_1 z\Big[[P^{x+(n+1)l}z^{-1}]^m - \dots + (-1)^r P^{r[x+(n+1)l-(m-r)l]}\left\{\begin{matrix} m \\ r \end{matrix}\right\} \\
&\qquad\qquad \times [P^{x+(n+1)l}z^{-1}]^{m-r}[P^{y-l}z]^r z^r + \dots\Big] \\
&\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\quad + u_r z^r\Big[[P^{x+(n+r)l}z^{-1}]^m - \dots + (-1)^r P^{r[x+(n+r)l-(m-r)l]}\left\{\begin{matrix} m \\ r \end{matrix}\right\} \\
&\qquad\qquad \times [P^{x+(n+1)l}z^{-1}]^{m-r} P^{y-rl}z]^r z^r + \dots\Big] \\
&\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned} \tag{34}
$$

Each large bracket contains $m+1$ terms.

Collecting the terms according to the powers of z, we see that

$$u_0[P^{x+nl}z^{-1}]^m = [P^x z^{-1}]^{-n}[P^{x+nl}z^{-1}]^m = [P^x z^{-1}]^{m-n}.$$

The second term is the sum of

$$u_1 z[P^{x+(n+1)l}z^{-1}]^m - u_0 P^{x+nl-(m-1)l}\frac{P^{ml}-1}{P^l-1}[P^{x+nl}z^{-1}]^{m-1}[P^y z]^1 z;$$

and, since $\quad u_1 z \equiv -P^{x+(n+1)l}\dfrac{P^{-nl}-1}{P^l-1}[P^x z^{-1}]^{-n-1}[P^y z]^1 z,$

the pair of terms may be written

$$\equiv -P^{x-(m-n-1)l}\frac{P^{(m-n)l}-1}{P^l-1}[P^x z^{-1}]^{m-n-1}[P^y z]^1 z.$$

And, in general, since

$$[P^x z^{-1}]^{-n}[P^{x+nl}z^{-1}]^{m-s} = [P^x z^{-1}]^{m-n-s},$$

the coefficient of z^r reduces to

$$
\begin{aligned}
&(-1)^r P^{r[x-(m-n-r)l]}[P^x z^{-1}]^{m-n-r}[P^y z]^r \\
&\qquad \times\left[\left\{\begin{matrix} m \\ r \end{matrix}\right\} + P^{m-rl}\left\{\begin{matrix} m \\ r-1 \end{matrix}\right\}\left\{\begin{matrix} -n \\ 1 \end{matrix}\right\} + \dots + P^{rml}\left\{\begin{matrix} -n \\ r \end{matrix}\right\}\right].
\end{aligned} \tag{35}
$$

The expression within the large brackets has been shown equal to $\left\{\begin{matrix} m-n \\ r \end{matrix}\right\}$; so

$$
\begin{aligned}
&f(-n)\{P^{x+y+nl}1\}^m \\
&= [P^x z^{-1}]^{m-n} + \Sigma(-1)^r P^{r(m-n-r)l+rx}\left\{\begin{matrix} m-n \\ r \end{matrix}\right\}[P^x z^{-1}]^{m-n-r}[P^y z]^r z^r.
\end{aligned} \tag{36}
$$

All terms in the series on the right after the $(m-n+1)$-th vanish since

$m-n$ is a positive integer, and the series is identically equal to $\{P^{x+y}1\}^{m-n}$. Therefore $f(-n)$ when convergent is equivalent to $\{P^{x+y}1\}^{-n}$, and the theorem is established for negative integral indices.

13. When n is unrestricted suppose that $\{P^{x+y}1\}^n$ is capable of expansion in the form

$$\{P^{x+y}1\}^n = A_0+A_1[P^y z]^1+\ldots+A_r[P^y z]^r+\ldots.$$

Then, since $$[P^y z]^r \equiv \frac{(P^y+z)\ \ldots\ (P^{y-(r-1)l}+z)}{z^r}$$

and $$\{P^{x+y}1\}^n = \underset{\kappa=\infty}{\mathrm{L}} \frac{(1-P^{x+y-(n-1)l})\ \ldots\ (1-P^{x+y-(n-\kappa)l})}{(1-P^{x+y+l})\ \ldots\ (1-P^{x+y+\kappa l})},$$

if we successively substitute $P^y=-z$, $P^{y-l}=-z$, $P^{y-2l}=-z$, $\ldots$, we obtain

$$[P^x z^{-1}]^n = A_0,$$

$$[P^{x+l} z^{-1}]^n = A_0-(P^l-1)A_1,$$

$$[P^{x+2l} z^{-1}]^n = A_0-(P^{2l}-1)A_1+(P^{2l}-1)(P^l-1)A_2,$$

$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

$$[P^{x+rl} z^{-1}]^n = A_0-(P^{rl}-1)A_1+(P^{rl}-1)(P^{(r-1)l}-1)A_2-\ldots$$
$$+(-1)^r(P^{rl}-1)(P^{(r-1)l}-1)\ \ldots\ (P^l-1)A_r.$$

$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

So $$A_0=[P^x z^{-1}]^n,\quad A_1=\frac{1}{P^l-1}\{[P^x z^{-1}]^n-[P^{x+l} z^{-1}]^n\},\quad \ldots,$$

and by induction (Art. 6, p. 479, Vol. XXVIII., *Proc. London Math. Soc.*)

$$A_r \equiv (-1)^r \frac{1}{P^{rl}-1\,.\,P^{(r-1)l}-1\ \ldots\ P^l-1}$$
$$\times\Big[[P^{x+rl} z^{-1}]^n - \left\{\begin{matrix} r \\ 1 \end{matrix}\right\}[P^{x+(r-1)l} z^{-1}]^n+\ldots+(-1)^s P^{\frac{1}{2}(s\,.\,s-1)l}$$
$$\times \left\{\begin{matrix} r \\ s \end{matrix}\right\}[P^{x+(r-s)l} z^{-1}]^n+\ldots+(-1)^r P^{\frac{1}{2}(r\,.\,r-1)l}[P^x z^{-1}]^n\Big]. \quad (38)$$

Now

$$[P^{x+(r-s)l} z^{-1}]^n = [P^x z^{-1}]^{n-r}[P^{x-(n-r)l} z^{-1}]^s[P^{x+(r-s)l} z^{-1}]^{r-s}$$
$$= P^{\frac{1}{2}(r-s)[rx+(r-s+1)l]}[P^x z^{-1}]^{n-r}[P^{x-(n-r)l} z^{-1}]^s[P^{-x-l} z]^{r-s} z^{r-s},$$

as can easily be seen from the index law

$$[P^x z^{-1}]^m[P^{x-ml} z^{-1}]^n = [P^x z^{-1}]^{m-n}.$$

Also $$[P^{x+(r-s)l}z^{-1}]^{r-s} \equiv \frac{(P^{x+(r-s)l}+z^{-1})(P^{x+(r-s-1)l}+z^{-1})\ldots(P^{x+l}+z^{-1})}{z^{-r+s}}$$

$$\equiv P^{\frac{1}{2}(r-s)[2x+(r-s+1)l]}\,[P^{-x-l}z]^{r-s}z^{r-s}.$$

Therefore A_r may be written

$$(-1)^r\frac{[P^x z^{-1}]^{n-r}}{P^l-1\ldots P^{rl}-1}\Bigg[[P^{-x-l}z]^r z^r - P^{-x-rl}\begin{Bmatrix} r \\ 1 \end{Bmatrix}[P^{-x-l}z]^{r-1}$$

$$\times[P^{x+(n-r)l}z^{-1}]^1 z^{r-1}+\ldots+(-1)^r P^{-rx-rl}[P^{x-(n-r)l}z^{-1}]^r\Bigg]P^{\frac{1}{2}(r^2l+2rx+rl)}.$$

The expression within the large brackets has been shown [(32), § 11] equal to
$$z^r(P^{nl}-1)(P^{(n-1)l}-1)\ldots(P^{(n-r+1)l}-1)\,P^{-\frac{1}{2}r(2nl-rl-l)}.$$

$$A_r = (-1)^r\begin{Bmatrix} n \\ r \end{Bmatrix}[P^x z^{-1}]^{n-1}P^{r[x-(n-r)l]}z^r, \tag{39}$$

which establishes the suitability of the series

$$[P^x z^{-1}]^n+\Sigma(-1)^r P^{r[x-(n-r)l]}\begin{Bmatrix} n \\ r \end{Bmatrix}[P^x z^{-1}]^{n-r}[P^y z]^r z^r.$$

14. To transform this series we require the following lemma:—

$$[P^x z^{-1}]^n = 1+\Sigma P^{r[x-(n-r)l]-\frac{1}{2}(r.r-1)l}\begin{Bmatrix} n \\ r \end{Bmatrix}z^r; \tag{40}$$

for $$[P^x z^{-1}]^n = \mathop{\mathrm{L}}_{\kappa=\infty}\frac{(P^{x-(n-1)l}+z^{-1})(P^{x-(n-2)l}+z^{-1})\ldots(P^{x-(n-\kappa)l}+z^{-1})}{(P^{x+l}+z^{-1})(P^{x+2l}+z^{-1})\ldots(P^{x+\kappa l}+z^{-1})}$$

$$= \mathop{\mathrm{L}}_{\kappa=\infty}\frac{(P^{x-(n-1)l}z+1)(P^{x-(n-2)l}z+1)\ldots(P^{x-(n-\kappa)l}z+1)}{(P^{x+l}z+1)(P^{x+2l}z+1)\ldots(P^{x+\kappa l}z+1)},$$

which has been shown equal to

$$1+\sum_{r=1}^{r=\infty}P^{rx-\frac{1}{2}(r.r-1)l}\frac{P^{-nl}-1.P^{(-n+1)l}-1\ldots P^{(-n+r-1)l}-1}{P^{-l}-1.P^{-2l}-1\ldots P^{-rl}-1}z^r,$$

being series (11) with l changed to $-l$; the above is easily transformed into

$$1+\sum_{r=1}^{r=\infty}P^{r[x-(n-r)l]-\frac{1}{2}(r.r-1)l}\begin{Bmatrix} n \\ r \end{Bmatrix}z^r. \tag{41}$$

In the series

$$[P^x z^{-1}]^n+\sum_{r=1}^{r=\infty}(-1)^r P^{r[x-(n-r)l]}\begin{Bmatrix} n \\ r \end{Bmatrix}[P^x z^{-1}]^{n-r}[P^y z]^r z^r$$

replace $[P^x z^{-1}]^n$, $[P^x z^{-1}]^{n-1}$, …, $[P^x z^{-1}]^{n-r}$, … by series of form (41) which represent them. We have

$$\left[1+P^{x-(n-1)l}\begin{Bmatrix}n\\1\end{Bmatrix}z+\ldots+P^{r[x-(n-r)l]-\frac{1}{2}(r.r-1)l}\begin{Bmatrix}n\\r\end{Bmatrix}z^r+\ldots\right]$$

$$-P^{x-(n-1)l}\begin{Bmatrix}n\\1\end{Bmatrix}[P^y z]^1 z\left[1+P^{x-(n-2)l}\begin{Bmatrix}n-1\\1\end{Bmatrix}z+\ldots\right.$$

$$\left.+P^{r[x-(n-r-1)l]-\frac{1}{2}(r.r-1)l}\begin{Bmatrix}n-1\\r\end{Bmatrix}z^r+\ldots\right]$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$+(-1)^r P^{r[x-(n-r)l]}\begin{Bmatrix}n\\r\end{Bmatrix}[P^y z]^r z^r\left[1+P^{x-(n-r-1)l}\begin{Bmatrix}n-r\\1\end{Bmatrix}z+\ldots\right.$$

$$\left.+P^{r[x-(n-2r)l]-\frac{1}{2}(r.r-1)l}\begin{Bmatrix}n-r\\r\end{Bmatrix}z^r+\ldots\right]$$

$$\ldots\quad\ldots,\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots. \tag{42}$$

Collecting according to ascending powers of z, we obtain a series of which the first two terms are

$$1+\left(P^{x-(n-1)l}\begin{Bmatrix}n\\1\end{Bmatrix}-P^{x-(n-1)l}\begin{Bmatrix}n\\1\end{Bmatrix}[P^y z]^1\right)z+\ldots$$

$$=1+P^{x-(n-1)l}\begin{Bmatrix}n\\1\end{Bmatrix}\left(1-\frac{P^y+z}{z}\right)z+\ldots=1-P^{x+y-(n-1)l}\begin{Bmatrix}n\\1\end{Bmatrix}+\ldots. \tag{43}$$

The $(r+1)$-th term is the expression

$$P^{r[x-(n-r)l]}\begin{Bmatrix}n\\r\end{Bmatrix}z^r\left[P^{-\frac{1}{2}(r.r-1)l}-P^{-\frac{1}{2}(r-1.r-2)l}\frac{P^{rl}-1}{P^l-1}[P^y z]^1+\ldots+(-1)^r[P^y z]^r\right].$$

The expression within the large brackets $\equiv P^{ry-\frac{1}{2}(r.r-1)l}z^{-r}$; so the series has been transformed into

$$1+\Sigma(-1)^r P^{r(x+y)-r(n-r)l-\frac{1}{2}(r.r-1)l}\begin{Bmatrix}n\\r\end{Bmatrix},$$

which is $\{P^{x+y}1\}^n$, being a particular case of theorem (40), viz., $z=-1$ ($x+y$ for x).

15. Writing, then,

$$\{P^{x+y}1\}^n=[P^x z^{-1}]^n+\sum_{r=1}^{r=\infty}(-1)^r P^{r[x-(n-r)l]}\begin{Bmatrix}n\\r\end{Bmatrix}[P^x z^{-1}]^{n-r}[P^y z]^r z^r, \tag{44}$$

dividing both sides by $[P^x z^{-1}]^n$, the general term of the series becomes

$$(-1)^r P^{r[x-(n-r)l]}\frac{P^{nl}-1.P^{(n-1)l}-1\ldots P^{(n-r+1)l}-1}{P^l-1.P^{2l}-1\ldots P^{rl}-1}$$

$$\times\frac{\dfrac{P^y+z.P^{y-l}+z\ldots P^{y-(r-1)l}+z}{z^r}}{\dfrac{P^{x-(n-1)l}+z^{-1}.P^{x-(n-2)l}+z^{-1}\ldots P^{x-(n-r+1)l}+z^{-1}}{z^{-r}}}z^r.$$

Making the substitutions z^{-1} for z,

$$\left.\begin{array}{lll} \alpha_1 \text{ for } -nl_1, & -\beta_1 \text{ for } y_1, & \gamma_1 \text{ for } x_1-(n-1)\,l_1, \\ \alpha_2 \text{ for } -nl_2, & -\beta_2 \text{ for } y_2, & \gamma_2 \text{ for } x_2-(n-1)\,l_2, \\ \dots \quad \dots, & \dots \quad \dots, & \dots \quad \dots \quad \dots, \end{array}\right\} \tag{45}$$

the series becomes

$$1+P^{\gamma-\alpha-\beta}\frac{P^{\alpha}-1\,.\,P^{\beta}+z}{P^{l}-1\,.\,P^{\gamma}+z}+P^{2(\gamma-\alpha-\beta)}\frac{P^{\alpha}-1\,.\,P^{\alpha+l}-1\,.\,P^{\beta}+z\,.\,P^{\beta+l}+z}{P^{l}-1\,.\,P^{2l}-1\,.\,P^{\gamma}+z\,.\,P^{\gamma+l}+z}+\dots, \tag{46}$$

and

$$\frac{\{P^{x+y}1\}^n}{[P^x z^{-1}]^n}=\underset{\kappa=\infty}{\mathrm{L}}\frac{(1-P^{x+y-(n-1)l})\dots(1-P^{x+y-(n-\kappa)l})(P^{x+l}+z^{-1})\dots(P^{x+\kappa l}+z^{-1})}{(1-P^{x+y+l})\dots(1-P^{x+y+\kappa l})(P^{x-(n-1)l}+z^{-1})\dots(P^{x-(n-\kappa)l}+z^{-1})}.$$

After making the substitutions (45), z for z^{-1}, γ for $x-(n-1)\,l$, $-\beta$ for y, $-\alpha$ for nl, the infinite product becomes

$$\underset{\kappa=\infty}{\mathrm{L}}\frac{(P^{\gamma-\beta}-1)\dots(P^{\gamma-\beta+(\kappa-1)l}-1)(P^{\gamma-\alpha}+z)\dots(P^{\gamma-\alpha+(\kappa-1)l}+z)}{(P^{\gamma-\beta-\alpha}-1)\dots(P^{\gamma-\beta-\alpha+(\kappa-1)l}-1)(P^{\gamma}+z)\dots(P^{\gamma+(\kappa-1)l}+z)}, \tag{47}$$

which we write
$$\frac{\Gamma(P^{\gamma-\beta-\alpha})\,\Gamma(P^{\gamma}z)}{\Gamma(P^{\gamma-\beta})\,\Gamma(P^{\gamma-\alpha}z)}. \tag{48}$$

16. At this point it will be best to give a few particular cases of the series

$$[P^x z]^n = 1+\Sigma P^{rx+\frac{1}{2}(r\,.\,r-1)l}\left\{\begin{matrix} n \\ r \end{matrix}\right\} z^r.$$

Putting $x=1$, $l=1$, n positive and integral, $P=p$, a simple element, we have

$$(1+pz)(1+p^2z)\dots(1+p^nz)\equiv 1+\Sigma p^{\frac{1}{2}(r\,.\,r+1)}\frac{p^n-1\,.\,p^{n-1}-1\dots p^{n-r+1}-1}{p-1\,.\,p^2-1\dots p^r-1}z^r, \tag{49}$$

$$(1+pz)(1+p^2z)\dots\text{ad inf.}=1+\Sigma(-1)^r\frac{p^{\frac{1}{2}(r\,.\,r+1)}}{p-1\,.\,p^r-1\dots p^r-1}z^r. \tag{50}$$

Putting $x=0$, $l=-1$, $n=-m$, we have

$$\frac{1}{(1+pz)(1+p^2z)\dots(1+p^mz)}=1+\Sigma(-1)^r p^r z^r\frac{p^{m+r-1}-1\dots p^m-1}{p-1\dots p^r-1}; \tag{51}$$

changing z to $-z$, making m infinite,

$$\frac{1}{(1-pz)(1-p^2z)\dots\text{ad inf.}}=1+\Sigma\frac{p^r z^r}{1-p\,.\,1-p^2\dots 1-p^r}. \tag{52}$$

Putting $x = 1$, $l = 2$, n positive and integral,

$$(1+pz)(1+p^3z)(1+p^5z)\dots(1+p^{2n-1}z)$$

$$\equiv 1+\Sigma p^{r^2}\frac{p^{2n}-1.p^{2n-2}-1\dots p^{2n-2r+2}-1}{p^2-1.p^4-1\dots p^{2r}-1}z^r \text{ (Cauchy)}$$

$$(x=-0,\ l=-2,\ n=-m); \qquad (53)$$

$$\frac{1}{(1-pz)(1-p^3z)\dots(1-p^{2m-1}z)} = 1+\Sigma p^r\frac{p^{2m}-1\dots p^{2m+2r-2}-1}{p^2-1\dots p^{2r}-1}z^r, \qquad (54)$$

$$\frac{1}{(1-pz)(1-p^3z)\dots\text{ad inf.}} = 1+\Sigma p^r\frac{z^r}{1-p^2.1-p^4\dots1-p^{2r}}, \qquad (55)$$

$$(1-p^2)(1-p^4)\dots\text{ad inf.} = 1-\frac{p^2}{1-p^2}+\frac{p^6}{1-p^2.1-p^4}$$

$$-\frac{p^{12}}{1-p^2.1-p^4.1-p^6}, \qquad (56)$$

$$(1-p)(1-p^2)(1-p^3)\text{ ad inf.} = 1-\frac{p}{1-p}+\frac{p^3}{1-p.1-p^2}-\dots. \qquad (57)$$

If $[P^xa+b]^n$ denotes $(p^xa+b)(p^{x+l}a+b)\dots(P^{x+(n-1)l}a+b)$, then

$$[P^{x+y}1-1]^n\frac{b^n}{P^{nx}}$$

$$\equiv[P^yb+a]^n-\frac{1}{P^x}\frac{P^{nl}-1}{P^l-1}[P^yb+a]^{n-1}[P^xa+b]^1+\dots+(-1)^n\frac{1}{P^{nx}}[P^xa+b]^n.$$

$$(58)$$

17. If ${}^1[x]$ denotes $\frac{p^x-1}{p-1}$, then ${}^1[x+y] = {}^1[x]+p^x\,{}^1[y]$. Let ${}^2[x]$ denote

$$\frac{p_2^{\frac{p_1^x-1}{p_1-1}}-1}{p_2-1};$$

then $${}^2[x+y]\equiv\frac{p_2^{[x+y]}-1}{p_2-1}\equiv\frac{p_2^{[x]}-1}{p_2-1}+p_2^{[x]}\frac{p_2^{p^x[y]}-1}{p_2-1}\equiv{}^2[x]+{}^2[P^xy].$$

Similarly $${}^3[x+y]\equiv\frac{p_3^{{}^2[x+y]}-1}{p_3-1}\equiv\frac{p_3^{{}^2[x]+{}^2[P^xy]}-1}{p_3-1}$$

$$\equiv\frac{p_3^{{}^2[x]}-1}{p_3-1}+p_3^{{}^2[x]}\frac{p_3^{{}^2[P^xy]}-1}{p_3-1}\equiv{}^3[x]+{}^3[P^xy].$$

In general, if ${}^{n}[x]$ denotes

$$\frac{p_1^x-1}{p_1-1}$$

$$\frac{p_2 \qquad -1}{p_2 \qquad -1}$$

$$\ldots \quad \ldots \quad \ldots$$

$$\frac{p_n \qquad\qquad -1}{p_n \qquad\qquad -1} \tag{59}$$

and ${}^{n}[P^x y]$ denotes

$$p_1^x \frac{p_1^y-1}{p_1-1}$$

$$p_2^{{}^1[P^x y]} \frac{p_2 \qquad\quad -1}{p_2 \qquad\quad -1}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$p_n^{{}^{n-1}[P^x y]} \frac{p_n \qquad\qquad\qquad -1}{p_n \qquad\qquad\qquad -1}, \tag{60}$$

then $${}^{n}[x+y] = {}^{n}[x] + {}^{n}[P^x y]. \tag{61}$$

Denote $${}^{m}[x]\,{}^{m}[x-1]\,{}^{m}[x-2] \ldots {}^{m}[x-n+1] \text{ by } {}^{m}[x]_n; \tag{62}$$

then $${}^{m}[x+y]_n = {}^{m}[x]_n + A_1{}^{m}[x]_{n-1} + A_2{}^{m}[x]_{n-2} + \ldots + {}^{m}[P^x y]_n; \tag{63}$$

for $${}^{m}[x+y]_1 \equiv {}^{m}[x]_1 + {}^{m}[P^x y], \tag{64}$$

and ${}^{m}[x+y-1]$ may be written in two forms:

$${}^{m}[x-1] + {}^{m}[P^{x-1} y] \tag{i.}$$

and $${}^{m}[x] + {}^{m}[P^x y-1]. \tag{ii.}$$

Multiply the first term of (64) by (i.) and the second term by (ii.); then we obtain

$${}^{m}[x+y]_2 \equiv {}^{m}[x]_2 + {}^{m}[x]_1 \{{}^{m}[P^x y] + {}^{m}[P^{x-1} y]\} + {}^{m}[P^x y]_2.$$

Proceeding in this way, we shall obtain

$${}^{m}[x+y]_n = {}^{m}[x]_n + {}^{m}[x]_{n-1} A_1 + {}^{m}[x]_{n-2} A_2 + \ldots + {}^{m}[x]_{n-r} A_r + \ldots + A_n,$$

in which A_r is the sum of $\dfrac{n!}{r!\,(n-r)!}$ terms of the type

$${}^{m}[P^{x-s_1} y]\,{}^{m}[P^{x-s_2} y-1]\,{}^{m}[P^{x-s_3} y-2] \ldots {}^{m}[P^{x-s_r} y-r+1], \tag{65}$$

the $\dfrac{n!}{r!\,(n-r)!}$ terms being formed by giving in the typical term all positive integral values from 0 to $n-r$ to each of $s_1, s_2, s_3, \ldots, s_r$; in any one set s_1 not to be $> s_2$, s_2 not $> s_3$, ..., s_{r-1} not $> s_r$. The following table shows the values of s_1, s_2, s_3 when $n = 6$:—

s_1	s_2	s_3	s_1	s_2	s_3
0	0	0	0	0	3
0	0	1	0	1	3
0	1	1	1	1	3
1	1	1	0	2	3
0	0	2	1	2	3
0	1	2	2	2	3
1	1	2	0	3	3
0	2	2	1	3	3
1	2	2	2	3	3
2	2	2	3	3	3

(66)

so that, when $n = 6$, A_3 will be the sum of the twenty terms

$$ {}^m[P^x y]\,{}^m[P^x y-1]\,{}^m[P^x y-2]+{}^m[P^x y]\,{}^m[P^x y-1]\,{}^m[P^{x-1} y-2] $$
$$ +{}^m[P^x y]\,{}^m[P^{x-1} y-1]\,{}^m[P^{x-1} y-2]+17 \text{ similar terms}; \quad (67) $$

when m is zero or $p = 1$ each of the $\frac{n!}{r!\,n-r!}$ terms in A_r is $y\,.\,y-1\,\ldots\,y-r+1$; therefore

$$ A_r = \frac{n!}{r!\,(n-r)!}\,(y_r) \quad \text{(Vandermonde's theorem)}; $$

when $m = 1$, $$ A_r \equiv p^{r(x-n-r)}\,\frac{p^n-1\,\ldots\,p^{n-r+1}-1}{p\;-1\,\ldots\,p^r\quad-1}\,[y]_r, $$

which is a generalized form of Vandermonde's theorem (*Proc. Lond. Math. Soc.*, Vol. XXVII., p. 347, § 25).

In the case $m = 2$ the $\frac{r!}{n!\,(n-r)!}$ terms whose sum gives A_r cannot be expressed as a product. It only appears possible to express A_r as a product in the cases $m = 0$, $m = 1$.

18. Denoting $\frac{p^n-1}{p-1}$ by $[n]$, if $y = x^{[n]}$,

$$ \frac{dy}{dx} = [n]\,x^{[n]-1}; $$

but $$ [n]-1 = \frac{p^n-1}{p-1}-1 = p\,\frac{p^{n-1}-1}{p-1} = p\,[n-1]; $$

therefore
$$\frac{dy}{dx} = [n]\,(x^{p})^{[n-1]}.$$

Now differentiate, regarding x^p as the independent variable;
$$\frac{d}{d\,(x^p)}\left\{\frac{dy}{dx}\right\} = [n][n-1]\,(x^{p})^{[n-1]-1} = [n][n-1]\,x^{p^2[n-2]}.$$

Write this
$$\frac{d^{(2)}y}{dx^{(2)}} = [n][n-1]\,(x^{p^2})^{[n-2]}.$$

Differentiating, regarding x^{p^2} as the independent variable,
$$\frac{d}{d\,(x^{p^2})}\left\{\frac{d^{(2)}y}{dx^{(2)}}\right\} = [n][n-1][n-2]\,x^{p^3[n-3]} = [n][n-1][n-2]\,x^{[n]-[3]},$$
$$\frac{d^{(3)}y}{dx^{(3)}} = [n][n-1][n-2]\,x^{[n]-[3]},$$

and, in general,
$$\frac{d^{(r)}y}{dx^{(r)}} = [n][n-1]\dots[n-r+1]\,x^{p^r[n-r]}$$
$$= [n][n-1]\dots[n-r+1]\,x^{[n]-[r]}. \tag{68}$$

19. We proceed to consider certain differential equations for the following series:—
$$y = 1+\frac{p^\alpha-1\,.\,p^\beta-1}{p-1\,.\,p^\gamma-1}\,x^{[1]}+\frac{p^\alpha-1\,.\,p^{\alpha+1}-1\,.\,p^\beta-1\,.\,p^{\beta+1}-1}{p-1\,.\,p^2-1\,.\,p^\gamma-1\,.\,p^{\gamma+1}-1}\,x^{[2]}+\dots, \tag{69}$$

which reduces when $p = 1$ to the hypergeometric series $F(\alpha\beta\gamma x)$. The differential equation is
$$\frac{(p^\alpha-1)(p^\beta-1)}{(p-1)^2}\,y+\frac{p^\alpha(p^{\beta+1}-1)+p^\beta(p^\alpha-1)}{p-1}\,x\,\frac{dy}{dx}+p^{\alpha+\beta+1}x^{[2]}\,\frac{d^{(2)}y}{dx^{(2)}}$$
$$-\frac{1}{x}\left\{\frac{p^\gamma-1}{p-1}\,x\,\frac{dy}{dx}+p^\gamma x^{[2]}\,\frac{d^{(2)}y}{dx^{(2)}}\right\} = f(x)-f(x^p), \tag{70}$$

reducing when $p = 1$ to
$$x\,(x-1)\,\frac{d^2y}{dx^2}+\{x\,(\alpha+\beta+1)-\gamma\}\,\frac{dy}{dx}+\alpha\beta y = 0.$$

The differential equation of the series
$$y = 1+\frac{[a_1][a_2]\dots[a_n]}{[1][\beta_1]\dots[\beta_{n_1}]}\,x^{[1]}+\frac{[a_1][a_1+1][a_2][a_2+1]\dots[a_n][a_n+1]}{[1][2][\beta_1]\beta_1+1]\dots[\beta_{n_1}][\beta_{n_1}+1]}\,x^{[2]} \tag{71}$$

will be found in the form

$$\sum_{r=1}^{r=n} A_r x^{[r]} \frac{d^{(r)}y}{dx^{(r)}} - \frac{1}{x} \sum_{r=0}^{r=n_1} B_r x^{[r+1]} \frac{d^{(r+1)}y}{dx^{(r+1)}} = f(x) - f(x^p), \tag{72}$$

A_r and B_r being functions of $\alpha_1 \alpha_2 \ldots \beta_1 \beta_2 \ldots$. The series and differential equation reduce when $p = 1$ to the hypergeometric series with n elements in the numerator and n' elements in the denominator of each term and its differential equation.

Consider

$$C_0 y + C_1 x \frac{dy}{dx} + C_2 x^{[2]} \frac{d^{(2)}y}{dx^{(2)}} + \ldots + C_s x^{[s]} \frac{d^{(s)}y}{dx^{(s)}} \equiv \phi\left[x \frac{dy}{dx}\right];$$

then, if y be a series of the form

$$y = a_1 x^{[m_1]} + a_2 x^{[m_2]} + \ldots + a_r x^{[m_r]} + \ldots, \tag{73}$$

performing the operations indicated by $\phi\left[x \frac{dy}{dx}\right]$, we have

$$\begin{aligned} \phi\left[x \frac{dy}{dx}\right] = \ & \{C_0 + C_1 [m_1] + C_2 [m_1][m_1 - 1] + \ldots \\ & \qquad + C_s [m_1][m_1 - 1] \ldots [m_1 - s + 1]\} \, a_1 x^{[m_1]} \\ & + \{C_0 + C_1 [m_2] + C_2 [m_2][m_2 - 1] + \ldots \\ & \qquad + C_s [m_2][m_2 - 1] \ldots [m_2 - s + 1]\} \, a_2 x^{[m_2]} + \ldots, \end{aligned}$$

that is,

$$\phi\left[x \frac{dy}{dx}\right] = a_1 \phi[m_1] x^{[m_1]} + a_2 \phi[m_2] x^{[m_2]} + \ldots + a_r \phi[m_r] x^{[m_r]} + \ldots, \tag{74}$$

in which $\phi[m]$ denotes

$$C_0 + C_1 [m] + C_2 [m][m-1] \ldots + C_s [m][m-1] \ldots [m-s+1].$$

Similarly, if

$$C_0' x \frac{dy}{dx} + C_1' x^{[2]} \frac{d^{(2)}y}{dx^{(2)}} + \ldots + C_t' x^{[t+1]} \frac{d^{(t+1)}y}{dx^{(t+1)}} \equiv \psi\left[x \frac{dy}{dx}\right],$$

then performing the operations indicated, we have

$$\frac{1}{x} \psi\left[x \frac{dy}{dx}\right] = a_1 \psi[m_1] x^{p[m_1 - 1]} + a_2 \psi[m_2] x^{p[m_2 - 1]} + \ldots \tag{75}$$

and $$\phi\left[x \frac{dy}{dx}\right] - \frac{1}{x} \psi\left[x \frac{dy}{dx}\right] = \Sigma a_r \phi[m_r] x^{[m_r]} - \Sigma a_r \psi[m_r] x^{p[m_r - 1]}. \tag{76}$$

A consideration of the indices $[m_1] \ldots [m_2] \ldots$, $p[m_1 - 1]$, $p[m_2 - 2]$, shows us that, putting

$$m_1 = m_2 - 1, \qquad m_2 = m_3 - 1, \qquad m_3 = m_4 - 1, \qquad \ldots,$$

and $$a_1 \phi[m_1] = a_2 \psi[m_2], \qquad a_2 \phi[m_2] = a_3 \psi[m_3], \qquad \ldots,$$

also choosing m_1 so as to make the coefficient of $x^{p[m_1-1]}$ vanish, that is choosing $m_1 = a$, where a is any root of $\psi[m_1] = 0$, then

$$y = A\left\{x^{[a]} + \frac{\phi[a]}{\psi[a+1]}x^{[a+1]} + \frac{\phi[a]\phi[a+1]}{\psi[a+1]\psi[a+2]}x^{[a+2]} + \dots\right], \quad (77)$$

which satisfies the equation

$$\phi\left[x\frac{dy}{dx}\right] - \frac{1}{x}\psi\left[x\frac{dy}{dx}\right] = f(x) - f(x)^p, \quad (78)$$

in which

$$f(x) = A\phi[a]\left\{x^{[a]} + \frac{\phi[a+1]}{\psi[a+1]}x^{[a+1]} + \frac{\phi[a+1]\phi[a+2]}{\psi[a+1]\psi[a+2]}x^{[a+2]} + \dots\right\}. \quad (79)$$

20. Denoting $\dfrac{(p^{a_1+m}-1)(p^{a_2+m}-1)\dots(p^{a_n+m}-1)}{(p-1)^n}$ by $\Pi_n[a+m]$, then

$$\Pi[a+m] = A_0 + A_1[m] + A_2[m][m-1] + \dots + A_n[m][m-1]\dots[m-n+1],$$

in which the coefficients A_0, A_1, ... are independent of m and are given by

$$A_r = \sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}s\cdot s-1}\frac{\Pi[a+r-s]}{[r-s]!\,[s]!}, \quad (80)$$

$$[r]! \equiv [r][r-1][r-2]\dots[2][1]$$

[*Proc. London Math. Soc.*, Vol. XXVIII., p. 479, (6)].

Consider a differential equation

$$\sum_{r=0}^{r=n} A_r x^{[r]}\frac{d^{(r)}y}{dx^{(r)}} - \frac{1}{x}\sum_{r=0}^{r=n_1} B_r x^{[r+1]}\frac{d^{(r+1)}y}{dx^{(r+1)}} = f(x) - f(x^p), \quad (81)$$

$$A_r = \sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}s\cdot s-1}\frac{\Pi_n[a+r-s]}{[r-s]!\,[s]!}, \quad B_r = \sum_{s=0}^{s=r}(-1)_s p^{\frac{1}{2}s\cdot s-1}\frac{\Pi_{n_1}[\beta+r-s]}{[r-s]!\,[s]!};$$

then, if this equation be satisfied by a series of the form

$$y = a_1 x^{[m_1]} + a_2 x^{[m_2]} + \dots + a_r x^{[m_r]} + \dots,$$

we see that in the notation of § 19

$$\phi[m_1] = A_0 + A_1[m_1] + A_2[m_1][m_1-1] + \dots$$
$$+ A_n[m_1][m_1-1]\dots[m_1-n+1],$$

A_0, A_1, ... having the values given above; therefore, by (80),

$$\phi[m_1] \equiv \Pi[a+m_1].$$

In the same way,

$$\psi[m_1] = B_0[m_1] + B_1[m_1][m_1-1] + \dots + B_{n_1}[m_1][m_1-1]\dots[m_1-n]$$
$$= [m_1]\Pi_{n_1}[\beta+m_1-1].$$

The values of m_1 which make $\psi[m_1] = 0$ are $0, 1-\beta_1, 1-\beta_2, 1-\beta_3, \ldots, 1-\beta_n$.

The zero value of m_1 gives

$$y = 1 + \frac{\Pi_n[a]}{[1]\,\Pi_{n_1}[\beta]} x^{[1]} + \frac{\Pi_n[a]\,\Pi_n[a+1]}{[1][2]\,\Pi_{n_1}[\beta]\,\Pi_{n_1}[\beta+1]} x^{[2]} + \ldots, \tag{82}$$

that is,

$$y = 1 + \frac{[a_1][a_2]\ldots[a_n]}{[1][\beta_1]\ldots[\beta_{n_1}]} x^{[1]} + \frac{[a_1][a_1+1]\ldots[a_n][a_n+1]}{[1][2][\beta_1][\beta_1+1]\ldots[\beta_{n_1}][\beta_{n_1}+1]} x^{[2]} + \ldots, \tag{83}$$

as a solution of equation (81), in which

$$f(x) = [a_1][a_2]\ldots[a_n]\left\{1 + \frac{[a_1+1][a_2+1]\ldots[a_n+1]}{[1][\beta_1]\ldots[\beta_{n_1}]} x^{[1]} + \ldots\right\}. \tag{84}$$

For the root $1-\beta_s$ of $\psi[m_1] = 0$ we shall have

$$y = x^{[1-\beta_s]} + \frac{[a_1-\beta_s+1][a_2-\beta_s+1]\ldots[a_n-\beta_s+1]}{[1][2-\beta_s][1-\beta_s+\beta_1]\ldots[1-\beta_s+\beta_n]} x^{[2-\beta_s]} + \ldots,$$

$$\begin{aligned} f(x) &= \Pi_n[a-\beta_s+1] \\ &\times \left\{x^{[1-\beta_s]} + \frac{[a_1-\beta_s+2]\ldots[a_n-\beta_s+2]}{[1]\,[2-\beta_s][1-\beta_s+\beta_1]\ldots[1-\beta_s+\beta_n]} x^{[2-\beta_s]} + \ldots\right\}. \end{aligned} \tag{85}$$

In the particular case of the series

$$y = 1 + \frac{p^\alpha-1\,.\,p^\beta-1}{p-1\,.\,p^\gamma-1} x^{[1]} + \frac{p^\alpha-1\,.\,p^{\alpha+1}-1\,.\,p^\beta-1\,.\,p^{\beta+1}-1}{p-1\,.\,p^2-1\,.\,p^\gamma-1\,.\,p^{\gamma+1}-1} x^{[2]} + \ldots \tag{86}$$

we have

$$\begin{aligned} [\alpha][\beta]y + \Big[\{p^\alpha[\beta+1] + p^\beta[\alpha]\}\,x - [\gamma]\Big]\frac{dy}{dx} + \{p^{\alpha+\beta+1}x^{[2]} - p^\gamma x^p\}\,\frac{d^{(2)}y}{dx^{(2)}} \\ = f(x) - f(x^p), \end{aligned} \tag{87}$$

$$f(x) = [\alpha][\beta]\left\{1 + \frac{p^{\alpha+1}-1\,.\,p^{\beta+1}-1}{p-1\,.\,p^\gamma-1} x^{[1]} + \ldots\right\},$$

also a similar series and equation from $m_1 = 1-\gamma$,

$$y = x^{[1-\gamma]}\left\{1 + \frac{p^{\alpha+1-\gamma}-1\,.\,p^{\beta+1-\gamma}-1}{p-1\,.\,p^{2-\gamma}-1} x^{p^{1-\gamma}[1]} + \ldots\right\}, \tag{88}$$

$$f(x) = x^{[1-\gamma]}\left\{1 + \frac{p^{\alpha+2-\gamma}-1\,.\,p^{\beta+2-\gamma}-1}{p-1\,.\,p^{2-\gamma}-1} x^{p^{1-\gamma}[1]} + \ldots\right\}.$$

Another interesting equation is

$$\sum_{r=1}^{r=n} A_r x^{[r]} \frac{d^{(r)}y}{dx^{(r)}} - \sum_{r=1}^{r=m} B_r x^{[r+1]} \frac{d^{(r+1)}y}{dx^{(r+1)}} = f(x) - f(x^p), \tag{89}$$

$$A_r = p^{r(\alpha-n+r)} \frac{p^n-1 . p^{n-1}-1 \dots p^{n-r+1}-1}{p-1 . p^2-1 \dots p^r-1} [p^\alpha z]^{n-r} z^r,$$

$$B_r = p^{r(\beta-m+r)} \frac{p^m-1 . p^{m-1}-1 \dots p^{m-r+1}-1}{p-1 . p^2-1 \dots p^r-1} [p^\beta z]^{m-r} z^r;$$

then $$y = A \left\{1+\frac{[p^\alpha z]^m}{[1][p^\beta z]^m} x^{[1]} + \frac{[p^\alpha z]^m [p^{\alpha+1} z]^m}{[1][2][p^\beta z]^m [p^{\beta+1} z]^m} x^{[2]} + \dots\right\}, \tag{90}$$

$$f(x) = A [p^\alpha z]^n \left\{1+\frac{[p^{\alpha+1} z]^n}{[1][p^\beta z]^m} x^{[1]} + \dots\right\} \quad [\S\ 1, (4)]. \tag{91}$$

The series of Legendre and Bessel can be generalized in the form

$$P_{[n]}(x) = A \left\{x^{[n]} - \frac{[n][n-1]}{[2][2n-1]} p^3 x^{[n-2]} \right.$$
$$\left. + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]} p^8 x^{[n-4]} + \dots\right\}, \tag{92}$$

the relation between successive coefficients being

$$A_{r+1} = -A_r p^{2r+1} \frac{[n-2r+1][n-2r+2]}{[2r][2n-2r+1]}, \tag{93}$$

$Q_{[n]}(x)$ as a series in which the relation between coefficients is

$$A_{r+1} = A_r \frac{[n+2r][n+2r-1]}{[2r][2n+2r+1]} p^{-3}. \tag{94}$$

$$J_{[n]}(x) = A \left\{x^{[n]} + \frac{x^{[n+2]}}{[2][2n+2]} + \frac{x^{[n+4]}}{[2][4][2n+2][2n+4]} + \dots\right\}. \tag{95}$$

ON NON-UNIFORM CONVERGENCE AND TERM-BY-TERM INTEGRATION OF SERIES

By W. H. YOUNG.

[Received January 6th, 1903.—Read January 8th, 1903].

1. *Introductory.*—In the present paper the problem of the classification and distribution of the points of non-uniform convergence of a convergent series of functions

$$F(x) = f_1(x) + f_2(x) + \ldots + f_n(x) + \ldots \text{ ad inf.}$$

and of the integration of this series term by term is discussed in its most general form, *i.e.*, when the only restriction laid on the functions $F, f_1, f_2, \ldots$ is that they are at most pointwise discontinuous.[*] Without this restriction, one at least of the functions being totally discontinuous, the integration would be impossible.

The results obtained, (§§ 12, 13), are generalizations of those enunciated and proved for continuous functions by Prof. Osgood in the fundamental paper on this subject in the *American Journal of Mathematics*, Vol. XIX., and recently extended by Dr. Hobson,[†] by the application of Baire's methods, to the case when only one of the functions is discontinuous; that is, when the sum of a series of continuous functions, not being itself continuous, is, by a theorem of Baire's,[‡] pointwise discontinuous.

It should be noted that the results previously obtained are not assumed in this paper. With certain simplifications, alluded to in their proper place, what follows constitutes a second proof of Dr. Hobson's results, and in the case of continuous functions reduces in all essentials to Prof. Osgood's original proof.

2. Let $f_1(x)$, $f_2(x)$, ..., $f_n(x)$, ... be at most pointwise discontinuous functions, and let their sum, denoted by $F(x)$, be also at most pointwise discontinuous. We denote the sum of the first n terms of the

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[*] In Part II., of course, they must also belong the class of *integrable* pointwise discontinuous functions.

[†] *Proc. London Math. Soc.*, Vol. XXXIV., pp. 254 *et seq.* Dr. Hobson communicated some of his results to me before his paper was ready for presentation, and I succeeded at that time in extending Prof. Osgood's results by the use of the latter's original methods. At Prof. Osgood's suggestion I undertook, before publishing my work, to attempt the further extension of these theorems: the present paper is the result.

[‡] "Dissertation," *Ann. di Mat.*, Vol. III., 1899.

series by $s_n(x)$, and the residue by $R_n(x)$, so that

$$s_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x), \quad R_n(x) = f_{n+1}(x) + f_{n+2}(x) + \ldots \text{ ad inf.},$$
$$F(x) = s_n(x) + R_n(x).$$

3. We begin by defining afresh some of the terms we shall have reason to employ.

Uniform Convergence at a Point.—A series of functions of x is said to be "uniformly convergent at a point ξ" if, given any positive quantity A, however small, an interval δ can be described, having ξ as internal point, so that, for all points x within the interval δ, $|R_n(x)| < A$ for all values of $n \geqslant m$ where m is an integer, independent of x, which can always be determined.

Similarly we may define the expression *right-handed*, (or *left-handed*), *uniform convergence* at ξ. In this case, ξ must be, left-hand, (or right-hand), end-point of δ.

Uniform Convergence throughout a Closed Interval or Closed Set.—A series of functions of x is said to be "uniformly convergent throughout a closed interval or closed set" if it is uniformly convergent at every point of the interval or set; the uniform convergence at the extreme left-hand point being in general right-handed only, and at the extreme right-hand point left-handed.

The Heine-Borel theorem and its extension* enable us to put this definition in a form sometimes more convenient for application. Let us take the case of a closed set, as including the other. Let C and D be the extreme points, (C, C') and (D, D') the corresponding intervals. Then the points of the given set in (C', D') form a closed set, every point of which is interior to an interval. Hence, by the theorems referred to, there are a finite number k of the intervals having the same property, and to each of these corresponds a definite integer m. If we take m_A to be the largest of these integers, and of those corresponding to (C, C') and (D, D'), then it is plain that m_A possesses the property that, for all values of $n \geqslant m_A$, $|R_n(x)| < A$ for all points x of the $k+2$ intervals containing the closed set.

FIG. 1.

Thus we have *a second (the usual) definition of uniform convergence throughout a closed interval.* A series of functions of x is said to be "uniformly convergent throughout a closed interval" when, given any positive quantity A, however small, an integer m can be found, independent of x, such that, for all values of $n \geqslant m$, $|R_n(x)| < A$ for all values of x in the interval.

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* W. H. Young, *Proc. London Math. Soc.*, Vol. xxxv., p. 387.

A similar definition might be given in the case of the closed set, the words after "$|R_n(x)| < A$" being replaced by the following:—"for all values of x within a finite number of intervals, having the points of the given set as internal points."

Uniform Convergence throughout an Open Interval or Set.—A series of functions of x is said to be "uniformly convergent throughout an open interval or an open set" if it be uniformly convergent at every point of the open interval or open set; or, which is the same thing, if it be uniformly convergent throughout every closed set contained in the open interval or set.

4. It will be noticed that uniform convergence at a point does not imply, in general, uniform convergence in any interval, however small, containing the point.

Given any interval in which the series is convergent, the points of the interval will consist of the set of points at which the series is uniformly convergent, and those at which it is not; these latter are called *the points of non-uniform convergence.*

A point of non-uniform convergence must, then, be such that when A is chosen sufficiently small, but finite, there is a set of points $P_1, P_2, \ldots, P_i, \ldots$ all lying above the line $y = A$, the coordinates of P_i being x_i, $|R_{n_i}(x_i)|$, where $n_1, n_2, \ldots$ is a series of positive integers, constantly increasing without limit, and the limiting value of x_i, for $i = \infty$, is the abscissa of P.

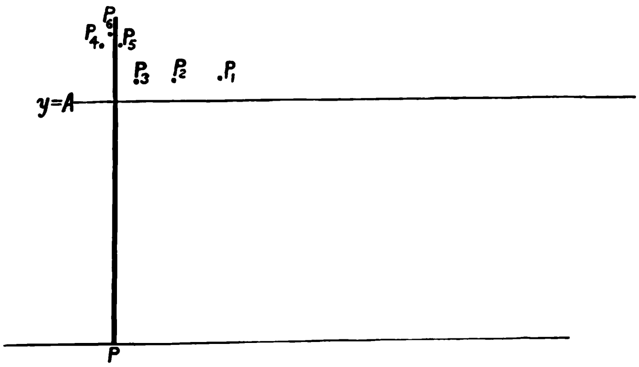

FIG. 2.

Osgood uses the graphic expression:—"A point P of non-uniform convergence is one in the neighbourhood of which the loci $y = s_n(x)$ have peaks." This expression, which, in the case when $y = s_n(x)$ is a graphic curve, is at once evident, can still be used *cum grano salis* in the general case: we shall see later what modifications have to be borne in mind.

5. *Measure of Non-uniform Convergence.*—It is convenient to classify the points of non-uniform convergence in two different ways: (1) quantitatively, and (2) qualitatively. First, quantitatively: Suppose P to be a point of non-uniform convergence, either left-handed or right-handed, or both. Then a smallest quantity B must exist, such that, for all values of $A \leqslant B$, it is impossible to find an interval δ, however small, having P as right-hand, left-hand, or interior point respectively, corresponding to a given integer m, so that, for all values of x in the interval δ, and for all values of $n \geqslant m$, $|R_n(x)| < A$. This quantity B may be called *the measure of convergence* at the point, the words "right-handed" or "left-handed" being added, if we are dealing with those cases. Notice that at a point at which the non-uniform convergence is both right- and left-handed the measure of non-uniform convergence obtained by this rule is the larger of the two measures of right- and left-handed non-uniform convergence.

It may happen that the measure of non-uniform convergence, so defined, is infinite at some points. Such a point may be, but is not of necessity, an infinite discontinuity of one or more of the functions.

6. We now proceed to a convenient qualitative classification of the points of non-uniform convergence.

It was first pointed out by Stokes[*] that, when the sum of a series of continuous functions is discontinuous at a point P, that point is necessarily a point of non-uniform convergence of the series. It was at first supposed that the converse was true, *i.e.*, that there were no other points of non-uniform convergence. Prof. Osgood's paper, however, deals with the sum of continuous functions only when it is itself continuous; so that no point of the type considered by Stokes occurs at all.

The following examples, due to Osgood, illustrate this point:—

Ex. 1.— $$s_n(x) = \frac{n^2x+a}{1+n^3x^2},$$

$$F(0) = a, \quad F(x) = 0, \qquad (0 < x \leqslant 1).$$

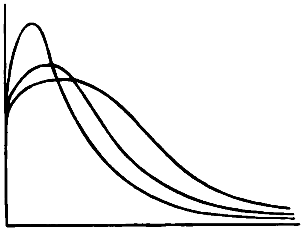

FIG. 3.

When a is not zero we have a Stokes point at the origin. When a is zero the sum is continuous. In any case, however, the origin is a point of non-uniform convergence, with what Osgood calls "infinite peaks," that is to say, the measure of non-uniform convergence at the origin is infinite.

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[*] *Camb. Phil. Trans.*, Vol. VII., 1847.

Ex. 2.— $s_n(x) = \dfrac{nx+a}{1+n^2x^2}$,

$$F(0) = a, \quad F(x) = 0, \quad (0 < x \leqslant 1).$$

Here again when a is not zero we have a Stokes point at the origin, while when a is zero the sum is continuous. The origin is, however, for all values of a, a point of non-uniform convergence with peaks of height $B = \dfrac{\sqrt{1+a^2}+a}{2}$, B being the measure of convergence (Fig. 4).

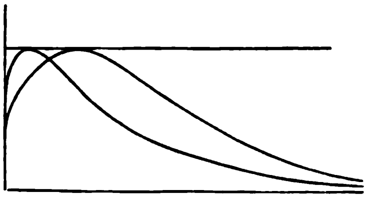

FIG. 4.

7. The set of points discovered by Stokes is, so to speak, visible; it consists of the discontinuities of $F(x)$; $f_1(x), f_2(x), \ldots$ being continuous. The remaining points of non-uniform convergence might be said to be invisible.

When we come to consider the case when $f_1(x), f_2(x), \ldots$ are pointwise discontinuous as well as $F(x)$, the matter becomes slightly more complicated. We have, however, still two classes of points of non-uniform convergence: (1) visible ones, due to discontinuities of the functions involved; and (2) invisible ones, which have no particular connection with discontinuities.

Referring to the diagram of Fig. 2, P may be a discontinuity of the functions $R_n(x)$ for an infinite number of values of n, that is to say, a discontinuity of one or more of the functions $F(x)$, $f_1(x)$, $f_2(x)$, ..., or else the points P_i may be points of the corresponding loci, due to discontinuities of the successive functions $R_{n_i}(x)$. In this case P would be among the limiting points of the set formed by the discontinuities of all the functions $R_n(x)$. Such a point P would be a visible point of non-uniform convergence.

On the other hand, P may be a point of continuity of all the functions, and the points P_i may belong to what may almost be called connected parts of their corresponding loci, due to the fact that, P_i being a point of continuity, we can assign a definite interval containing P_i, throughout which the oscillation of the function R_{n_i} is as small as we please. We shall then have, as in the former case, an invisible point of non-uniform convergence.

The following example illustrates the theory of the invisible points in the case when the functions $f_1(x)$, $f_2(x)$, ..., are discontinuous.

Ex. 3.— $$s_n(x) = \frac{n^2x+a}{1+n^3x^2}$$

at all points except the points $1/n$, $2/n$, $3/n$, ..., $n-1/n$, and at all these points has the value zero.

$$F(0) = a, \quad F(x) = 0, \quad (0 < x \leqslant 1).$$

The origin is an invisible point of non-uniform convergence when a is zero, and a visible one when a is not zero. There are no other visible points of non-uniform convergence at all, the discontinuities of the functions $s_n(x)$ and their limiting points giving rise always to points of uniform convergence of the sum $F(x)$.

Ex. 4.—To illustrate the second type of visible points of non-uniform convergence consider the following series of functions :—f_1 is zero at all the points between 0 and 1 except the point $\frac{1}{2}$, at which it is 1 ; f_2 is -1 at the point $\frac{1}{2}$, 1 at $\frac{1}{4}$ and $\frac{3}{4}$, and zero at all other points ; f_3 is -1 at $\frac{1}{4}$ and $\frac{3}{4}$, 1 at $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, and zero at all the other points ; and so on. Thus s_1 is zero at all points except $\frac{1}{2}$, where it is 1 ; s_2 is zero at all points except $\frac{1}{4}$ and $\frac{3}{4}$, where it is 1 ; s_3 is zero at all points except $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, where it is 1 ; and so on. F is zero everywhere, and is therefore continuous.

Given any point, we can always assign an integer m such that, for all values of $n > m$, the value of $R_n(x)$ there is zero. Hence the series is always convergent. At all their points of continuity all the functions f_r vanish ; hence there are no invisible points of non-uniform convergence. The series is, however, *non-uniformly convergent at every point of the continuum* (0, 1), since in the neighbourhood of every assigned point there are discontinuities, equal to 1, of an infinite number of the functions $s_n(x)$, and therefore, since F is continuous, of $R_n(x)$. The points of discontinuity are therefore all visible and belong to the limiting points of the discontinuities of all the functions $R_n(x)$, and are, in fact, identical with them.

8. Two things are to be noted with respect to the visible points of non-uniform convergence. The first is that such a point is not necessarily itself a discontinuity of any of the functions concerned ; but, if not, it is necessarily a *limiting* point of discontinuities of the functions $f_1, f_2, \ldots$. The other is that all the discontinuities of F are not necessarily points of non-uniform convergence ; some might be due to discontinuities of one or more of the functions $f_1(x), f_2(x), \ldots$. To take a simple example, we might have $F = u+f_1+f_2+\ldots = u+s_n+R_n$ where u is zero everywhere

except at the point $\frac{1}{2}$, where it is 1, and s_n has the value $(n^2x+a)/(1+n^3x^2)$. Here the point $\frac{1}{2}$ is a discontinuity of F, but is not a point of non-uniform convergence. Similarly we see from *Ex.* 3 that *all* the limiting points of discontinuities of an infinite number of the functions $f_1(x), f_2(x), \ldots$ are not of necessity points of non-uniform convergence, but only (*of necessity*) those where the value of each $R_n(x)$ at its particular discontinuity remains always greater than some positive quantity.

9. It is evident that there is nothing to prevent our writing the equation in the form

$$-f_r(x) = f_1(x)+\ldots+f_{r-1}(x)-F(x)+f_{r+1}(x)+\ldots,$$

and we see that points of discontinuity of $f_r(x)$ may be points of non-uniform convergence of the equation so written.

All the series so obtained have the same $R_n(x)$, for sufficiently large n, and this remains the case if we replace any finite number of the functions by any other at most pointwise discontinuous functions, in such a way that the equation holds. The points of non-uniform convergence are in all these cases the same. Thus we see that the discontinuities of the function $f_r(x)$, for any value of r, have just the same importance from this point of view as those of F itself. It is evident from the above that, in general, a discontinuity of a single function $f_r(x)$ will only cause a discontinuity of $F(x)$, and will not coincide with a point of non-uniform convergence. The two important classes of points to be investigated are those in the neighbourhood of which there are peaks at points of continuity of the successive residues $R_n(x)$, [it may happen casually that one of these points coincides with a discontinuity of one or more of the functions $f_r(x)$ or $F(x)$]; and those points which are limiting points of the discontinuities of an infinite number of the functions $f_r(x)$, such that the value of each $R_n(x)$ at its particular discontinuity is always greater than some positive quantity.

In the cases disposed of by Prof. Osgood and Dr. Hobson, this second class of points cannot come in, since at most one of the functions is discontinuous. Hence in these, and a variety of other cases, only the former class of points of non-uniform convergence can occur.

10. *A-points.*—The former class of points we subdivide into classes of points which we call A-points, the formal definition being as follows:—

P is said to be an A-point if, assigned any integer m and any interval δ, however small, containing P as internal point, we can find*

---

* In Osgood's case this is the same as what he calls a γ_A-point. In the general case all the invisible points of non-uniform convergence are A-points, but some of the A-points may be discontinuities of one or more of the functions $F, f_1, f_2, \ldots$.

inside δ a point x which is, at least on one side, a point of continuity of $R_n(x)$, at which $|R_n(x)| > A$, n being some integer $\geqslant m$.

That is to say, P is a point in the neighbourhood of which *the locus of the points of continuity* of $y = |R_n(x)|$ has a *peak* greater than or equal to A. These points being, as we know, dense everywhere, the corresponding figure is of the type given in Fig. 5.

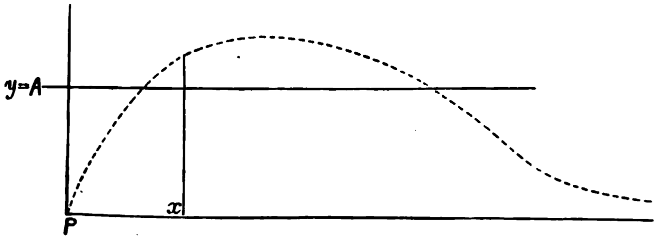

Fig. 5.

In the case disposed of by Prof. Osgood $R_n(x)$ is continuous, and therefore we do not need to postulate that the point x is a point of continuity. In Dr. Hobson's case $R_n(x)$ is not continuous, but the position of the discontinuities as well as their magnitude is the same for all values of n, depending only on $F(x)$. They will certainly be points of non-uniform convergence, but their presence does not seriously complicate affairs.

11. *The set G_A is closed and dense nowhere.*

The set composed of all the A-points, for assigned A, we denote by G_A.

These A-points necessarily form a closed set. For let T be any limiting point, and let us assign an integer m and an interval δ containing T as internal point. Then, as near as we please to T, we can find an A-point P,

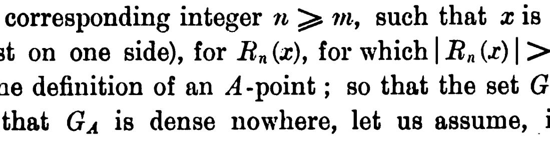

Fig. 6.

and we can assign an interval δ', lying inside δ, and containing P as internal point. Inside δ', and therefore inside δ, there is, since P is an A-point, a point x, and a corresponding integer $n \geqslant m$, such that x is a point of continuity (at least on one side), for $R_n(x)$, for which $|R_n(x)| > A$. Hence T also satisfies the definition of an A-point; so that the set G_A is closed.

To prove that G_A is dense nowhere, let us assume, if possible, the contrary, and let (C, D) be an interval in which, the closed set G_A being dense everywhere, every point is an A-point. Then in the neighbourhood of any point we please of (C, D) we can assign a point x_1 and an integer n_1, greater than any assigned m, such that x_1 is a point of continuity of R_{n_1}, at which $|R_{n_1}(x_1)| > A$. Since x_1 is a point of continuity of R_{n_1}, we can assign an interval (C_1, D_1), having x_1 as internal point or as end-point, such that the oscillation of $R_{n_1}(x)$ in (C_1, D_1) is less than $\frac{A}{2}$.

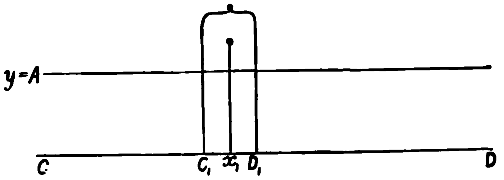

FIG. 7.

This being so, and the value of $|R_{n_1}(x_1)|$ being greater than A, $|R_{n_1}(x)| > \frac{A}{2}$ at *every* point of (C_1, D_1).

We can now proceed with (C_1, D_1) as we did with (C, D), since every point of (C_1, D_1) is an A-point, and determine an integer $n_2 > n_1$ and a point x_2, such that $|R_{n_2}(x_2)| > A$, x_2 being a point of continuity of $|R_{n_2}(x)|$, and then we can determine an interval (C_2, D_2) interior to (C_1, D_1), and containing x_2, such that, for *every* value of x inside (C_2, D_2), $|R_{n_2}(x)| > \frac{A}{2}$. In this way we can determine a sequence of intervals (C, D), (C_1, D_1), ..., (C_r, D_r), ..., and a sequence of integers $n_1, n_2, \ldots, n_r, \ldots$, such that, at *every* point of (C_r, D_r),

$$|R_{n_r}(x)| > \frac{A}{2}. \tag{1}$$

These intervals have, of course, at least one point ξ interior to all of them, and at that point the limit of $R_n(\xi)$ for n infinite is zero. Hence we can assign a definite integer m, such that

$$|R_n(\xi)| < \frac{A}{2}, \tag{2}$$

for all values of n after the assigned m, and therefore for all values $n = n_r$ greater than m. But this is in contradiction to (1). Hence, by a *reductio ad absurdum*, we are constrained to deny the possibility of finding an interval (C, D) consisting entirely of A-points, which proves the assertion that the set G_A is closed and dense nowhere.

12. *Distribution of Points of non-Uniform Convergence.*—It is evident, from what was pointed out at the end of § 9, that the points of non-

uniform convergence fall into two classes: (1) the A-points for all values of A, and (2) points which are limiting points of discontinuities of an *infinite* number of the functions f_r, including points, if any, which are themselves discontinuities of an *infinite* number of these functions. By what has been proved *the most general distribution of the points of the first of the above classes is when they form a set of the first category,*[*] *those for which A has any definite value*[†] *forming, like the discontinuities of a pointwise discontinuous function, a closed set, dense nowhere.*

The points of the second of the above classes may, however, as has been shown by an example (§ 7, Ex. 4), fill up the whole continuum.

It will be noticed that in Osgood's and in Hobson's cases the second class of points cannot occur; so that the distribution of the points of non-uniform convergence is strictly limited to form a set of the first category.

II. *On Term-by-term Integration.*

13. Let us now suppose that $F(x)$, $f_1(x)$, ... are all integrable functions. Then, since the sum of any finite number of integrable functions is integrable, s_n will be integrable for all values of n, and therefore $R_n(x) = F(x) - s_n(x)$ will be integrable also.

We shall now be able to prove that, *in any interval* (C, D) *in which, for all values of n greater than some assignable integer,* $R_n(x)$ *has a definite upper limit, the series may be integrated term by term.*[‡]

Thus, if in the interval considered there are (1) no points of infinite discontinuity of any of the functions, (2) no points in the neighbourhood of which the discontinuities of the functions become larger than any assignable quantity, and (3) no points of non-uniform convergence at which the measure of non-uniform convergence is infinite, we may certainly integrate term by term.

14. The following lemma is an immediate consequence of the definition of integration, and corresponds to the theorem that, when $y = f(x)$ is a graphic curve, $\int_C^D f(x)\,dx$ is the area bounded by the curve, the axis of x, and the ordinates through C and D.

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[*] Baire, *loc. cit.*

[†] This may be expressed otherwise by saying: all for which the measure of non-uniform convergence is greater than or equal to a given quantity A, forming, &c.

[‡] This result may be deduced as a special case of an important theorem due to Arzelà ("Sulle serie di funzioni," *Rend. di Bologna*, 1900) involving the generalization of the idea of uniform convergence.

LEMMA.—*Of Integrals.*

If $f(x)$ *be* $\geqslant 0$ *and finite throughout the closed interval* (C, D), *and we draw any graphic continuous line whatever not cutting itself, so that no point* $[x, f(x)]$ *lies above it, the area bounded by that graphic line, the axis of* x, *and the ordinates through* C *and* D *is greater than or equal to* $\int_C^D f(x)\,dx$.

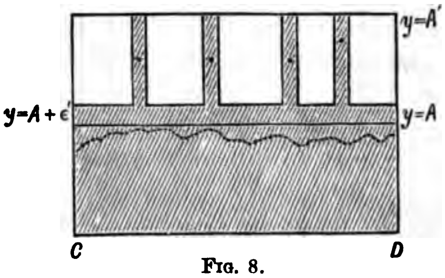

FIG. 8.

15. THEOREM 1.—*If* (C, D) *be a closed interval of length* L, *entirely free of* A*-points (for assigned* A*), in which* $|R_n(x)|$ *has a definite upper limit* A', *for all values of* x *from* C *to* D, *both inclusive, and for all values of* n *from and after an assignable integer, then we can find an integer* m, *so that, for all values of* $n \geqslant m$,

$$\int_C^D |R_n(x)|\,dx < AL + \epsilon,$$

ϵ *being as small as we please.*

For, since there are no A-points in the closed interval (C, D), we can assign an integer m, such that, for all values of $n \geqslant m$, the locus $y = |R_n(x)|$ only arises above the line $y = A$ at points of discontinuity of $R_n(x)$. If $A' \leqslant A$, the theorem now follows, by the Lemma; if not, we divide these discontinuities into two classes: (1) those at which the discontinuity is less than some small quantity ϵ', and (2) those for which the discontinuity $\geqslant \epsilon'$.

Then all the points $[x, |R_n(x)|]$, except those at which the discontinuity is greater than or equal to ϵ', lie below the line $y = A + \epsilon'$. Since, however, $R_n(x)$ is an integrable function, *all* its discontinuities $\geqslant \epsilon'$ form a closed set of content zero. Hence we can enclose all the remaining points in small rectangles, bounded by the lines $y = A + \epsilon'$, $y = A'$, the sum of whose breadths is $< \epsilon'$.

We have, in this way, enclosed all the points $[x, |R_n(x)|]$ below a broken line, and the area bounded by this line, the axis of x, and the ordinates through C and D, is less than $(A+\epsilon')L + (A'-A)\epsilon'$.

Now assigning an ϵ, however small, we can choose ϵ' so that

$$\epsilon' = \frac{\epsilon}{L+A'-A}.$$

We shall then have, by the lemma of integrals,

$$\int_C^D |R_n(x)|\,dx < AL+\epsilon.$$

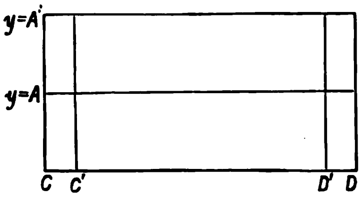

FIG. 9.

COROLLARY.—If (C, D) be as in Theorem 1, only C and D are themselves A-points, the theorem still holds. For we can cut off from each end a length ϵ', and for the remaining interval (C', D'), we can, as in the proof of Theorem 1, determine m so that, for all values of $n \geqslant m$, all the points $[x, |R_n(x)|]$ are enclosed within an area $(A+\epsilon')L'+(A'-A)\epsilon'$, L' being the length of (C', D'). Adding on to this the area of the two small rectangles, of height A' on (C, C') and (D', D) as bases, we have included all the points $[x, |R_n(x)|]$ within an area less than $(A+\epsilon')L+3A'\epsilon'-5A\epsilon'$.

Choosing therefore $\epsilon' = \dfrac{\epsilon}{L+3A'-5A}$, the corollary is proved.

16. It will now be convenient to express the closed set G_A as the limit of a sequence of closed sets, as Prof. Osgood does in his paper. The sets of the sequence we denote by $G_{A,m}$, and define them in the following way:—

$G_{A,m}$ consists of all the A-points, such that, for all values of $n \geqslant m$, $|R_n(x)| \leqslant A$, and their limiting points.

The limiting points, if any, of $G_{A,m}$ which do not satisfy the above conditions are still A-points, since G_A is a closed set. They must therefore, by the above definition and that of an A-point, be discontinuities of $R_n(x)$ for an infinite series of values of n, and, if σ be the discontinuity of $R_n(x)$ at such a point Q, the value of $|R_n(x)|$ at Q will be less than or equal to $A+\sigma$, since there are certainly points in the neighbourhood below the line $y = A$. Hence:—

Those of the limiting points of the closed set $G_{A,m}$ for which $|R_n(x)| \geqslant A+\sigma$, where σ is any assigned positive quantity, and n any assigned integer $\geqslant m$, belong to a closed set of content zero.

Since at *any* particular point x we can *always* assign an integer m, such that, for all values of $n \geqslant m$, $|R_n(x)| < A$, it is evident that $G_{A,m}$ *contains* $G_{A,m-1}$, *and the limiting set is* G_A.

17. To these closed sets $G_{A,m}$ and their limiting set G_A, we propose to apply the Lemma of Content, proved in a paper published in the Society's *Proceedings* (Vol. xxxv., p. 269). The enunciation of the theorem is as follows :—

If G_1, G_2, ... *be a sequence of closed sets of points nowhere dense, and the limiting set* G *be closed also, and if we assign any small quantity* σ, *we can determine an integer* r *and a small quantity* ϵ, *so that, for all values of* $n \geqslant r$, *all the black intervals* $\geqslant \epsilon$ *of* G_n *are identical with all the black intervals* $\geqslant \epsilon$ *of* G, *and the sum of the remaining black intervals of* $G_n < \sigma$.

18. THEOREM 2.—*If* (C, D) *be any closed interval of length* L, *in which* $|R_n(x)|$ *has a finite upper limit* A', *for all values of* x *from* C *to* D, *both inclusive, and for all values of* n *from and after an assignable integer, we can determine an integer* m, *such that, for all values of* $n \geqslant m$,

$$\int_C^D |R_n(x)|\, dx < AL + \epsilon,$$

where A *and* ϵ *are any assigned small positive quantities.*

If $A' \leqslant A$, the theorem is an immediate consequence of the Lemma of Integrals. Let then $A' > A$.

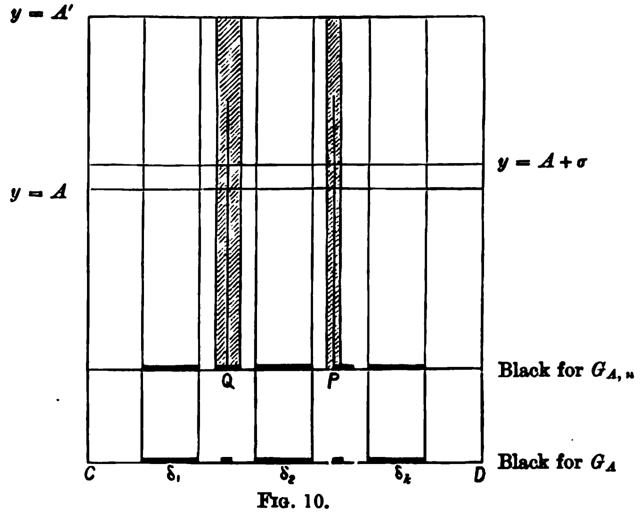

FIG. 10.

Then, by the Lemma of Content, we can determine an integer m' and a small quantity ϵ', so that, for all values of $n \geqslant m'$, all the black intervals $\geqslant \epsilon'$ of $G_{A,n}$ are identical with all the black intervals $\geqslant \epsilon'$ of G_A, and the sum of the remaining black intervals of $G_{A,n} <$ any assigned small σ.

Denoting the black intervals $>\epsilon'$ of G_A by $\delta_1, \delta_2, \dots, \delta_k$, it is an immediate deduction from the corollary to Theorem 1, that we can determine $m \geqslant m'$, so that, for all values of $n \geqslant m$, the part of the integral due to these k intervals will be less than $A(\delta_1+\delta_2+\dots\delta_k)+\sigma$. In the remaining segments the locus $y=|R_n(x)|$ can only rise above any given line $y=A+\sigma$ at (1) a point P belonging to $G_{A,n}$, (2) a point Q in a black interval $<\epsilon'$ of $G_{A,n}$.

By § 16 we know that P must be a discontinuity $>\sigma$ of $R_n(x)$, and belongs therefore to a set of content zero. Hence these points $[x, |R_n(x)|]$ can all be enclosed in small rectangles of height A', the sum of whose breadths is less than σ. The points Q can be enclosed in small rectangles of height A', the sum of whose lengths, being not greater than that of all the black intervals $<\epsilon'$ of $G_{A,n}$, is certainly less than σ.

Thus the part of the integral due to the complementary segments of $\delta_1, \delta_2, \dots, \delta_k$ is less than $(A+\sigma)(L-\delta_1-\delta_2\dots\delta_k)+2\sigma(A'-A)$; hence the whole integral is certainly less than $(A+\sigma)L+\sigma+2\sigma(A'-A)$. Choosing now

$$\sigma = \frac{\epsilon}{L+1+2(A'-A)},$$

the theorem follows.

19. The theorem enunciated in § 13 is an immediate result of Theorem 2. For, since $\int_C^D |R_n(x)|\,dx$ can be made as small as we please, by properly choosing A and ϵ, n being sufficiently large, therefore

$$\operatorname*{Lt}_{n=\infty} \int_C^D |R_n(x)|\,dx = 0;$$

a fortiori,

$$\operatorname*{Lt}_{n=\infty} \int_C^D R_n(x)\,dx = 0,$$

that is, since

$$\operatorname*{Lt}_{n=\infty} R_n(x) = 0,$$

$$\int_C^D \operatorname*{Lt}_{n=\infty} [R_n(x)]\,dx = \operatorname*{Lt}_{n=\infty} \int R_n(x)\,dx,$$

that is,

$$\int_C^D \operatorname*{Lt}_{n=\infty} [F(x)-s_n(x)]\,dx = \operatorname*{Lt}_{n=\infty} \int [F(x)-s_n(x)]\,dx,$$

or, since

$$\operatorname*{Lt}_{n=\infty} s_n(x) = F(x),$$

$$\int_C^D F(x)\,dx = \operatorname*{Lt}_{n=\infty} \int_C^D s_n(x)\,dx,$$

that is, in words: *The integral of the sum is the limit of the sum of the integrals of the separate terms.*

ON THE MINORS OF A SKEW-SYMMETRICAL DETERMINANT

By J. BRILL, M.A.

[Received December 30th, 1902.—Read January 8th, 1903.]

1. The most general type of determinant that can arise as a minor of a skew-symmetrical determinant may be expressed in the form

$$\begin{vmatrix} [1x_1], & [1x_2], & \dots, & [1x_n] \\ [2x_1], & [2x_2], & \dots, & [2x_n] \\ \dots & \dots & \dots & \dots \\ [nx_1], & [nx_2], & \dots, & [nx_n] \end{vmatrix},$$

where we have, generally, $[rx_s]+[x_s r] = 0$.

In the most general type the two sets of numbers 1, 2, 3, ..., n and $x_1, x_2, x_3, \dots, x_n$ are distinct, but all special cases may be deduced by making pairs out of the two sets identical.

Now we have

$$\begin{vmatrix} [1x_1], & [1x_2] \\ [2x_1], & [2x_2] \end{vmatrix} = [1x_1][2x_2]-[1x_2][2x_1] = -[12x_1x_2]+[12][x_1x_2].$$

Further, the determinant

$$\begin{vmatrix} [1x_1], & [1x_2], & [1x_3] \\ [2x_1], & [2x_2], & [2x_3] \\ [3x_1], & [3x_2], & [3x_3] \end{vmatrix}$$

is equal to

$$[1x_1]\{-[23x_2x_3]+[23][x_2x_3]\}-[1x_2]\{-[23x_1x_3]+[23][x_1x_3]\}$$
$$+[1x_3]\{-[23x_1x_2]+[23][x_1x_2]\}$$
$$= -[123x_1x_2x_3]+[12][3x_1x_2x_3]-[13][2x_1x_2x_3]+[23][1x_1x_2x_3].$$

Similarly, the determinant of four rows and columns is equal to

$$\begin{aligned}
&[1x_1]\{-[234x_2x_3x_4]+[23][4x_2x_3x_4]-[24][3x_2x_3x_4]+[34][2x_2x_3x_4]\}\\
&-[1x_2]\{-[234x_1x_3x_4]+[23][4x_1x_3x_4]-[24][3x_1x_3x_4]+[34][2x_1x_3x_4]\}\\
&+[1x_3]\{-[234x_1x_2x_4]+[23][4x_1x_2x_4]-[24][3x_1x_2x_4]+[34][2x_1x_2x_4]\}\\
&-[1x_4]\{-[234x_1x_2x_3]+[23][4x_1x_2x_3]-[24][3x_1x_2x_3]+[34][2x_1x_2x_3]\}\\
&=[1234x_1x_2x_3x_4]-\{[12][34x_1x_2x_3x_4]-[13][24x_1x_2x_3x_4]\\
&\qquad+[14][23x_1x_2x_3x_4]+[23][14x_1x_2x_3x_4]-[24][13x_1x_2x_3x_4]\\
&\qquad+[34][12x_1x_2x_3x_4]\}+[1234][x_1x_2x_3x_4].
\end{aligned}$$

Proceeding in this manner, we obtain, for the determinant of five rows and columns, the expression

$$\begin{aligned}
&[12345x_1x_2x_3x_4x_5]-\{[12][345x_1x_2x_3x_4x_5]-[13][245x_1x_2x_3x_4x_5]+\dots\\
&\qquad\qquad+[45][123x_1x_2x_3x_4x_5]\}\\
&\qquad+\{[1234][5x_1x_2x_3x_4x_5]-[1235][4x_1x_2x_3x_4x_5]+\dots\\
&\qquad\qquad+[2345][1x_1x_2x_3x_4x_5]\}.
\end{aligned}$$

Also, for that of six rows and columns, we should have

$$\begin{aligned}
&-[123456x_1x_2x_3x_4x_5x_6]\\
&+\{[12][3456x_1x_2x_3x_4x_5x_6]-\dots+[56][1234x_1x_2x_3x_4x_5x_6]\}\\
&-\{[1234][56x_1x_2x_3x_4x_5x_6]-\dots+[3456][12x_1x_2x_3x_4x_5x_6]\}\\
&+[123456][x_1x_2x_3x_4x_5x_6].
\end{aligned}$$

We are now in a position to state the general rule deducible from these special cases. For the determinant of n rows and columns we have an expression of the form

$$\begin{aligned}
&\pm\{[123\dots nx_1x_2\dots x_n]\\
&\quad-\Sigma\pm[rs][12\dots(r-1)(r+1)\dots(s-1)(s+1)\dots nx_1x_2\dots x_n]\\
&\quad+\Sigma\pm[pqrs][12\dots(p-1)(p+1)\dots(q-1)(q+1)\dots(r-1)(r+1)\dots\\
&\qquad\dots(s-1)(s+1)\dots nx_1x_2\dots x_n]-\dots\}.^{*} \quad (1)
\end{aligned}$$

---

[*] To obtain the sign attached to any particular product under one of the Σ's, we have the following rule:—Write down the numbers in the first Pfaffian of the product, and after them write down such of the numbers 1, 2, ..., n as occur in the second Pfaffian, preserving in each case the order in which the said numbers stand in the Pfaffians. Then, according as the number of displacements requisite to restore the numbers so obtained to their natural order be odd or even, so is the sign of the product negative or positive.

This expression needs some further explanation. Firstly, with regard to the ambiguous sign outside the double bracket, we have the following rule:—If n be even, the negative or positive sign is to be chosen according as $\frac{1}{2}n$ is odd or even; if n be odd, the negative or positive sign is to be chosen according as $\frac{1}{2}(n-1)$ is odd or even.*

For the typical sum-term within the double bracket, we have

$$(-1)^k \Sigma \pm [p_1 p_2 \ldots p_{2k}]\, C[p_1 p_2 \ldots p_{2k}],$$

where $p_1, p_2, \ldots, p_{2k}$ are a set of $2k$ numbers, standing in their natural order, selected from the set $1, 2, \ldots, n$, and the symbol $C[p_1 p_2 \ldots p_{2k}]$ denotes the Pfaffian whose symbolical expression can be formed from the set of numbers $1, 2, \ldots, n, x_1, x_2, \ldots, x_n$ by leaving out the set $p_1, p_2, \ldots, p_{2k}$.

If n be an even number, the last term within the bracket is

$$(-1)^{\frac{1}{2}n}[123 \ldots n][x_1 x_2 x_3 \ldots x_n].$$

If n be an odd number, then we have, for the last sum-term, the expression

$$(-1)^{\frac{1}{2}(n-1)} \Sigma \pm [123 \ldots (p-1)(p+1) \ldots n][p x_1 x_2 x_3 \ldots x_n].$$

I propose to give a justification of the generality of this result by the method of induction. Thus, supposing the result to be true in the case of the determinant of n rows and columns, we proceed to establish its correctness for the case of the determinant of $n+1$ rows and columns. All the minors of this latter determinant will be expressible in the form (1); and, as the sign outside the double bracket will be the same for all, we may leave this out of account for the present.

Now

$$\begin{aligned}
&[1x_1][234 \ldots (n+1) x_2 x_3 x_4 \ldots x_{n+1}] \\
&-[1x_2][234 \ldots (n+1) x_1 x_3 x_4 \ldots x_{n+1}] \\
&+\ldots \\
&+(-1)^n [1x_{n+1}][234 \ldots (n+1) x_1 x_2 x_3 \ldots x_n] \\
= \; &(-1)^n \{ \quad [123 \ldots (n+1) x_1 x_2 x_3 \ldots x_{n+1}] \\
&\qquad -[12][345 \ldots (n+1) x_1 x_2 x_3 \ldots x_{n+1}] \\
&\qquad +[13][245 \ldots (n+1) x_1 x_2 x_3 \ldots x_{n+1}] \\
&\qquad -\ldots \\
&-(-1)^{n+1} [1(n+1)][234 \ldots n x_1 x_2 x_3 \ldots x_{n+1}] \}. \qquad (2)
\end{aligned}$$

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* The referee points out that this sign may be accounted for by the inclusion of the factor $(-1)^{\frac{1}{2}n(n-1)}$.

This supplies us with the first term of the new expression, and with those portions of the first sum-term for which $r = 1$. Further, considering the typical term out of those composing the first sum-term, we have

$$\begin{aligned}
&[1x_1][234 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_2 x_3 x_4 \ldots x_{n+1}] \\
&-[1x_2][234 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_1 x_3 x_4 \ldots x_{n+1}] \\
&+\ldots \\
&+(-1)^n[1x_{n+1}][234 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_1 x_2 x_3 \ldots x_n] \\
= \quad &(-1)^n \{ \qquad [123 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_1 x_2 x_3 \ldots x_{n+1}] \\
&\qquad -[12][345 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_1 x_2 x_3 \ldots x_{n+1}] \\
&\qquad +[13][245 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_1 x_2 x_3 \ldots x_{n+1}] \\
&\qquad -\ldots \\
&-(-1)^{n-1}[1(n+1)] \\
&\qquad \times [234 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots n\, x_1 x_2 x_3 \ldots x_{n+1}]\}. \quad (3)
\end{aligned}$$

In this expression there will be no terms containing $[1r]$ or $[1s]$. The alternation of the positive and negative signs attached to the terms will, however, be steady throughout the expression.

It will thus be seen that, taking the latter terms on the right-hand side of equation (2), and all those included under the typical first term of the right-hand side of equation (3), we shall have the first sum-term of our new expression complete, viz.,

$$\begin{aligned}
&(-1)^n \{ -\Sigma \pm [rs][123 \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots \\
&\qquad\qquad \ldots (n+1)\, x_1 x_2 x_3 \ldots x_{n+1}]\}.
\end{aligned}$$

The remaining terms on the right-hand side of equation (3) will go towards the formation of the second sum-term. Thus, collecting all the terms containing

$$[234 \ldots (q-1)(q+1) \ldots (r-1)(r+1) \ldots (s-1)(s+1) \ldots (n+1)\, x_1 x_2 x_3 \ldots x_{n+1}],$$

we have, for the coefficient of this expression,

$$\begin{aligned}
&(-1)^{n+1} \{(-1)^{q-1}[1q](-1)^{r+s-5}[rs] + (-1)^{r-2}[1r](-1)^{q+s-5}[qs] \\
&\qquad\qquad + (-1)^{s-3}[1s](-1)^{q+r-5}[qr]\} \\
&\qquad = (-1)^{n+1}(-1)^{q+r+s-6}\{[1q][rs]-[1r][qs]+[1s][qr]\} \\
&\qquad = (-1)^n(-1)^{q+r+s-5}[1qrs] = (-1)^n(-1)^{q+r+s-9}[1qrs].
\end{aligned}$$

Thus these terms give us that portion of our next sum-term for which $p = 1$, viz.,

$$(-1)^n \{ +\Sigma \pm [1qrs][123 \dots (q-1)(q+1) \dots (r-1)(r+1) \dots (s-1)(s+1) \dots \\ \dots (n+1)\, x_1 x_2 x_3 \dots x_{n+1}] \}.$$

Now, dealing with the typical sum-term, and making a slight and obvious modification of our notation, we have, on consideration of the coefficient of $[p_1 p_2 \dots p_{2k}]$,

$$\begin{aligned} &[1x_1]\, C[1p_1 p_2 \dots p_{2k} x_1] - [1x_2]\, C[1p_1 p_2 \dots p_{2k} x_2] + \dots \\ &\qquad + (-1)^n [1x_{n+1}]\, C[1p_1 p_2 \dots p_{2k} x_{n+1}] \\ &= (-1)^n \{ C[p_1 p_2 \dots p_{2k}] - [12]\, C[12p_1 p_2 \dots p_{2k}] \\ &\quad + [13]\, C[13p_1 p_2 \dots p_{2k}] - \dots \\ &\qquad + (-1)^{n-2k} [1(n+1)]\, C[1p_1 p_2 \dots p_{2k}(n+1)] \}. \end{aligned}$$

The contribution of the typical sum-term of expression (1) to the sum-term of the same order in our new expression is thus clear. It only remains to consider its contribution to the sum-term of the next order. Considering the portion containing $C[p_1 p_2 \dots p_{2k+1}]$, we have, for the coefficient of this expression,

$$\begin{aligned} &(-1)^n (-1)^k \{ \quad (-1)^{p_1 - 1} [1p_1] (-1)^{h - p_1} [p_2 p_3 \dots p_{2k+1}] \\ &\qquad + (-1)^{p_2 - 2} [1p_2] (-1)^{h - p_2} [p_1 p_3 p_4 \dots p_{2k+1}] \\ &\qquad + (-1)^{p_3 - 3} [1p_3] (-1)^{h - p_3} [p_1 p_2 p_4 p_5 \dots p_{2k+1}] \\ &\qquad + \dots \\ &\qquad + (-1)^{p_{2k+1} - (2k+1)} [1p_{2k+1}] (-1)^{h - p_{2k+1}} [p_1 p_2 \dots p_{2k}] \} \\ &= (-1)^n (-1)^k (-1)^{h-1} \{ [1p_1][p_2 p_3 \dots p_{2k+1}] - [1p_2][p_1 p_3 p_4 \dots p_{2k+1}] \\ &\qquad\qquad + \dots \\ &\qquad\qquad + [1p_{2k+1}][p_1 p_2 \dots p_{2k}] \} \\ &= (-1)^n (-1)^{k+1} (-1)^{h-2} [1p_1 p_2 \dots p_{2k+1}], \end{aligned}$$

where $$h = p_1 + p_2 + \dots + p_{2k+1} - k(2k+3).$$

Further, $$h - 2 = p_1 + p_2 + \dots + p_{2k+1} - (2k^2 + 3k + 2),$$

and therefore

$$h - 2(k+1) = p_1 + p_2 + \dots + p_{2k+1} - (2k+1)(k+2).$$

Also $$(-1)^{h-2(k+1)} = (-1)^{h-2}(-1)^{2k} = (-1)^{h-2}.$$

Thus our expression becomes

$$(-1)^n(-1)^{k+1}(-1)^{p_1+p_2+\dots+p_{2k+1}-(2k+1)(k+2)}[1p_1p_2\dots p_{2k+1}].$$

Thus we obtain the contribution of these terms to the next sum-term, and observe that it is of the required form.

The only point remaining to be considered is the ambiguous sign outside the double bracket. We have shown that, for the determinant of $n+1$ rows and columns, we have an expression of the proper type affected with the sign $(-1)^n$. Further, if n be an even number, the ambiguous sign outside the double bracket in the expression (1) may be replaced by $(-1)^{\frac{1}{2}n}$. Thus our new expression is affected with the sign

$$(-1)^n(-1)^{\frac{1}{2}n} = (-1)^{\frac{1}{2}n} = (-1)^{\frac{1}{2}(m-1)},$$

where $m = n+1$, since n is even. As m is an odd number, this is of the proper form.

Taking n for an odd number, the above mentioned ambiguous sign may be replaced by $(-1)^{\frac{1}{2}(n-1)}$, and in this case our new expression will be affected with the sign

$$(-1)^n(-1)^{\frac{1}{2}(n-1)} = (-1)^{\frac{1}{2}(n+1)} = (-1)^{\frac{1}{2}m},$$

since n is odd. Further, m now denoting an even number, this also will be of the requisite form.

2. Having established the law for the general case, we proceed to obtain a few verifications, by application to special cases. It is a general rule that, if in the series of numbers involved in the symbolical expression of a Pfaffian any number be repeated, the corresponding expression vanishes. Thus, in some of the special cases our general expression becomes much reduced. Taking, for example, the case of the skew-symmetrical determinant of n rows and columns, we obtain this case by putting $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, ..., $x_n = n$; and we see that, if n be odd, it vanishes entirely. Further, if n be even, it reduces to $(-1)^n\{[123\dots n]\}^2$, that is, to $\{[123\dots n]\}^2$.

We next apply our result to the first minors of a skew-symmetrical determinant of n rows and columns, considering the minor of $[rs]$. In this case we shall have a determinant of $n-1$ rows and columns, obtained from the typical form by assuming $x_1 = 1$, $x_2 = 1$, ..., $x_{s-1} = s-1$,

$x_s = s+1$, $x_{s+1} = s+2$, ..., $x_{n-1} = n$, and by replacing the numbers r, $r+1$, ..., $n-1$ by the respective numbers $r+1$, $r+2$, ..., n.

If n be an even number, the expression for our minor becomes

$$(-1)^{\frac{1}{2}(n-2)}(-1)^{\frac{1}{2}(n-2)}(-1)^{n-s}[123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$\times [s123 \dots (s-1)(s+1) \dots n]$$

or
$$(-1)^{\frac{1}{2}(n-2)}(-1)^{\frac{1}{2}(n-2)}(-1)^{n-s}[123 \dots (s-1)(s+1) \dots (r-1)(r+1) \dots n]$$
$$\times [s123 \dots (s-1)(s+1) \dots n].$$

Now
$$[s123 \dots (s-1)(s+1) \dots n] = (-1)^{s-1}[123 \dots n]$$
and
$$(-1)^{\frac{1}{2}(n-2)}(-1)^{\frac{1}{2}(n-2)}(-1)^{n-s}(-1)^{s-1} = (-1)^{2n-3}.$$

Thus our minor is equivalent to either

$$-[123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n][123 \dots n]$$
or
$$-[123 \dots (s-1)(s+1) \dots (r-1)(r+1) \dots n][123 \dots n].$$

If n be an odd number, we obtain for our minor

$$(-1)^{\frac{1}{2}(n-1)}(-1)^{\frac{1}{2}(n-1)}[123 \dots (r-1)(r+1) \dots n][123 \dots (s-1)(s+1) \dots n]$$
$$= [123 \dots (r-1)(r+1) \dots n][123 \dots (s-1)(s+1) \dots n].$$

As a further illustration we will consider the minor of

$$\begin{vmatrix} [pr], & [ps] \\ [qr], & [qs] \end{vmatrix}.$$

In this case we have a determinant of $n-2$ rows and columns, obtained by putting $x_1 = 1$, $x_2 = 2$, ..., $x_{r-1} = r-1$, $x_r = r+1$, $x_{r+1} = r+2$, ..., $x_{s-2} = s-1$, $x_{s-1} = s+1$, $x_s = s+2$, ..., $x_{n-2} = n$, and by replacing the numbers $p, p+1, \dots, q-2$ by the respective numbers $p+1, p+2, \dots, q-1$, and the numbers $q-1$, q, ..., $n-2$ by the respective numbers $q+1$, $q+2$, ..., n.

For simplicity we will take the case in which p, q, r, s are in order of magnitude. In this case, if n be even, we have

$$(-1)^{\frac{1}{2}(n-2)}\{(-1)^{\frac{1}{2}(n-4)}(-1)^{n-(r+s)+3}$$
$$\times [123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$\times [rs123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$+(-1)^{\frac{1}{2}(n-2)}[123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots n]$$
$$\times [123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]\}$$
$$= -[123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$\times [123 \dots n] + [123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots n]$$
$$\times [123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n].$$

On the other hand, if n be odd, we have

$$\begin{aligned}(-1)^{n-3}\{(-1)^{n-2}&[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots n]\\ &\qquad\times[123\ldots(s-1)(s+1)\ldots n]\\ +(-1)^{n-1}&[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(s-1)(s+1)\ldots n]\\ &\qquad\times[123\ldots(r-1)(r+1)\ldots n]\\ =-&[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots n]\\ &\qquad\times[123\ldots(s-1)(s+1)\ldots n]\\ +&[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(s-1)(s+1)\ldots n]\\ &\qquad\times[123\ldots(r-1)(r+1)\ldots n].\end{aligned}$$

In a similar manner we could determine those cases in which either or both of the numbers p and q are greater than either or both of the numbers r and s.

3. In the thirty-fourth volume of the *Proceedings* of the Society[*] I gave the following theorem :—

$$\begin{aligned}&[12y_1z_1\ldots y_mz_m][345\ldots(2n)y_1z_1\ldots y_mz_m]\\ &-[13y_1z_1\ldots y_mz_m][245\ldots(2n)y_1z_1\ldots y_mz_m]\\ &+\ldots\\ &+(-1)^{2n}[1(2n)y_1z_1\ldots y_mz_m][234\ldots(2n-1)y_1z_1\ldots y_mz_m]\\ =\;&[y_1z_1\ldots y_mz_m][123\ldots(2n)y_1z_1\ldots y_mz_m].\end{aligned}$$

This theorem leads directly to a generalization of a theorem given by Baker in a former volume of the *Proceedings*.[†] Thus, if we expand the Pfaffian $[123\ldots(2n)]$, and replace each element of the type $[rs]$ by an element of the form $[rsy_1z_1\ldots y_mz_m]$, then the value of the resulting expression will be

$$[y_1z_1\ldots y_mz_m]^{n-1}[123\ldots(2n)y_1z_1\ldots y_mz_m].$$

We can also apply our first quoted theorem to our typical determinant, and obtain an expression for the determinant derived from it by replacing each element of the type $[rx_s]$ by an element of the form $[rx_s y_1 z_1 \dots y_m z_m]$. The resulting expression will be of the form

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[*] P. 149. I find that I had been anticipated in some of the theorems of this paper by Vivanti. See *Rendiconti del Circolo Matematico di Palermo*, t. XII., pp. 1–20.

[†] Vol. XXIX., p. 141.

$$\begin{aligned}\pm\Big[&[123 \dots n x_1 x_2 \dots x_n y_1 z_1 \dots y_m z_m]\,\{[y_1 z_1 \dots y_m z_m]\}^{n-1}\\ &+\{[y_1 z_1 \dots y_m z_m]\}^{n-2}\,\{-\Sigma \pm [rs y_1 z_1 \dots y_m z_m]\\ &\times[12 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n x_1 x_2 \dots x_n y_1 z_1 \dots y_m z_m] + \dots\}\Big].\end{aligned}$$

ON AN ARITHMETICAL THEOREM CONNECTED WITH ROOTS OF UNITY, AND ITS APPLICATION TO GROUP-CHARACTERISTICS*

By W. BURNSIDE.

[Communicated February 12th, 1903.—Received February 20th, 1903.]

I.

The arithmetical theorem in question may be stated as follows :—

Let ω be a primitive m-th root of unity, χ a rational integral function of ω with real integers as coefficients, and χ' the conjugate imaginary of χ. Let S ($= \Sigma\chi\chi'$) be formed by taking for ω each of the $\phi(m)$ distinct primitive m-th roots of unity in turn, constructing the product $\chi\chi'$, and adding the $\phi(m)$ products so obtained. Then the absolutely least value of S, other than zero, is $\phi(m)$.

The case in which m is the power of a prime, say p^a, will be first considered.

Since ω is the root of an irreducible equation of degree $\phi(m)$, with unity for the leading coefficient, χ can be expressed as a rational integral function of ω, of degree not exceeding $\phi(m)-1$, with integral coefficients. Hence

$$\chi = a_0+a_1\omega+a_2\omega^2+\ldots+a_{\phi(m)-1}\omega^{\phi(m)-1}.$$

Since χ and χ' are conjugate imaginaries, $\chi\chi'$ must be positive or zero, and S can only be zero when χ is zero. Now

$$\chi\chi' = \sum_{i=0}^{i=\phi(m)-1} a_i^2+\sum_{i,j} a_i a_j(\omega^{i-j}+\omega^{j-i}),$$

and therefore $$S = \Sigma\chi\chi' = s_0 \sum_{i=0}^{i=\phi(m)-1} a_i^2+2\sum_{i,j} s_{i-j}a_i a_j,$$

where $s_0 = \phi(m) =$ sum of the m-th powers of the primitive m-th roots, $s_{i-j} =$ sum of the $(i-j)$-th powers of the primitive m-th roots, and the latter sum in the expression for S is taken for each distinct pair of suffixes i and j.

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* [The proof of the theorem in Section I. originally given was incomplete; it has been replaced by that now printed.—*October* 17*th*, 1903.]

Unless $i-j$ is a multiple of m/p, or p^{a-1}, s_{i-j} is zero; and, when $i-j$ is a multiple of p^{a-1}, s_{i-j} is equal to $-p^{a-1}$, *i.e.*, $-s_0/(p-1)$. Hence the product $a_i a_j$ will occur in S only when i and j differ by a multiple of p^{a-1}. The a's may therefore be divided into $s_0/(p-1)$ sets of $p-1$ each, so that no product term occurs except among the a's belonging to the same set. Any such set will have suffixes

$$i+x\frac{s_0}{p-1} \quad (x=0,\,1,\,\ldots,\,p-2),$$

while the values $i=0,\,1,\,\ldots,\,\frac{s_0}{p-1}-1$ give the different sets.

The part of S which arises from the first set will be

$$S_0=\frac{s_0}{p-1}\Big[(p-1)\sum_x a^2_{xp^{a-1}}-2\sum_{x,\,y} a_{xp^{a-1}}a_{yp^{a-1}}\Big]. \tag{i}$$

From this the part of S which arises from the i-th set will be given by increasing each suffix by $i-1$.

Equation (i) may be written

$$S_0=\frac{s_0}{p-1}\Big[\overset{(p-3)}{}\sum_x a^2_{xp^{a-1}}+\sum_{x,\,y}(a_{xp^{a-1}}-a_{yp^{a-1}})^2\Big].$$

Since each of the $p-1$ a's is an integer or zero, it is quite obvious that the number in the square brackets cannot be less than $p-1$, unless it is zero. Hence S_0, if not zero, is at least equal to s_0. The same is true of the part of S which arises from any other set of a's; and therefore S itself, if not zero, must be equal to or greater than s_0.

It will now be assumed that the theorem has been proved when m has not more than $n-1$ different prime factors. Suppose then that

$$m=p^a q,$$

where q is the product of powers of $n-1$ distinct primes, and let ω be a primitive p^a-th root of unity and ω' a primitive q-th root of unity; so that $\omega\omega'$ is a primitive m-th root of unity. Here χ may be expressed in the form

$$\chi=\psi_0+\omega\psi_1+\omega^2\psi_2+\ldots+\omega^{p^{a-1}(p-1)-1}\,\psi_{p^{a-1}(p-1)-1},$$

where each ψ is a rational function of ω' with integral coefficients. Also

$$\chi\chi'=\sum_i\psi_i\psi_i'+\sum_{i,\,j}(\omega^{i-j}\psi_i\psi_j'+\omega^{j-i}\psi_i'\psi_j).$$

From this expression S is obtained by summing, after replacing ω by each primitive p^a-th root of unity and ω' by each primitive q-th root.

Let Σ denote the partial sum, when ω is replaced by each primitive p^a-th root of unity. Then from the previous case

$$\Sigma = p^{a-1}(p-1)\sum_i \psi_i \psi'_i - p^{a-1}\sum_{i,j}(\psi_i \psi'_j + \psi'_i \psi_j),$$

where in the second term on the right-hand side only those products occur in which i and j differ by a multiple of p^{a-1}. Hence Σ can be expressed as the sum of p^{a-1} parts, of which the leading part is given by

$$\Sigma_0 = p^{a-1}(p-1)\sum_x \psi_{xp^{a-1}}\psi'_{xp^{a-1}} - p^{a-1}\sum_{x,y}(\psi_{xp^{a-1}}\psi'_{yp^{a-1}} - \psi'_{xp^{a-1}}\psi_{yp^{a-1}})$$

$$= p^{a-1}\left[\sum_x \psi_{xp^{a-1}}\psi'_{xp^{a-1}} + \sum_{x,y}(\psi_{xp^{a-1}} - \psi_{yp^{a-1}})(\psi'_{xp^{a-1}} - \psi'_{yp^{a-1}})\right],$$

while the other parts are obtained by increasing each suffix by the same number.

The part of S which arises from Σ_0, say S_0, is obtained by summing when ω' is replaced in turn by each primitive q-th root of unity. For each ψ that occurs, $\Sigma\psi\psi'$, if not zero, is not less than $\phi(q)$. Now, exactly as before, it is clear that, of the $\frac{1}{2}p(p-1)$ products in the square brackets of the last equation, at least $p-1$ must be different from zero, unless all are zero. Hence, if S_0 is not zero, it cannot be less than $p^{a-1}(p-1)\phi(q)$, *i.e.*, than $\phi(m)$. The same is true for the part of S which arises from each part of Σ. Hence S, if greater than zero, is not less than $\phi(m)$; and the theorem is true when m has n distinct prime factors, if it is true when m has $n-1$. The theorem is therefore completely proved.

The quantity S_0 can be represented in a remarkable form, which is perhaps worth reproducing. It is given here for the case

$$m = p^a q^\beta r^\gamma,$$

which immediately suggests the form in the general case. S_0 is a quadratic function of $(p-1)(q-1)(r-1)$ a's. If these are

$$a_{ijk} \quad (1 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant q-1,\ 1 \leqslant k \leqslant r-1),$$

then $S_0/p^{a-1}q^{\beta-1}r^{\gamma-1}$

$$= pqr\sum_{ijk} a_{ijk}^2 - qr\sum_{ik}\Big(\sum_i a_{ijk}\Big)^2 - rp\sum_{ik}\Big(\sum_j a_{ijk}\Big)^2 - pq\sum_{ij}\Big(\sum_k a_{ijk}\Big)^2 + p\sum_i\Big(\sum_{jk} a_{ijk}\Big)^2$$

$$+ q\sum_j\Big(\sum_{ik} a_{ijk}\Big)^2 + r\sum_k\Big(\sum_{ij} a_{ijk}\Big)^2 - \Big(\sum_{ijk} a_{ijk}\Big)^2.$$

II.

Let S be any operation of order m belonging to a group G of finite order N. The $\phi(m)$ powers of S of order m will in general belong to a number of different conjugate sets. If t and no more belong to one set, t must be a factor of $\phi(m)$, and the powers of S will be distributed among $\phi(m)/t$, $= t'$, distinct conjugate sets, each of which will contain t of them, while the number of conjugate operations, h, in each of the sets is the same. Suppose that in an irreducible representation of G the characteristic of S is χ_1. If n is the number of symbols in the representation, χ_1 is the sum of n m-th roots of unity; and, if S^x is conjugate to S, χ_1 is unaltered when ω, the primitive m-th root of unity in terms of which it is expressed, is replaced by ω^x. If, on the other hand, S^y, where y is relatively prime to m, is not conjugate to S, then, when ω is replaced by ω^y, χ_1 becomes the characteristic of the conjugate set to which S^y belongs. Hence, if $\chi_1, \chi_2, \dots, \chi_{t'}$ are the characteristics of the t' conjugate sets to which S and its powers of order m belong, then, when ω is replaced in turn by each of the primitive m-th roots of unity, χ_1 takes each of the values $\chi_1, \chi_2, \dots, \chi_{t'}$ t times. Therefore

$$\chi_1\chi_1'+\chi_2\chi_2'+\dots+\chi_{t'}\chi_{t'}' = \frac{1}{t}\,\Sigma\chi\chi',$$

where $\Sigma\chi\chi'$ is the sum for all primitive m-th roots of unity. But $\Sigma\chi\chi'$ is either zero or not less than $\phi(m) (= tt')$. Hence $h(\chi_1\chi_1'+\chi_2\chi_2'+\dots+\chi_{t'}\chi_{t'}')$. if not zero, is not less than ht'.

Now $\Sigma h\chi\chi' = N$, where the sum is extended to all conjugate sets of G, and $\Sigma h = N$; moreover, for the identical operation $\chi\chi'$ is n^2, Therefore, if $n > 1$, for at least one conjugate set $\chi\chi'$, and therefore χ itself, must be zero. Hence:—

In any irreducible group of linear substitutions of finite order, other than a cyclical group in a single variable, at least one of the characteristics is zero.

To this property of group-characteristics I hope to return on another occasion. It may be here pointed out that, in general, it considerably facilitates the calculation of the characteristics for any given group. As an example the table of characteristics of the simple group of order 504 is given with a short indication of the manner in which it has been arrived at.

The group, which may be represented as a triply-transitive group of degree 9 and class 7, contains the identical operation, represented by S_1;

63 operations of order 2, forming the set S_2; 56 operations of order 3, forming the set S_3; 3·72 operations of order 7, forming the sets S_7', S_7'', S_7'''; 3·56 operations of order 9, forming the sets S_9', S_9'', S_9'''.

For any irreducible representation, χ_2 and χ_3 are rational, as also are $\Sigma\chi_7\chi_7'$, for the three sets of order 7 and $\Sigma\chi_9\chi_9'$ for the three sets of order 9; and, except for the set in which each characteristic is unity, one of these four quantities must vanish. Further $\Sigma h\chi\chi' = 504$ and $\Sigma h\chi = 0$.

If, for instance, χ_7 vanishes, χ_1 must be a multiple of 7. Taking then as a first trial $\chi_1 = 7$,

$$9\chi_2^2+8(\chi_3^2+\Sigma\chi_9\chi_9') = 65, \quad 9\chi_2+8(\chi_3+\Sigma\chi_9) = -1,$$

and the only possible solutions of these, subject to the above conditions, are

$$\chi_2 = -1, \quad \chi_3 = -2, \quad \chi_9 = 1, \quad \text{or} \quad \chi_2 = -1, \quad \chi_3 = 1, \quad \chi_9 = -\beta-\beta^{-1},$$

where β is a primitive ninth root of unity. If α represents a primitive seventh root of the unity, the complete table is that given below; and the check on its accuracy is that it satisfies the complete system of relations[*] that a set of group-characteristics must in any case obey.

	G_1	G_7	G_7'	G_7''	G_7'''	G_8	G_9'	G_9''	G_9'''	h
S_1	1	7	7	7	7	8	9	9	9	1
S_2	1	-1	-1	-1	-1	0	1	1	1	63
S_3	1	-2	1	1	1	-1	0	0	0	56
S_7'	1	0	0	0	0	1	$\alpha+\alpha^{-1}$	$\alpha^2+\alpha^{-2}$	$\alpha^4+\alpha^{-4}$	72
S_7''	1	0	0	0	0	1	$\alpha^2+\alpha^{-2}$	$\alpha^4+\alpha^{-4}$	$\alpha+\alpha^{-1}$	72
S_7'''	1	0	0	0	0	1	$\alpha^4+\alpha^{-4}$	$\alpha+\alpha^{-1}$	$\alpha^2+\alpha^{-2}$	72
S_9'	1	1	$-\beta-\beta^{-1}$	$-\beta^2-\beta^{-2}$	$-\beta^4-\beta^{-4}$	-1	0	0	0	56
S_9''	1	1	$-\beta^2-\beta^{-2}$	$-\beta^4-\beta^{-4}$	$-\beta-\beta^{-1}$	-1	0	0	0	56
S_9'''	1	1	$-\beta^4-\beta^{-4}$	$-\beta-\beta^{-1}$	$-\beta^2-\beta^{-2}$	-1	0	0	0	56

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[*] "Group-Characteristics," *Proc. Lond. Math. Soc.*, Vol. xxxiii., p. 154.

ON THE REPRESENTATION OF A GROUP OF FINITE ORDER AS AN IRREDUCIBLE GROUP OF LINEAR SUBSTITUTIONS AND THE DIRECT ESTABLISHMENT OF THE RELATIONS BETWEEN THE GROUP-CHARACTERISTICS

By W. BURNSIDE.

[Received February 28th, 1903.—Read March 12th, 1903.]

IT is hardly too much to say that the methods by which new ideas are introduced and new results obtained in any branch of mathematical science are in general indirect. In fact the historical development of a subject rarely follows the lines of the logical presentment that can be given when a sufficient number of results and their connexion with each other have become familiar.

A case in point is the theory of group-characteristics and the allied subject of the representation of a group of finite order as a group of linear substitutions. In the remarkable series of memoirs, published in the *Berliner Sitzungsberichte*, in which Herr Frobenius developed this theory, the actual starting point is an algebraical theorem connected with permutable matrices; and the methods made use of in the subsequent discussion turn largely on the properties of matrices. In the paper on "Group-Characteristics"[*] in which I have given an account of Herr Frobenius's theory the methods used, though quite distinct from his, are at least as indirect. They involve the consideration of a continuous group from whose properties those of the group-characteristics are deduced.

The present paper is an attempt to deal with the theory of the representation of a group of finite order as a group of linear substitutions and the properties of the group-characteristics from what may be described as a self-contained point of view. No considerations or ideas are introduced foreign to those involved in the conceptions of an abstract group of finite order and of a group of linear substitutions of finite order. Moreover, the theorems made use of are those which lie at the foundations of the subject. It will be seen that the order in which the various parts of the subject are considered differs materially from that in which they occur in Herr Frobenius's memoirs. First, the complete reducibility (defined below) of a group of linear substitutions of finite order is taken; secondly, the number of distinct irreducible representations; thirdly, the composition of the irreducible representations; and, lastly, the group-characteristics and their properties.

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[*] *Proc. Lond. Math. Soc.*, Vol. XXXIII., pp. 146-162. This paper will be referred to as "G.-C."

1. Let $S_1, S_2, \ldots, S_i, \ldots$ be the operations of an abstract group G, and let $s_1, s_2, \ldots, s_i, \ldots$ be a set of linear substitutions in n variables such that, for every pair of suffixes i and j, if $S_iS = S_k$, then $s_i s_j = s_k$. The set of linear substitutions then form a group g which is isomorphic with G. This group of linear substitutions is said to constitute a *representation* of G, the substitution s_i being the representative of the abstract operation S_i. One and the same group of linear substitutions may thus give two or more representations of a given abstract group.

Let $s'_1, s'_2, \ldots, s'_i, \ldots$ be a second set of linear substitutions, in the same number n of variables, which gives a representation of G, s'_i being the representative of S_i. If it is possible to find a linear substitution t in the n variables, such that, for each suffix i, $t^{-1}s_i t = s'_i$, the two representations of G given by the first and second sets of linear substitutions are said to be *equivalent*. If, on the other hand, no such substitution as t exists, the representations are called *distinct*.

A group of linear substitutions in n variables is called *reducible* when it is possible to find n' $(< n)$ linear functions of the variables which are transformed among themselves by every substitution of the group. I propose to call a group of linear substitutions *completely reducible* when it is possible to choose the variables in such a way that they fall into sets with the properties (i) that the variables of each set are transformed among themselves by every substitution of the group; and (ii) that the group of linear substitutions in each separate set is irreducible.

2. In a memoir to appear shortly in the *Acta Mathematica* I have dealt with the questions of the reducibility of a group of linear substitutions of finite order, and of the number of distinct representations of a group of finite order as an irreducible group of linear substitutions. The discussion is based solely (i) on the possibility of representing any group of finite order as a regular group of permutations, and (ii) on the existence of at least one invariant Hermitian form for any group of linear substitutions of finite order.* These two theorems are fundamental in connexion with groups of finite order, and the results arrived at in the memoir referred to are deduced from them without introducing any extraneous considerations, and in particular without any reference to group-characteristics.

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* Loewy, *Comptes Rendus*, Vol. CXXIII., pp. 168–171 (1896); and Moore, *Math. Ann.*, Vol. L., pp. 213–219 (1898).

These results are (i) that every group of linear substitutions of finite order is completely reducible,[*] and (ii) that the number of distinct representations of a group of finite order as an irreducible group of linear substitutions is equal to the number of sets of conjugate operations contained in the group.

I propose here, starting from these results, as acquired by direct and comparatively simple reasoning, to deduce by a further direct process the properties of the group-characteristics.

3. Let G be any abstract group of finite order n with r sets of conjugate operations, and denote by $g_1, g_2, \ldots, g_r$ the r distinct representations of G as an irreducible group of linear substitutions. One of these, which will always be denoted by g_1, necessarily operates on a single variable and consists of the identical operation only. If g is any representation of G as a group of linear substitutions, the complete reducibility of g will be represented by the equation

$$g = \sum_1^r a_i g_i,$$

where the a's are positive integers or zeros; and the equation expresses that the variables in g may be so chosen that they fall into sets; there being, for each i, a_i sets each of which undergoes an irreducible group of linear substitutions equivalent to g_i.

If in each of the substitutions $s_1, s_2, \ldots, s_i, \ldots$ of g_i every coefficient is replaced by its conjugate imaginary, a new set of substitutions $s'_1, s'_2, \ldots, s'_i, \ldots$ arises for which $s'_i s' = s'_k$ if $S_i S_j = S_k$. This new set of substitutions therefore constitute a group, which is obviously irreducible, and gives rise to an irreducible representation of G, which must be included among $g_1, g_2, \ldots, g_r$. If this representation is equivalent to g_i, then g_i is called *self-inverse*. If it is distinct, it may be denoted by $g_{i'}$, and then g_i and $\breve{g}_i$ are called *inverse* representations.

Since g_i is a group of finite order, any one of its substitutions s_j is of finite order. Now a linear substitution of finite order can always be transformed so that, if m is the number of variables operated on, it takes the form[†]

$$x'_1 = \omega_1 x_1, \quad x'_2 = \omega_2 x_2, \quad \ldots, \quad x'_m = \omega_m x_m,$$

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[*] Independently of the investigations of Herr Frobenius, this result has been directly proved by Maschke, *Math. Ann.*, Vol. LII., pp. 363-368 (1899); and more recently, as the particular case of a more general theorem, by Loewy, *American Transactions*, Vol. IV., pp. 44-64 (1903).

[†] Many proofs have been given of this theorem; but, in the present connexion, I may refer to one I have given (*Messenger of Mathematics*, Vol. XXVIII., pp. 111-114) which involves no reference to the characteristic determinant, and depends directly and solely on s being of finite order.

where $\omega_1, \omega_2, \ldots, \omega_m$ are roots of unity. The ω's are called the *multipliers* of the substitution, and their sum $\omega_1+\omega_2+\ldots+\omega_m$ is defined to be the *characteristic* of S_j in the irreducible representation g_i. Any other substitution s_k of g_i may similarly be transformed to the form

$$x_1' = \varpi_1 x_1, \quad x_2' = \varpi_2 x_2, \quad \ldots, \quad x_m' = \varpi_m x_m.$$

If S_j and S_k are conjugate operations of G, then s and s_k are conjugate substitutions of g_i; and therefore

$$x_1' = \omega_1 x_1, \quad x_2' = \omega_2 x_2, \quad \ldots, \quad x_m' = \omega_m x_m$$

can be transformed into

$$x_1' = \varpi_1 x_1, \quad x_2' = \varpi_2 x_2, \quad \ldots, \quad x_m' = \varpi_m x_m.$$

It may be immediately verified that this is only possible when the two sets of roots of unity $\omega_1, \omega_2, \ldots, \omega_m$, and $\varpi_1, \varpi_2, \ldots, \varpi_m$ differ only in the sequence in which they are written. Hence two conjugate substitutions of g_i have the same multipliers; and the characteristics of two conjugate operations of G in g_i are equal to each other.

To the r conjugate sets of operations in G there will therefore correspond r characteristics in g_i which may be denoted by $\chi_1^i, \chi_2^i, \ldots, \chi_j^i, \ldots, \chi_r^i$. Taking, as will be done throughout, the first conjugate set to be that which consists of the identical operation only, χ_1^i is simply the number of variables operated on by g_i, and, if p is the order of any operation of the j-th set, χ_j^i is the sum of χ_1^i p-th roots of unity. To each irreducible representation of G there belongs such a set of characteristics, the characteristics in the set belonging to g_1 each being unity. Further, it follows at once from the definition of inverse representations that, if g_i and $g_{i'}$ are inverse representations, then χ_j^i and $\chi_j^{i'}$ are conjugate imaginaries for each j; while, if g_i is a self-inverse representation, χ_j^i is real for each j. So also, if the j-th and the j'-th are inverse conjugate sets (*i.e.*, if S belongs to one when S^{-1} belongs to the other), then χ_j^i and $\chi_{j'}^i$ are conjugate imaginaries (or are real and equal); while, if the j-th is a self-inverse set, χ_j^i is real for each i.

If the two irreducible representations g_i and g_j are set up in two distinct sets of χ_1^i and χ_1^j variables, the $\chi_1^i\chi_1^j$ products of the variables undergo a linear substitution for every operation of G. This group may be represented by $g_i g_j$, and the equation which expresses its complete reduction is

$$g_i g_j = \sum_1^r d_{ijk} g_k.$$

This equation implies ("G.-C.," p. 161) that for the characteristics of each conjugate set the equation

$$\chi_s^i \chi_s^j = \sum_1^r d_{ijk} \chi_s^k$$

holds.

If d_{ij1} is not zero, the above equation implies that there are d_{ij1} invariant bilinear functions of the two sets of χ_1^i and χ_1^j variables. Denote these variables by $x_1, x_2, \ldots, x_{\chi_1^i}$ and $y_1, y_2, \ldots, y_{\chi_1^j}$, and let χ_1^i be not greater than χ_1^j. Any bilinear function of the x's and y's can then be expressed in the form $x_1 Y_1 + x_2 Y_2 + \ldots + x_{\chi_1^i} Y_{\chi_1^i}$. If this is invariant when the x's and y's undergo simultaneously the corresponding substitutions of g_i and g_j, then $Y_1, Y_2, \ldots, Y_{\chi_1^i}$ must be transformed among themselves by every operation of g_j. Since g_j is irreducible, this is only possible if $\chi_1^j = \chi_1^i$; and when this condition is satisfied there can only be one such invariant function. In fact, if both

$$x_1 y_1 + x_2 y_2 + \ldots + x_{\chi_1^i} y_{\chi_1^i} \quad \text{and} \quad x_1 Y_1 + x_2 Y_2 + \ldots + x_{\chi_1^i} Y_{\chi_1^i}$$

were invariant, then $y_1' = Y_1,\ y_2' = Y_2,\ \ldots,\ y'_{\chi_1^i} = Y_{\chi_1^i}$ would be permutable with every operation of g_j, which is impossible since g_j is irreducible. When g_i is assigned the corresponding substitutions of g_j are completely defined by the condition that $x_1 y_1 + x_2 y_2 + \ldots + x_{\chi_1^i} y_{\chi_1^i}$ shall be invariant; so that corresponding to a given i there is just one j such that $d_{ij1} = 1$.

On the other hand, the theorem of the existence of an invariant Hermitian form for g_i is equivalent to the assertion that $g_i g_{i'}$ has an invariant bilinear form. Hence

$$d_{ii'1} = 1, \quad d_{ij1} = 0 \quad (j \neq i').$$

4. Let now

$$x_s = \sum_t a_{stk_a} x_t \quad (s, t = 1, 2, \ldots, \chi_1^i)$$

be the representative in g_i of one of the operations of G belonging to the j-th conjugate set, the h_j different operations of the set being given by the values $1, 2, \ldots, h_j$ of the suffix a. When transformed by any substitution of g_i, these h_j substitutions are permuted among themselves. Hence the linear substitution

$$x_s' = \sum_t x_t \sum_a a_{stk_a} \quad (s, t = 1, 2, \ldots, \chi_1^i)$$

is permutable with every substitution of g_i. But, since g_i is irreducible, the only substitutions permutable with every substitution of g_i are those which multiply each variable by the same number. Hence

$$\sum_a a_{stk_a} = 0 \ (s \neq t) \quad \text{and} \quad \sum_a a_{ssk_a} = \lambda.$$

But $$\sum_s a_{ssk_a} = \chi_j^i,$$

and therefore $$\lambda = \frac{h_j\chi_j^i}{\chi_1^i}.$$

Hence, if C_j is the sum of the operations of G which belong to the j-th conjugate set, then, in the representation of G given by g_i, C_j is represented by the substitution

$$x_s' = \frac{h_j\chi_j^i}{\chi_1^i}x_s \quad (s = 1, 2, \ldots, \chi_1^i).$$

Now the sums of the operations of G which belong to the different conjugate sets satisfy the equations ("G.-C.," p. 148)

$$C_j C_k = \sum_s c_{jks} C_s,$$

where c_{jks} are positive constants (or zeros) which can be calculated from the multiplication table of the group. These relations, being in fact identities, must hold between the expressions which correspond to the C's in any concrete representation. But in the concrete representation here considered $C_j C_k$ is the substitution which multiplies each variable by $\frac{h_j h_k \chi_j^i \chi_k^i}{(\chi_1^i)^2}$; while $\sum_s c_{jks} C_s$ is the substitution which multiplies each variable by $\sum_s c_{jks}\frac{h_s\chi_s^i}{\chi_1^i}$. Hence, for all values of i, j, k,

$$h_j h_k \chi_j^i \chi_k^i = \chi_1^i \sum_s c_{jks} h_s \chi_s^i. \tag{i}$$

From this system of relations, and those given by $d_{ii'1} = 1$, $d_{ij1} = 0$, the complete system may be deduced algebraically.

If each side of (i) is multiplied by $\chi_{k'}^l$, while $c_{jks}h_s$ is replaced by $c_{js'k'}h_{k'}$ ("G.-C.," p. 149), it becomes

$$h_j\chi_j^i h_k \chi_k^i \chi_{k'}^l = \chi_1^i \sum_s \chi_s^i c_{js'k'} h_{k'} \chi_{k'}^l.$$

Summing each side with respect to k,

$$h_j\chi_j^i \sum_k h_k \chi_k^i \chi_{k'}^l = \chi_1^i \sum_s \chi_s^i \frac{h_j h_{s'} \chi_j^l \chi_{s'}^l}{\chi_1^l}.$$

Hence $$(\chi_1^l\chi_j^i - \chi_1^i\chi_j^l)\sum_k h_k \chi_k^i \chi_{k'}^l = 0$$

for all values of i, l, j.

In this put $l = 1$. Then, since χ_1^i and χ_j^i cannot be equal for all values of j, $$\sum_k h_k \chi_k^i = 0$$

for all values of i except unity.

Now in the equation $\chi_k^i \chi_k^{i'} = 1 + \sum_s{}' d_{ii's} \chi_k^s$,

Σ' extending to all values of s except unity, multiply each side by h_k, and sum with respect to k. Then

$$\sum_k h_k \chi_k^i \chi_k^{i'} = \sum_k h_k + \sum_s{}' d_{ii's} \sum_k h_k \chi_k^s = n.$$

Similarly, from $\chi_k^i \chi_k^j = \sum_s{}' d_{ijs} \chi_k^s, \quad (j \neq i')$,

it follows that $\sum_k h_k \chi_k^i \chi_k^j = 0 \quad (j \neq i')$.

The two sets of relations

$$\sum_k h_k \chi_k^i \chi_k^{i'} = n, \tag{ii}$$

$$\sum_k h_k \chi_k^i \chi_k^j = 0 \quad (j \neq i') \tag{iii}$$

give the ratio of the first minors in the determinant of the χ's; and thence the two remaining sets of relations

$$\sum_i \chi_k^i \chi_{k'}^i = \frac{n}{h_k}, \tag{iv}$$

$$\sum_i \chi_k^i \chi_l^i = 0 \quad (l \neq k') \tag{v}$$

immediately follow. The relations (i), (ii), (iii), (iv), and (v) are, in fact, the relations which connect the group-characteristics ("G.-C.," pp. 151, 153).

ON THE CONVERGENCE OF CERTAIN MULTIPLE SERIES

By G. H. HARDY.

[Received February 27th, 1903.—Read March 12th, 1903.—Revised April, 1903.]

1. The most important of the few known tests for the conditional convergence of simple series are derived from an elementary theorem generally known as Abel's lemma. In this paper I propose to extend this theorem in such a way as to derive similar tests for the conditional convergence of multiple series. So far as I am aware, no one has yet proved the convergence of any general class of multiple series whose terms are not all positive, though the general theory of such series has been worked out in considerable detail by Pringsheim.*

2. Abel's lemma may be written in the form

$$\sum_{i=1}^{p} a_i u_i = \sum_{i=1}^{p-1} (a_i - a_{i+1}) \sum_{k=1}^{i} u_k + a_p \sum_{k=1}^{p} u_k.$$

This is an almost obvious identity. Hence

$$\sum_{i=1}^{p} \sum_{j=1}^{q} a_{i,j} u_{i,j} = \sum_{j=1}^{q} \left[\sum_{i=1}^{p-1} \beta_{i,j} v_{i,j} + a_{p,j} v_{p,j} \right], \tag{1}$$

where

$$\beta_{i,j} = a_{i,j} - a_{i+1,j},$$

$$v_{i,j} = \sum_{k=1}^{i} u_{k,j}.$$

But

$$\sum_{j=1}^{q} \beta_{i,j} v_{i,j} = \sum_{j=1}^{q-1} (\beta_{i,j} - \beta_{i,j+1}) \sum_{l=1}^{j} v_{i,l} + \beta_{i,q} \sum_{l=1}^{q} v_{i,l} \tag{2}$$

and

$$\sum_{j=1}^{q} a_{p,j} v_{p,j} = \sum_{j=1}^{q-1} \gamma_{p,j} \sum_{l=1}^{j} v_{p,l} + a_{p,q} \sum_{l=1}^{q} v_{p,l}, \tag{3}$$

where

$$\gamma_{i,j} = a_{i,j} - a_{i,j+1}.$$

It is convenient to put $\Delta = a_{p,q}$,

$$\Delta_i = \beta_{i,q} = a_{i,q} - a_{i+1,q}, \qquad \Delta_j = \gamma_{p,j} = a_{p,j} - a_{p,j+1},$$

$$\Delta_{i,j} = \beta_{i,j} - \beta_{i,j+1} = \gamma_{i,j} - \gamma_{i+1,j} = a_{i,j} - a_{i+1,j} - a_{i,j+1} + a_{i+1,j+1}.$$

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* *Sitzungsberichte d. Ak. d. Wiss. zu München*, Vol. XXVII. See also later papers by Pringsheim and London in the *Math. Annalen*.

Then, from (1), (2), and (3),

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_{i,j}u_{i,j} = \sum_{i=1}^{p-1}\sum_{j=1}^{q-1}\Delta_{i,j}\sum_{k=1}^{i}\sum_{l=1}^{j} u_{k,l}+\sum_{i=1}^{p-1}\Delta_i\sum_{k=1}^{i}\sum_{l=1}^{q} u_{k,l}$$
$$+\sum_{j=1}^{q-1}\Delta_j\sum_{k=1}^{p}\sum_{l=1}^{j} u_{k,l}+\Delta\sum_{k=1}^{p}\sum_{l=1}^{q} u_{k,l}. \qquad \text{(A)}$$

3. The corresponding equation for n-ple series may be written in the form

$$\sum_{i_1=1}^{p_1}\sum_{i_2=1}^{p_2}\dots\sum_{i_n=1}^{p_n} a_{i_1, i_2, \dots, i_n} u_{i_1, i_2, \dots, i_n} = \Sigma[\Sigma\Delta(\Sigma\Sigma\dots\Sigma u_{k_1, k_2, \dots, k_n})]. \qquad \text{(A}')$$

To form the right-hand side we proceed as follows. We take any selection of the suffixes $i_1, i_2, \dots, i_n$ as a suffix for Δ. If i_1 does not occur in this selection, we put $i_1 = p_1$ in $a_{i_1, \dots, i_n}$. If it does, we substitute

$$a_{i_1, i_2, \dots, i_n} - a_{i_1+1, i_2, \dots, i_n} \quad \text{for} \quad a_{i_1, i_2, \dots, i_n}.$$

We repeat this for each suffix, and the result is the corresponding Δ; thus, *e.g.*,

$$\Delta_{i_{n-1}, i_n} = a_{p_1, p_2, \dots, i_{n-1}, i_n} - a_{p_1, p_2, \dots, i_{n-1}+1, i_n} - a_{p_1, p_2, \dots, i_{n-1}, i_n+1}$$
$$- a_{p_1, p_2, \dots, i_{n-1}+1, i_n+1}.$$

In the summation in round brackets the limits for k_ν are 1 and i_ν if i_ν is a suffix of Δ; 1 and p_ν otherwise. The summation within the square brackets applies to every i_ν which is a suffix of Δ, and the limits are 1 and $p_\nu - 1$. The outside summation applies to all selections of the suffixes, including that in which no suffix is selected. In this case $\Delta = a_{p_1, p_2, \dots, p_\nu}$.

It is easy to prove (2) by induction. We assume it for n indices, and suppose each a and u affected with a new suffix i_{n+1}. We then sum from $i_{n+1} = 1$ to $i_{n+1} = p_{n+1}$, and apply Abel's lemma to each of the terms on the right. Then it is almost obvious that we obtain

$$\Sigma[\Sigma\Delta(\Sigma\Sigma\dots\Sigma u_{k_1, k_2, \dots, k_n, k_{n+1}})]. \qquad \text{(A}'')$$

For the term of (A′) whose characteristic suffixes are $i_\theta, i_\phi, \dots, i_\omega$ gives the two terms of (A″) whose characteristic suffixes are $i_\theta, i_\phi, \dots, i_\omega, i_{n+1}$ and $i_\theta, i_\phi, \dots, i_\omega$ respectively.

The theorem expressed by the equation (A′) is therefore true generally.

4. Now suppose that $\Delta'_{i_\theta, i_\phi, \dots, i_\omega}$ is the quantity formed in the same way as $\Delta_{i_\theta, i_\phi, \dots, i_\omega}$, except that the suffixes i_κ which are not suffixes of Δ'

are *not* put equal to p_κ. And suppose that the quantities a, u satisfy the following conditions:—

(i.) All the quantities Δ' are positive;

(ii.) $\lim\limits_{i_r=\infty} a_{i_1,\ldots,i_r,\ldots,i_n} = 0$ $(i = 1, 2, \ldots, n)$ *uniformly* for all values of $i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_n$;

(iii.) $\left|\sum\limits_1^{i_1}\sum\limits_1^{i_2}\ldots\sum\limits_1^{i_n} u_{k_1,k_2,\ldots,k_n}\right|$ is less than a constant C for all values of $i_1, i_2, \ldots, i_n$.

Then *the series* $\sum\limits_1^{\infty}\sum\limits_1^{\infty}\ldots\sum\limits_1^{\infty} a_{i_1,i_2,\ldots,i_n} u_{i_1,i_2,\ldots,i_n}$ *is convergent.*

It is evident that (ii.) implies that the s-ple limit obtained by keeping any $n-s$ suffixes constant and making the remaining s tend simultaneously to infinity is zero.

5. To prove the convergence of the series we have to show that however small be σ we can so choose M that

$$\left|\sum_{i_1=1}^{m_1+p_1}\sum_{i_2=1}^{m_1+p_2}\ldots\sum_{i_n=1}^{m_n+p_n} a_{i_1,i_2,\ldots,i_n} u_{i_1,i_2,\ldots,i_n} - \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\ldots\sum_{i_n=1}^{m_n} a_{i_1,i_2,\ldots,i_n} u_{i_1,i_2,\ldots,i_n}\right| < \sigma \qquad (4)$$

for any values of $m_1, m_2, \ldots, m_n$ all $> M$ and all positive values of $p_1, p_2, \ldots, p_n$.

Now let us take any selection of the i's (including at least one) and form the sum $\Sigma\Sigma\ldots\Sigma a_{i_1,i_2,\ldots,i_n} u_{i_1,i_2,\ldots,i_n}$, in which the limits for i_ν are $m_\nu+1$ to $m_\nu+p_\nu$ if i_ν is selected, 1 to m_ν if not. We can form 2^n-1 such sums, and their sum is the quantity whose modulus figures in (4).

Consider, for example, the sum for which the selected i's are $i_1, i_2, \ldots, i_\mu$, and put

$$j_\nu = i_\nu - m_\nu \quad (\nu = 1, 2, \ldots, \mu),$$

$$j_\nu = i_\nu \quad (\nu = \mu+1, \ldots, n),$$

$$q_\nu = p_\nu \quad (\nu = 1, 2, \ldots, \mu),$$

$$q_\nu = m_\nu \quad (\nu = \mu+1, \ldots, n),$$

$$a'_{j_1,j_2,\ldots,j_n} = a_{j_1+m_1,j_2+m_2,\ldots,j_\mu+m_\mu,\ldots,j_n},$$

and similarly for the u's. Then the sum is

$$\sum_{j_1=1}^{q_1} \sum_{j_2=1}^{q_2} \dots \sum_{j_n=1}^{q_n} a'_{j_1, j_2, \dots, j_n} u'_{j_1, j_2, \dots, j_n},$$

which, by (A′), $$= \Sigma[\Sigma\Delta(\Sigma\Sigma \dots \Sigma u'_{k_1, k_2, \dots, k_n})],$$

the Δ's being now formed from a' instead of a. The modulus of this $< C\Sigma[\Sigma\Delta]$. But to find $\Sigma[\Sigma\Delta]$ we have only to suppose that $u = 1$ if all its suffixes are $= 1$, and $= 0$ otherwise. This gives

$$a'_{1, 1, \dots, 1} = \Sigma[\Sigma\Delta].$$

Thus the modulus of our sum is $< Ca'_{1, 1, \dots, 1}$, *i.e.*, $< Ca_{m_1+1, \dots, m_p+1, 1, \dots, 1}$, and can therefore be made $< \sigma/2^n$ by choice of M.

Exactly the same argument applies to the other 2^n-2 partial sums; and (4) follows. Therefore the series is convergent.

6. The most interesting case is that in which

$$a_{i_1, i_2, \dots, i_n} = \phi\left(\sum_{\nu=1}^{n} a_\nu i_\nu\right),$$

where $a_1, \dots, a_n$ are positive, and $\phi(u)$ is a function which has 0 as its limit for $u = \infty$ and has continuous derivates $\phi'(u)$, $\phi''(u)$, …, $\phi^{(n)}(u)$, such that $\phi'(u) < 0$, $\phi''(u) > 0$, $\phi'''(u) < 0$, …, and

$$u_{i_1, i_2, \dots, i_n} = \exp\left\{\left(\sum_{\nu=1}^{n} i_\nu \theta_\nu\right)\sqrt{(-1)}\right\},$$

where $\theta_1, \dots, \theta_n$ are any real quantities other than multiples of 2π. Then

$$(-)^s \Delta'_{i_1, i_2, \dots, i_s} = a_1 a_2 \dots a_s \int_{i_1}^{i_1+1} \int_{i_2}^{i_2+1} \dots \int_{i_s}^{i_s+1} \phi^{(s)}(a_1 x_1 + \dots + a_s x_s$$
$$+ a_{s+1} i_{s+1} + \dots + a_n i_n)\, dx_1\, dx_2 \dots dx_s,$$

which has the sign of $(-)^s$, so that $\Delta'_{i_1, i_2, \dots, i_s} > 0$, and similarly for each Δ'. Thus (i.) of § 4 is satisfied. Evidently (ii.) is satisfied. Finally,

$$\left| \sum_{1}^{i_1} \sum_{1}^{i_2} \dots \sum_{1}^{i_n} \exp\{(i_1\theta_1 + \dots + i_n\theta_n)\sqrt{(-1)}\} \right|$$
$$= \left| \Pi \frac{e^{\theta_\nu \sqrt{(-1)}} \{1 - e^{i_\nu \theta_\nu \sqrt{(-1)}}\}}{1 - e^{\theta_\nu \sqrt{(-1)}}} \right| \leqslant \operatorname{cosec} \tfrac{1}{2}\theta_1 \operatorname{cosec} \tfrac{1}{2}\theta_2 \dots \operatorname{cosec} \tfrac{1}{2}\theta_\nu;$$

so that (iii.) is satisfied.

In particular we may suppose

$$\phi(u) = \frac{1}{u^\rho} \quad (\rho > 0).$$

Hence *the series*

$$\sum_1^\infty \sum_1^\infty \dots \sum_1^\infty \frac{\overset{\cos}{\sin}(i_1\theta_1 + i_2\theta_2 + \dots + i_n\theta_n)}{(a_1 i_1 + a_2 i_2 + \dots + a_n i_n)^\rho}$$

are convergent if $\rho > 0$.

It was with the object of proving the convergence of these series, which form the natural generalization of some of the simplest single series considered in the books, that I undertook the preceding investigation.

[*Note added October 4th,* 1903.—A very interesting question is whether the multiple series written above is also convergent for all complex values of ρ whose real part is positive. The argument of §§ 4–6 fails when ρ is complex, but it can be proved directly from (A′) that the series is convergent if the right values of the complex powers are taken. A further question is how far the restriction that the a's are to be real and positive is necessary.

These questions, however, belong rather to the theory of zeta and allied functions than to the elementary theory of series, and would probably be answered most easily by totally different methods depending on Cauchy's theorem.]

THE INFLEXION CONIC OF A TRINODAL QUARTIC CURVE

By H. W. RICHMOND *and* T. STUART.

[Received March 13th, 1903.]

1. By means of the parameter equations of a trinodal quartic curve Brill* has shown that its six points of inflexion lie on a conic. In the following simple proof the equation of the conic is obtained in one of the systems of coordinates used by Salmon.† Suppose that on the trinodal quartic curve

$$\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}}=0, \qquad a\alpha+b\beta+c\gamma+d\delta=0 \tag{1}$$

the point $(\alpha, \beta, \gamma, \delta)$ is an inflexion, and $(\alpha', \beta', \gamma', \delta')$ any point on the tangent there; the equation

$$(\alpha+\lambda\alpha')^{\frac{1}{2}}+(\beta+\lambda\beta')^{\frac{1}{2}}+(\gamma+\lambda\gamma')^{\frac{1}{2}}+(\delta+\lambda\delta')^{\frac{1}{2}}=0,$$

which when rationalized is of the fourth degree in λ, will have three zero roots. Therefore, by differentiation,

$$\alpha^{-\frac{1}{2}}\alpha'+\beta^{-\frac{1}{2}}\beta'+\ldots=0, \qquad \alpha^{-\frac{3}{2}}\alpha'^2+\beta^{-\frac{3}{2}}\beta'^2+\ldots=0,$$

also $$a\alpha'+b\beta'+c\gamma'+d\delta'=0.$$

All values of $\alpha', \beta', \gamma', \delta'$ consistent with the first and third of these conditions must satisfy the second condition; hence‡

$$\alpha^{\frac{3}{2}}(a+\mu\alpha^{-\frac{1}{2}})^2+\beta^{\frac{3}{2}}(b+\mu\beta^{-\frac{1}{2}})^2+\ldots$$

must vanish for all values of μ; so that at a point of inflexion, in addition to equations (1), we have

$$a^2\alpha^{\frac{3}{2}}+b^2\beta^{\frac{3}{2}}+c^2\gamma^{\frac{3}{2}}+d^2\delta^{\frac{3}{2}}=0. \tag{2}$$

Now consider the product

$$F(z)\equiv(z-a\alpha^{\frac{1}{2}})(z-b\beta^{\frac{1}{2}})(z-c\gamma^{\frac{1}{2}})(z-d\delta^{\frac{1}{2}})\equiv z^4+pz^3+qz^2+rz+s.$$

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* *Mathematische Annalen*, Bde. XII., XIII.; see also Stahl, *Crelle*, Vol. CI., and Meyer, *Apolarität*, p. 283. Stahl mentions also a dissertation by Grassmann, Berlin, 1875.

† *Higher Plane Curves*, pp. 257, 254.

‡ The conditions are the same as those which in geometry of three dimensions express that a straight line lies wholly upon a quadric, viz., that every plane containing the line touches the quadric.

It is clear that $s = abcd\,\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}}$, and also that

$$a^3\alpha^2+b^3\beta^2+c^3\gamma^2+d^3\delta^2+p\,(a^2\alpha^{\frac{3}{2}}+b^2\beta^{\frac{3}{2}}+\ldots)$$
$$+q\,(a\alpha+b\beta+\ldots)+r\,(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\ldots)+s\,(a^{-1}+b^{-1}+c^{-1}+d^{-1})$$
$$= a^{-1}F(a\alpha^{\frac{1}{2}})+b^{-1}F(b\beta^{\frac{1}{2}})+c^{-1}F(c\gamma^{\frac{1}{2}})+d^{-1}F(d\delta^{\frac{1}{2}}) = 0. \qquad (3)$$

Therefore at the inflexions of the curve (1) we have, from (1), (2), and (3),

$$a^3\alpha^2+b^3\beta^2+c^3\gamma^2+d^3\delta^2+(bcd+cda+dab+abc)\,\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}} = 0.$$

Also, as in Salmon, we deduce from $\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}} = 0$ that

$$\alpha^2+\beta^2+\ldots-2\alpha\beta-\ldots = -\,8\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}}.$$

Hence the six inflexions of the trinodal quartic (1) lie on

$$8\,(a^3\alpha^2+b^3\beta^2+c^3\gamma^2+d^3\delta^2) = (bcd+cda+dab+abc)\,(\alpha^2+\beta^2+\ldots-2\alpha\beta-\ldots),$$

a conic.

2. We can also prove the theorem and find the equation of the inflexion conic, when the equation of the curve, referred to the nodal triangle, is given.*

If the fixed conic $(a, b, c, f, g, h\between X, Y, Z)^2 = 0$ is osculated by the variable conic $\lambda YZ+\mu ZX+\nu XY = 0$, the six points of osculation lie on the cubic

$$\frac{aX}{aX+hY+gZ}+\frac{bY}{hX+bY+fZ}+\frac{cZ}{gX+fY+cZ} = 0.$$

This cubic does not pass through the vertices of the triangle of reference, but the cubic

$$aX(hX+bY+fZ)(gX+fY+cZ)+\ldots$$
$$-(LX+MY+NZ)(aX^2+bY^2+\ldots+2fYZ+\ldots),$$

which also passes through the six inflexions, will do so, provided $L = gh$, $M = hf$, $N = fg$, and its equation is then

$$AYZ(hY+gZ)+BZX(fZ+hX)+CXY(gX+fY)+DXYZ = 0,$$

where A, B, C denote the corresponding minors of

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \qquad \text{and} \qquad D \equiv \tfrac{1}{2}(af^2+bg^2+ch^2+3abc-6fgh).$$

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[*] Salmon, *Higher Plane Curves*, p. 254.

Transforming this equation by the relations

$$X : Y : Z = yz : zx : xy,$$

it follows that the six points of inflexion of the trinodal quartic $(a, b, c, f, g, h \between 1/x, 1/y, 1/z)^2 = 0$ lie on the cubic

$$Ax^2(hz+gy)+By^2(fx+hz)+Cz^2(gy+fx)+Dxyz = 0. \qquad (4)$$

The conic $1/fx+1/gy+1/hz = 0$ has evidently triple contact with (4) at the three nodes, and hence the remaining intersections of cubic and quartic, viz., the six inflexions, must also lie on a conic.

To find its equation, we express that

$$(ay^2z^2+\ldots+2fx^2yz+\ldots)+[Ax^2(hz+gy)+\ldots+Dxyz][\alpha x+\beta y+\gamma z]$$
$$\equiv (ghyz+\ldots)(ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy).$$

We easily find that

$$u = \frac{A\alpha}{f}, \ \ldots, \qquad \alpha = \frac{f(gG+hH)}{(Ag^2h^2+\ldots)-fghD}, \quad \ldots,$$

$$u' = -\frac{fF^2+\frac{1}{2}\Delta(2gh+3af)}{(Ag^2h^2+\ldots)-fghD}, \quad \ldots .$$

Hence the equation of the inflexion conic is

$$A(gG+hH)x^2+\ldots-[2fF^2+\Delta(gh+\tfrac{3}{2}af)]\,yz-\ldots = 0,^*$$

and the two other points where this conic meets the quartic lie on the line

$$f(gG+hH)\,x+\ldots = 0,$$

and hence they are easily seen to be the points where we have

$$\frac{af}{x}+\frac{bg}{y}+\frac{ch}{z} = 0.$$

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[*] This form is given by Brill, *Math. Ann.*, Vol. XII., p. 106, lines 10-12.

ON 4-IC RESIDUACITY AND RECIPROCITY

By Lt.-Col. ALLAN CUNNINGHAM, R.E., Fellow of King's College, London.

[Received and Read February 12th, 1903.—Received, in revised form, May 8th, 1903.]

[The author's acknowledgments are due to Mr. A. E. Western for help in revising this paper, and for part of Arts. 3*c* and 15, *q.v.*]

Notation.—In this paper all symbols denote *integers* (usually +). p denotes an *odd prime*. ω, ϵ denote *odd* or *even* numbers respectively.

[*References* to Mathews's *Theory of Numbers*, Part I., 1892, are made by quoting the author's name and the number of the article required.]

1. *Introductory.*—The short notation (Dirichlet's)

$$\left(\frac{q}{p}\right)_4 \quad \text{or} \quad (q/p)_4 = \pm 1 \quad \text{denotes} \quad q^{\frac{1}{4}(p-1)} \equiv \pm 1 \pmod{p}. \tag{1}$$

The preliminary conditions, that $(q/p)_4 = \pm 1$ may be possible, are

i. *Linear*, $p = 4\varpi+1$, which involves $p = \mathrm{a}^2+\mathrm{b}^2$, (2)

ii. *Quadratic*, $(q/p)_2 = +1$, which involves (with i.) $(-q/p)_2 = +1$. (3)

The further condition, distinguishing the (±) sign in equation (1), *i.e.*, whether $(q/p)_4 = +1$ or -1, and (at same time) whether $(-q/p)_4 = +1$ or -1, is called the *quartic criterion*. Its research and reduction to a convenient form, and the development of the properties of the symbol $(q/p)_4$, are the objects of this paper. The values of $(\pm\, q/p)_4$ are connected by the relation

$$(+q/p)_4 . (-q/p)_4 = (-1/p)_4 = -1, \quad \text{when } p = 4\omega+1 \tag{4a}$$

$$= +1, \quad \text{when } p = 8\varpi+1; \tag{4b}$$

so that either may be derived immediately from the other.

2. *Previous 4-ic Criteria.*—It has been shown by Gauss[*], Eisenstein[†], and others that the 4-ic relation of an *odd prime* $q = 4\kappa \pm 1$ to an *odd prime* $p = 4\varpi+1$ depends on the 4-ic relation of the ratio of the complex factors $(\mathrm{a}+\mathrm{b}\iota) : (\mathrm{a}-\mathrm{b}\iota)$ of p to q, and may be expressed in the form[‡]

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[*] *See* H. J. S. Smith's (British Association) *Report on the Theory of Numbers*, 1859, Arts. 24–28.

[†] *Ibid.*, Arts. 29, 30.

[‡] This is the form given by Father Th. Pépin in his *Mémoire sur les lois de réciprocité relatives aux résidus de puissances*, Rome, 1878, pp. 46, 55,

$$q^{\frac{1}{4}(p-1)} \equiv \pm 1 \pmod{p} \quad \text{when} \quad \begin{cases} \left(\dfrac{\mathrm{a}+\mathrm{b}\iota}{\mathrm{a}-\mathrm{b}\iota}\right)^{\frac{1}{4}(q-1)} \equiv \pm 1 \ (\mathrm{mod}\ q = 4\kappa+1), & (5a) \\ \left(\dfrac{\mathrm{a}-\mathrm{b}\iota}{\mathrm{a}+\mathrm{b}\iota}\right)^{\frac{1}{4}(q+1)} \equiv \pm 1 \ (\mathrm{mod}\ q = 4\kappa-1). & (5b) \end{cases}$$

These may be always reduced (by rationalization and expansion) so as finally to present reduced criteria as a simple linear relation between a, b (mod q),

$$\mathrm{a} \equiv \mu\mathrm{b} \quad \text{or} \quad \mathrm{b} \equiv \lambda\mathrm{a} \pmod{q}. \qquad (5c)$$

The reduction has the inconvenience of increasing in difficulty as q increases [because the exponent $\frac{1}{4}(q \mp 1)$ increases with q], and the application when q is *composite* is difficult and laborious. As this criterion depends on a, b only, a table of the partition $p = \mathrm{a}^2+\mathrm{b}^2$ enables the 4-ic character of any prime q and also (but with some difficulty) of any composite q to any prime p in the table to be determined.

3. *2-ic Forms of* $p = 4\varpi+1$.—Inasmuch as the 2-ic forms $p = t^2 \pm qu^2$ give a very close 2-ic relation between q, p, they may be expected to serve conveniently also for the 4-ic relation. A preliminary investigation will now be made of the conditions under which p may be expressed in some one or more of the 2-ic forms, wherein q may be *any integer*,

$$p = \frac{1}{\lambda}(t^2 \pm qu^2), \qquad p = \frac{1}{m^2}(t^2 \pm qu^2), \qquad p = t^2 \pm qu^2. \qquad (6)$$

3*a*. *Forms* $p = (t^2 \pm qu^2)/\lambda$.—Since, by the preliminary conditions (3), both $(-q/p)_2$ and $(+q/p)_2 = 1$, it follows that p is expressible in *some* 2-ic form of determinant $(-q)$, and also in *some* 2-ic form of determinant $(+q)$, (Mathews, Arts. 59, 90). Let

p be of form (a, b, c) when $b^2-ac = -q$ (a, c both $+$), (7*a*)

p be of form (a', b', c') when $b'^2-a'c' = +q$ (one of a', c' is $-$). (7*b*)

Then λp becomes of the *principal class, i.e.*,

$$\lambda p = t^2+qu^2, \text{ when } \lambda = a \text{ or } c; \quad \lambda p = t^2-qu^2, \text{ when } \lambda = a' \text{ or } c'. \qquad (8)$$

Thus a quantity λ *always* exists such that λp is of the required forms (8), and is *at once known* when p has been expressed in the forms (7*a*, *b*).

3*b*. *Forms* $p = t^2 \pm qu^2$.—The preliminary conditions (2), (3) involve

$$(p/q)_2 = +1 \ \textit{always}, \quad \text{and} \quad (-1)^{\frac{1}{2}(p-1)} = +1 \ \textit{always}, \qquad (9)$$

and

$$(-1)^{\frac{1}{8}(p^2-1)} = +1, \quad \text{when} \quad p = 8\varpi+1. \qquad (10)$$

These are the (necessary and sufficient) conditions (Mathews, Arts. 129, 130) that p should be expressible in *some class of the principal genus* of both determinants $-q$ and $+q$, as follows:

Conditions (9) suffice when $q =$ an *odd prime*, (9a)

Conditions (9), (10) suffice when $q = 2 \times$ an *odd prime*, with $p = 8\varpi+1$. ......(10a)

Next; when q is *composite*, and $= Q$ or $2Q$,

where $Q = a^{\alpha} b^{\beta} c^{\gamma} \ldots$ ($a, b, c, \ldots$ being *odd primes*).

Now, suppose $(p/a)_2 = +1$, $(p/b)_2 = +1$, &c., (for every odd factor of p). ......(11)

Then p is a 2-ic residue of Q, and $(-1)^{\frac{1}{2}(p-1)} = +1$ always, (12)

and $(-1)^{\frac{1}{8}(p^2-1)} = +1$, when $p = 8\varpi+1$. (13)

These are the (necessary and sufficient) conditions (Mathews, Arts. 129, 130) that p should be expressible in *some class of the principal genus* of both determinants $-q$ and $+q$, as follows:

Conditions (12) suffice when $q = Q$ (an odd number), (12a)

Conditions (12), (13) suffice when $q = 2Q$, with $p = 8\varpi+1$. (13a)

Hence, when either $+q$ or $-q$ is an *Idoneal*,[*] the above conditions *suffice* that p shall be expressible in the *principal class* of the determinant $-q$ or $+q$ respectively (because an Idoneal has *only one class* per genus), *i.e.*,

$$p = t^2+qu^2 \quad \text{or} \quad = t^2-qu^2, \text{ respectively.} \tag{14}$$

The following is a list of the Idoneals of form $q = \omega$ or $2\omega < 100$:

For (t^2+qu^2)[†] $\left\{\begin{array}{l} 3, 5, 7, 9, 13, 15, 21, 25, 33, 37, 45, 57, 85, 93; \\ 2, 6, 10, 18, 22, 30, 42, 58, 70, 78; \end{array}\right.$

For (t^2-qu^2)[‡] $\left\{\begin{array}{l} \text{All odd numbers except } 25, 37, 49, 79, 81, 99; \\ \text{Doubles of all odd numbers except } 34, 82; \end{array}\right.$

When the conditions (9)–(13) *necessary* for the expressibility of p in the *principal genus* are satisfied, p is in many cases expressible in the *principal class* (*i.e.*, as $p = t^2 \pm qu^2$), even when $\pm q$ is not idoneal; but the conditions *necessary* for this case do not appear to be known.

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[*] The determinant $(\mp q)$ of a form $(t^2 \pm qu^2)$ having *only one class in each genus* is, with changed sign, called a $(\pm)$ *Idoneal*. This is Euler's term. See a paper "On Idoneals" (by the present author and the Rev. J. Cullen jointly) in *Brit. Assoc. Report*, 1901, p. 552. It is hoped to publish this shortly.

[†] See Mathews, Art. 213, p. 263.

[‡] Cayley's *Tables de Formes quadratiques* (see *Collected Mathematical Papers*, Vol. v., pp. 148–153). Cayley omits squares (25, 49, 81); these have been supplied by Mr. Cullen.

8*c*. *Forms* $p = (t^2 \pm qu^2)/m^2$.—The above conditions (9)–(13), as to q and p, are also the (necessary and sufficient) conditions (Mathews, Arts. 129, 130) that m^2p shall be expressible *in some class of the principal genus* of both determinants $-q$ and $+q$ as follows:

m odd: Conditions (9) suffice when q is an *odd prime*, (15*a*)

Conditions (9), (10) suffice when $q = 2 \times$ an *odd prime*, with $p = 8\varpi + 1$, (15*b*)

Conditions (12) suffice when $q = Q$ (an *odd composite*), (15*c*)

Conditions (12), (13) suffice when $q = 2Q$, with $p = 8\varpi + 1$; (15*d*)

m even: Conditions (9), (10) suffice when q is an *odd prime* (as below), (15*e*)

Conditions (12), (13) suffice when q is an *odd composite* (as below), (15*f*)

$$\text{In } (15e, f) \left\{\begin{array}{l} p = \dfrac{1}{m^2}(t^2 + qu^2) \text{ requires } q = 8\kappa + 3 \\[2ex] p = \dfrac{1}{m^2}(t^2 - qu^2) \text{ requires } q = 4\omega + 1 \end{array}\right\}. \tag{15g}$$

Hence, when either $+q$ or $-q$ is an *Idoneal*, the above conditions *suffice* that m^2p shall be expressible in the *principal class* thereof, *i.e.*, that

$$m^2p = t^2 + qu^2 \quad \text{or} \quad = t^2 - qu^2, \quad \text{respectively.} \tag{16}$$

It will now be shown* that it is always possible [when the above conditions (15*a*–*g*) are fulfilled] to choose m so that m^2p shall be expressible in the *principal class*, *i.e.*, $m^2p = t^2 \pm qu^2$ (whether $\pm q$ be idoneal or not).

Using the symbolic notation for the class of a 2-ic form (Mathews, Art. 141), let K denote the class of the principal genus in which p is expressible. Then the *opposite* class K^{-1} is also in the *principal genus* (Mathews, Art. 145, iv.). Hence K^{-1} is a class which can be obtained from *some other* class (say X) *by duplication*, *i.e.*, $K^{-1} = X^2$ (Mathews, Art. 161); and therefore $X^2K = 1$, which denotes the *principal class* (Mathews, Art. 145).

And, since K denotes p, this shows that any number m which is of the class X gives the product m^2p expressible in the *principal class*, *i.e.*, in the form (16).

The *expressibility* of p in the form $(t^2 \pm qu^2)/m^2$ has alone been considered so far. The actual finding of the suitable number m *in general* seems difficult. The following process would often succeed.

Suppose some form (7*a*, *b*) in which p itself is expressible to have been found, and let a series of forms, say (a_r, b_r, c_r) or (a'_r, b'_r, c'_r), *properly equivalent* thereto be found by the ordinary rules. If any of the quantities of type a_r, c_r or a'_r, c'_r be a *square*, that quantity may be taken for λ in result (8), and is a *suitable value* of m^2.

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* The author is indebted to Mr. A. E. Western for this elegant proof.

The search for m in the above way may prove pretty tedious; but *possible* values of m may often be assigned at once from a table of quadratic forms. The table below contains a list* of the *lowest* values of $m > 1$, corresponding to the values of $q = \omega$ or $2\omega < 100$ stated alongside, which, when the conditions (15*a*–*g*) are satisfied, are suitable for bringing m^2p into the *principal class*, when p itself is not of the principal class (*i.e.*, when $m = 1$ is *impossible*). The table is in *two parts* (A, B).

Table A.—m^2p is *certainly* in the principal class.

Table B.—m^2p may *possibly* be in the principal class.

m	A		B
	q in (t^2+qu^2)	q in (t^2-qu^2)	q in (t^2+qu^2)
2	11, 19, 27, 35, 43, 51, 67, 75, 91, 99	37	59, 83
3	17, 35, 41, 53, 63, 65, 77, 81	79, 99	59, 71, 83, 89, 95
	26, 50, 54, 62	34, 82	74, 86, 98
4	... ...	... ...	47, 79
5	49, 99	... ...	74
7	97	... ...	98
3 or 5	74	... ...	
3 or 7	98	... ...	

It may happen, moreover, that p and m^2p are each expressible in one or in both of the forms $(t^2 \pm qu^2)$. Thus,

$$\text{If}\quad p = t^2 \pm qu^2 \quad\text{and}\quad m^2 = t_0^2 \pm qu_0^2 \quad\text{(same sign in both),}$$

then also, by known rules

$$m^2p = t'^2 \pm qu'^2 \quad \text{(same sign as for } p \text{ and } m^2\text{)}.$$

Useful cases of this kind are

$$p = A^2+3B^2,\quad 4 = 1^2+3.1^2,\quad \text{giving}\quad 4p = L^2+3(3m)^2;$$

$$p = x^2-5y^2,\quad 4 = 3^2-5.1^2,\quad \text{giving}\quad 4p = X^2-5Y^2.$$

4. *New* 4-*ic Criterion* (*mod p*).—From the 2-ic forms of p just given (6), the 4-ic relation of q to p may now be shown to depend on the 2-ic relation of the product tu to p. These forms may be included in the double general form (known to be *always possible*), [Art. (3*a*)],

$$p = \frac{1}{\lambda}(t^2 \pm qu^2), \tag{18}$$

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* Compiled from Cayley's *Tables des Formes quadratiques.*

which gives $$\mp qu^2 \equiv t^2 \pmod{p},$$

whence $$(\mp q/p)_4 . (u/p)_2 = (t/p)_2 \quad\text{and}\quad (\mp q/p)_4 = (t/p)_2 . (u/p)_2. \tag{19}$$

Thus $$p = \frac{1}{\lambda}(t^2 + qu^2) \quad\text{gives}\quad (-q/p)_4 = (tu/p)_2, \tag{19a}$$

and $$p = \frac{1}{\lambda}(t^2 - qu^2) \quad\text{gives}\quad (+q/p)_4 = (tu/p)_2. \tag{19b}$$

Either of these is the sought 4-ic criterion, giving the 4-ic character (± 1) of $(-q)$ or of $(+q)$ to p as being the *same* as the 2-ic character of the product tu to p. As this latter (2-ic) relation is easily calculable by known rules, it follows that *either* partition (18) suffices to determine the 4-ic character of q to p with sufficient ease. And note that this rule is *applicable to all values* of q, and that it is *independent* of λ (except in so far as t, u depend on λ).

5. 4-*ic Criteria* (mod T, Q).—A process will now be developed whereby the criteria (mod p) of Art. 4 are converted into equally simple 2-ic relations between T, Q (the *odd* factors of t, q), *i.e.*, with mod T or Q instead of mod p. This new result has the advantage of being *more easily calculable* than the former, when q is small, as is usually the case, compared with p.

The second of the 2-ic forms of p (Art. 3) will here be used,

$$p = \frac{1}{m^2}(t^2 \pm qu^2). \tag{20}$$

As m may be either odd or even, *each* of t, u may be odd or even; say

$$t = 2^\tau . T, \quad u = 2^\nu . U, \quad\text{and}\quad T,\ U \textit{ both odd}. \tag{21}$$

Then $$m^2 p = 2^{2\tau} . T^2 \pm q . 2^{2\nu} . U^2.$$

Both forms give $$(p/U)_2 = +1, \quad\text{whence}\quad (U/p)_2 = +1, \tag{22}$$

and give *respectively* $$\left.\begin{array}{l}(p/T)_2 = (\pm q/T)_2, \quad\text{whence}\quad (T/p)_2 = (\pm q/T)_2 \\ \text{[the } (\pm q) \text{ going with the } T \text{ of } (t^2 \pm qu^2) \text{ respectively]}\end{array}\right\}. \tag{23}$$

The 4-ic criteria (mod p), Art. 4, become

$$(\mp q/p)_4 = (t/p)_2 . (up)_2 = (2^{\tau+\nu}/p)_2 . (T/p)_2 . (U/p)_2. \tag{24}$$

Thus $$p = \frac{1}{m^2}(t^2 + qu^2) \quad\text{gives}\quad (-q/p)_4 = (2^{\tau+\nu}/p)_2 . (+q/T)_2, \tag{24a}$$

$$p = \frac{1}{m^2}(t^2 - qu^2) \quad\text{gives}\quad (+q/p)_4 = (2^{\tau+\nu}/p)_2 . (-q/T)_2. \tag{24b}$$

These general results may be reduced to one *simple* standard form

$$(q/p)_4 = \theta.(q/T)_2, \quad \text{where} \quad \theta = \pm 1, \tag{25}$$

and it suffices to tabulate the (± 1) value of θ as dependent on the linear forms of p, $\tau+\nu$, T; writing for shortness

$$\left.\begin{array}{l} \theta \text{ with } m^2p = t^2+qu^2; \quad \theta', \theta'' \text{ with } m^2p = t^2-qu^2 \\ \theta' \text{ when } T = 4t'+1, \quad \theta'' \text{ when } T = 4t''-1 \end{array}\right\}, \tag{26}$$

$$\left.\begin{array}{lllll;lllll;lllll} \end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccc} p, & (\tau+\nu), & \theta, & \theta', & \theta''; & p, & (\tau+\nu), & \theta, & \theta', & \theta''; & p, & (\tau+\nu), & \theta, & \theta', & \theta'' \\ 8\varpi+1, & \text{any}, & +1 & +1, & -1; & 4\omega+1, & \omega, & +1, & -1, & +1; & 4\omega+1, & \epsilon, & -1, & +1, & -1 \end{array}\right\}. \tag{26a}$$

Another way of presenting the above results (24*a*, *b*), which will be found convenient for future developments (Art. 12*a*), is to exhibit the final values of $(q/p)_4$, $(\bar{q}/p)_4$ separately [$\bar{q}$ is written for shortness for $(-q)$, and $\bar{1}$ for (-1)].

$$m^2p = t^2+qu^2 \quad \text{gives} \quad (q/p_4) = (\bar{1}/p)_4.(2^{\tau+\nu}/p)_2.(q/T)_2, \tag{27a}$$

$$m^2p = t^2-qu^2 \quad \text{gives} \quad (\bar{q}/p)_4 = (\bar{1}/p)_4.(2^{\tau+\nu}/p)_2.(\bar{q}/T)_2. \tag{27b}$$

Also

$$m^2p = t^2-qu^2 \quad \text{gives} \quad (q/p)_4 = (\bar{1}/T)_2.(2^{\tau+\nu}/p)_2.(q/T)_2, \tag{27c}$$

$$m^2p = t^2+qu^2 \quad \text{gives} \quad (\bar{q}/p)_4 = (\bar{1}/T)_2.(2^{\tau+\nu}/p)_2.(\bar{q}/T)_2. \tag{27d}$$

These show that

The forms $m^2p = t^2 \mp qu^2$ give formally the same results for $(\bar{q}/p)_4$ that $m^2p = t^2 \pm qu^2$ give for $(q/p)_4$. (27*e*)

Further, the symbol $(q/T)_2$ in (24*a*, *b*), (27*a–d*) may be converted so as to depend on the reciprocal $(T/Q)_2$ (where T, Q are the *odd* factors in t, q), and thereby be rendered more easily calculable, by the known relations

$$(q/T)_2 = (-1)^{(q-1)(T-1)}.(T/q)_2, \quad \text{when} \quad q = \omega \tag{28a}$$

$$= (-1)^{\frac{1}{4}(Q-1)(T-1)}.(2/T)_2.(T/Q)_2, \quad \text{when} \quad q = 2Q, \quad (Q = \omega). \tag{28b}$$

The general results (24*a*, *b*), (27*a–d*) may now all be reduced to the simple standard form

$$(q/p)_4 = \Theta.(T/Q)_2, \tag{28}$$

and a scheme similar to (26*a*) might now be drawn up giving the (± 1) value of Θ as dependent on the *linear* forms of p, q, τ, ν, T; but it was found to run to some length, so is not given here; a nearly similar form involving (t/q) instead of (T/Q) will be worked out in Arts. 10, 12.

Either of the results (25), (28) may be looked on as the sought 4-ic criterion (mod T, Q), giving the 4-ic relation of q to p in terms of the 2-ic relations of T, Q; any of the forms thereof are quite convenient for practical calculation. Note that all the results in this article are *applicable* *to all values of* q ($= \omega$ or 2ω), and are *independent of* m^2 (except in so far as t, u depend on m); they are less general than those of Art. 4, in that λ must here be a *square* ($\lambda = m^2$).

6. *Forms* p *and* $4p = t^2 \pm qu^2$.—These simple forms are so important that it is worth developing the 4-ic criterion for them in fuller detail, according to the *linear* forms of p, q, t, u, T, *i.e.*, according as

$$p = 8\varpi+1,\ 4\omega+1;\quad q = 8\kappa+1,\ 3,\ 5,\ 7,\ \text{or } = 2\omega;\quad t = \omega, \epsilon;\quad u = \omega, \epsilon;$$
$$T = 4t \pm 1.$$

It will be seen that

$$\left.\begin{array}{l} \tau = 0, \text{ when } t = T = \omega; \quad \upsilon = 0, \text{ when } u = U = \omega \\ \tau, \upsilon \text{ are } \textit{finite} \text{ when } t, u = \epsilon \text{ respectively} \end{array}\right\}. \tag{29}$$

Also

$$\left.\begin{array}{l} p = t^2 \pm qu^2 \\ \text{with } q = \omega \end{array}\right\} \left.\begin{array}{l} \text{require } (t, u) \text{ to be } (\omega, \epsilon) \text{ or } (\epsilon, \omega) \\ \text{so that } \tau, \upsilon \text{ are one } \textit{zero}, \text{ one } \textit{finite} \end{array}\right\}, \tag{29a}$$

$$\left.\begin{array}{l} p = t^2 \pm qu^2 \\ \text{with } q = 2\omega \end{array}\right\} \text{require } p = 8\varpi+1;\ t = \omega,\ u = \epsilon;\ \tau = 0,\ \upsilon \textit{ finite}, \tag{29b}$$

$$p = \tfrac{1}{4}(t^2 \pm qu^2) \text{ requires } q = 8\kappa+3, 5;\ t = \omega, u = \omega;\ \tau = 0,\ \upsilon = 0;$$
$$[t, u \text{ may } \textit{both} = \epsilon, \text{ but then } p = (\tfrac{1}{2}t)^2 \pm q(\tfrac{1}{2}u)^2]. \tag{29c}$$

The results may all be reduced to the standard form (25) of Art. 5; the (± 1) value of θ, θ', θ'' is given in the two schemes below: these schemes differ from the short scheme (26*a*) of Art. 5 only in their greater detail [they have the great advantage of showing clearly what forms of t ($= \omega, \epsilon$) and u ($= \omega, \epsilon$) do yield, or fail to yield, *possible* forms of p; the latter case is shown by a blank entry for θ, θ', θ''].

	$p =$	$8\varpi+1$		$4\omega+1$		$8\varpi+1$		$4\omega+1$		$8\varpi+1$
	$q =$	$8\kappa+1$	$4\omega+1$	$4\omega+1$	$8\kappa+1$	$8\kappa+7$	$8\kappa+3$	$8\kappa+3$	$8\kappa+7$	2ω
$p = t^2 \pm qu^2$	$t =$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$	$\omega,\ \epsilon$
	$u =$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$	$\epsilon,\ \omega$
	$\tau =$	$0,\ >1$	$0,\ 1$	$0,\ \omega>1,\ \epsilon$	$0,\ 1$	$0,\ >1$	$0,\ .$	$0,\ \omega,\ \epsilon$	$0,\ 1$	$0,\ .$
	$\upsilon =$	$>1,\ 0$	$>1,\ 0$	$1,\ 0,\ 0$	$1,\ 0$	$>1,\ 0$	$>1,\ .$	$1,\ 0,\ 0$	$1,\ 0$	$\omega, \epsilon,\ .$
	$\theta =$	$+1,\ +1$	$+1,\ +1$	$+1,\ +1,\ -1$	$+1,\ +1$	$+1,\ .$	$+1,\ .$	$+1,\ .\ .$	$+1,\ .$	$+1,\ .$
	$\theta' =$	$+1,\ .$	$+1,\ .$	$-1,\ .\ .$	$-1,\ .$	$+1,\ +1$	$+1,\ .$	$-1,\ -1,\ +1$	$-1,\ -1$	$+1,\ .$
	$\theta'' =$	$-1,\ .$	$-1,\ .$	$+1,\ .\ .$	$+1,\ .$	$-1,\ -1$	$-1,\ .$	$+1,\ +1,\ -1$	$+1,\ +1$	$-1,\ .$

	$p =$	$8\varpi+1$		$4\omega+1$		$8\varpi+1$		$4\omega+1$	
$p = \frac{1}{4}(t^2 \pm qu^2)$	$q =$	$8\kappa+1$	$4\omega+1$	$4\omega+1$	$8\kappa+1$	$8\kappa+7$	$8\kappa+3$	$8\kappa+3$	$8\kappa+7$
	$\theta =$	.	.	.	.	.	$+1$	-1	.
	$\theta' =$	.	$+1$	$+1$	.	.	.	.	.
	$\theta'' =$	.	-1	-1	.	.	.	.	.

The results θ, θ', θ'' in the above schemes may be all included in the following *simple* formulæ

$$q = 4\kappa \pm 1\,; \quad \theta = (-1)^{\frac{1}{4}(p-1)\,.\,\frac{1}{4}(q\mp1)\,.\,(\tau+v+1)}, \quad \text{[using } \tfrac{1}{4}(q \mp 1) \text{ in the exponent of } \theta \text{ when } q = 4\kappa \pm 1 \text{ respectively]} \tag{30a}$$

$$q = \omega\,; \quad \theta' = -\theta'' = (-1)^{\frac{1}{4}(p-1)\,.\,(\tau+v)}, \tag{30b}$$

$$q = 2\omega\,; \quad \theta = +1\,; \quad \theta' = +1\,; \quad \theta'' = -1. \tag{30c}$$

7. *Canonical 4-ic Criterion.*—The schemes of Art. 6 show that, when $q = 4\kappa - 1$ or 2ω,

$$p = t^2 + qu^2 \quad \text{gives} \quad (q/p)_4 = +(q/T)_2 \ \textit{always}, \tag{31a}$$

$$4p = t^2 + qu^2 \quad \text{gives} \quad (-q/p)_4 = +(q/T)_2 \ \textit{always}. \tag{31b}$$

This suggests the expression of T in a form exhibiting it as a *quadratic modulus* of q, which leads to a remarkable simplification of the 4-ic criterion. Let T be written

$$T\,(= t, \text{ when } odd) = 2q\,.\,\mu \pm r \text{ or } r'. \tag{32}$$

Then, by taking r, r', satisfying the following conditions (determined by the form of q, which is now supposed to contain *no square factor*),

$$\left.\begin{array}{lll} q = 4\kappa - 1\,; & q = 2Q = 2\,(4\kappa+1)\,; & q = 2Q = 2\,(4\kappa-1), \\ r = 4\rho+1,\ (r/q)_2 = +1\,; & r = 8\rho\pm1,\ (r/Q)_2 = +1\,; & r = 8\rho+1,\ 3, \quad (r/Q) = +1 \\ r' = 4\rho-1,\ (r'/q)_2 = -1\,; & r' = 8\rho\pm3,\ (r'/Q)_2 = -1\,; & r' = 8\rho-1,\ -3,\ (r'/Q) = -1 \end{array}\right\}, \tag{34}$$

then, it follows, as may be seen by slightly modifying the rules in Mathews, Art. 46, II., III., IV., that

$$(q/T)_2 = +1, \text{ when } \mu = \epsilon\,; \quad = -1, \text{ when } \mu = \omega. \tag{35}$$

Hence the 4-ic criterion derived from p and $4p = t^2 + qu^2$ takes (when $q = 4\kappa - 1$ or 2ω) what may be called the *canonical**† *form*,

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* The particular instance of this, when $q = 2$, was discovered by Gauss.

† Two particular instances of this (viz., when $q = 3$ or 7) were given by (the late) Mr. Chas. E. Bickmore in his paper "On the Numerical Factors of $(a^n - 1)$," see *Messenger of Mathematics*, Vol. xxv., 1876, pp. 23, 27; but the rule when $q = 3$ is unfortunately incorrectly printed; viz., on p. 23, l. 3: *for* "odd or even" *read* "even or odd."

$$\left.\begin{aligned} p \text{ gives } (q/p)_4 = (-1)^{\mu}; \quad & 4p \text{ gives } (-q/p)_4 = (-1)^{\mu} \\ = +1, \text{ when } \mu = \epsilon; \quad & = -1, \text{ when } \mu = \omega \end{aligned}\right\}. \qquad (36)$$

7*a*. *Table of* r, r'.—To enable the above to be applied *directly*, a table is subjoined giving the values of r, r' for all values of $q = 4\kappa-1$ or $2\omega \not> 35$. This table may easily be extended by Legendre's tables.*

	q	r	r'
$q = 4\kappa-1$	3	1;	
	7	1, 9;	3;
	11	1, 5, 9;	7, 19;
	15	1, 17;	7, 11;
	19	1, 5, 9, 17, 25;	3, 15, 27, 31;
	23	1, 9, 13, 25, 29, 41;	7, 11, 15, 19, 43;
	31	1, 5, 9, 25, 33, 41, 45, 49;	3, 11, 15, 23, 27, 43, 55;
	35	1, 9, 13, 17, 29, 33;	19, 23, 31, 43, 59, 67;
$q = 2(4\kappa+1)$	2	1;	
	10	1, 9;	3, 13;
	26	1, 9, 17, 23, 25, 49;	5, 11, 19, 21, 37, 45;
	34	1, 9, 15, 25, 33, 47, 49, 55;	3, 5, 11, 27, 29, 37, 45, 61
$q = 2(4\kappa-1)$	6	1;	5;
	14	1, 9, 11, 25;	5, 13;
	22	1, 3, 9, 25, 27;	7, 13, 21, 29, 39;
	30	1, 17, 19, 49;	7, 13, 29, 37;

8. *Law of* 4-*ic Reciprocity* $[(q/p)_4 : (p/q)_4]$.—In introducing the symbol $(p/q)_4$ it will now be supposed that $q = 4\kappa+1 =$ *prime*. This, with the conditions (2), (3) laid down for $(q/p)_4$ in Art. 7, involve both $(p/q)_2 = +1$ and $(-p/q)_2 = +1$; and it will be further supposed that

$$p, \text{ or } 4p, \text{ or } m^2p = t^2 \pm qu^2. \qquad (37)$$

These give respectively

$$(p/q)_4, \text{ or } (2/q)_2.(p/q)_4, \text{ or } (m/q)_2.(p/q)_4 = (t/q)_2 = (2^{\tau}/q)_2.(T/q)_2. \qquad (38)$$

Combining these with the previous results (24*a*, *b*),

$$p = t^2+qu^2 \text{ gives } (q/p)_4.(p/q)_4 = (\bar{1}/p)_4.(2^{\tau+\nu}/p)_2.(2^{\tau}/q)_2, \qquad (39a)$$

$$p = t^2-qu^2 \text{ gives } (q/p)_4.(p/q)_4 = (\bar{1}/T)_2.(2^{\tau+\nu}/p)_2.(2^{\tau}/q)_2. \qquad (39b)$$

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* *Théorie des Nombres*, third edition, Table III.

Also, from (38), comes the rule

When $m^2p = t^2 \pm qu^2$, multiply the dexter of (39*a*, *b*) by $(m/q)_2$, (39*c*)

and, since the formulæ for $4p$ require $q = 4\omega+1$ (see the scheme of Art. 6), which gives $(2/q)_2 = -1$, result (38) gives

When $4p = t^2 \pm qu^2$, change the sign of the dexter of (39*a*, *b*). (39*d*)

These results (39*a*–*d*) may be all included in one simple standard form

$$(q/p)_4 \cdot (p/q)_4 = \vartheta_m, \quad \text{where} \quad \vartheta_m = \pm 1, \tag{39}$$

which may be called the *Law of 4-ic reciprocity*. It suffices now to tabulate the ($\pm$ 1) value of ϑ_m, as in the scheme below, as dependent on the linear forms of p, q, t, u, τ, v, T; writing also, for shortness,

$$\left.\begin{array}{l} \vartheta_m \text{ when } m^2p = t^2+qu^2; \quad \vartheta'_m \text{ and } \vartheta''_m \text{ when } m^2p = t^2-qu^2 \\ \vartheta'_m \text{ when } T = 4t'+1, \quad \vartheta''_m \text{ when } T = 4t''-1 \end{array}\right\}. \tag{40}$$

	$p = t^2 \pm qu^2$				$4p = t^2 \pm qu^2$	
$p =$	$8\varpi+1$		$4\omega+1$		$8\varpi+1$	$4\omega+1$
$q =$	$8\kappa+1$	$4\omega+1$	$4\omega+1$	$8\kappa+1$	$4\omega+1$	$4\omega+1$
$\tau =$	$0, >1$	$0, >1$	$0, \omega>1, \epsilon$	$0, 1$	0	0
$v =$	$>1, 0$	$>0, 0$	$1, 0, 0$	$1, 0$	0	0
$\vartheta =$	$+1, +1$	$+1, -1$	$+1, -1, -1$	$+1, +1$	.	.
$\vartheta' =$	$+1, +1$	$+1, -1$	$-1, +1, +1$	$-1, -1$	-1	-1
$\vartheta'' =$	$-1, -1$	$-1, +1$	$+1, -1, -1$	$+1, +1$	$+1$	$+1$

This scheme may also be presented in the following condensed form which presents certain advantages.

$p =$	$p = t^2 \pm qu^2$				$4p = t^2 \pm qu^2$	
$q =$	$8\kappa+1$		$4\omega+1$		$4\omega+1$	
$p =$	$8\varpi+1$	$4\omega+1$	$4\omega+1$	$8\varpi+1$	$8\varpi+1$	$4\omega+1$
$\vartheta =$	$+1$	$+1$	$(-1)^u$	$(-1)^u$	.	.
$\vartheta' =$	$+1$	-1	$(-1)^t$	$(-1)^u$	-1	-1
$\vartheta'' =$	-1	$+1$	$(-1)^u$	(-1)	$+1$	$+1$

And it is now seen that the results (ϑ_m, ϑ'_m, ϑ''_m) of the above schemes may be all included in the following *simple* formulæ:—

$$p = t^2+qu^2 \text{ gives } \vartheta_1 = (-1)^{\frac{1}{4}(q-1)u}, \tag{41a}$$

$$p = t^2-qu^2 \text{ gives } \vartheta'_1 = -\vartheta''_1 = (-1)^{\frac{1}{4}(q-1)u+\frac{1}{4}(p-1)}, \tag{41b}$$

$$4p = t^2 \pm qu^2 \text{ give } \vartheta'_2 = -1,\ \vartheta''_2 = +1, \tag{41c}$$

$$m^2p = t^2 \pm qu^2 \text{ give } \vartheta_m = (m/q)_2 . \vartheta_1,\ \vartheta'_m = (m/q)_2 . \vartheta'_1,\ \vartheta''_m = (m/q)_2 . \vartheta''_1. \tag{41d}$$

Lastly, since the preliminary conditions are similar as to p and q, it follows that, if q be given in the quadratic form, (instead of p),

$$q, \text{ or } 4q, \text{ or } m^2q = t^2 \pm pu^2,$$

then p, q may be interchanged in all the above results [(38)–(41d)].

9. *Reduction of* $(p/q)_4$.—It will now be shown how to reduce the expression $(p/q)_4$, when $p > q$.

Let p be written in form $p = q\lambda+\pi$, so that $p \equiv \pi \pmod{q}$. Then

$$(\pi/q)_2 = (p/q)_2 = +1 \quad \text{[by (3) and (37)]}.$$

Now, let a multiple (say, $m'q$) of q be added to π, such that

$$m'q+\pi = r^2 \quad [\text{always possible, since } (\pi/q)_2 = +1];$$

therefore $$p \equiv \pi \equiv m'q+\pi \equiv r^2 \pmod{q}.$$

And, by definition, $(p/q)_4$ = residue of $p^{\frac{1}{4}(q-1)}$, (mod q); therefore

$$(p/q)_4 \equiv (r^2)^{\frac{1}{4}(q-1)} \equiv (\pm r)^{\frac{1}{2}(q-1)} \pmod{q}$$
$$= (r/q)_2, \tag{42}$$

and the ($\pm$ 1) value of $(r/q)_2$ can always be found by known rules.

Note that π may be *any* residue of p (mod q), not necessarily the lowest. Now, since p, $4p$, or m^2p is given by (37); therefore

$$p, \text{ or } 4p, \text{ or } m^2p \equiv t^2 \pmod{q}.$$

Hence r, or $2r$, or $mr \equiv t$ (mod q) [with p, $4p$, or m^2p respectively]. (43)

As t is supposed to be among the data (37), this is *usually* the most convenient value of r to work with; it leads of course only to the known result (38). But, when p is *given* in the form $p = q.\lambda+r^2$, this gives the result (42) at once.

Ex. $p = q.\lambda+1$ gives $(p/q)_4 = +1$ at once,

$p = q.\lambda-1$ gives $(p/q)_4 = (-1/q)_4 = (-1)^{\frac{1}{4}(q-1)}$.

10. *Use of Law of* 4-ic *Reciprocity.*—The reciprocal relation of $(q/p)_4$ to $(p/q)_4$ may now be used for finding the value of $(q/p)_4$ when $q < p$, in much the same way as the law of 2-ic reciprocity is used for finding $(q/p)_2$ when $q < p$; and, when q is small compared with p, this application is very useful.

Combining the standard form (39) of the reciprocal expression with the reduced form of $(p/q)_4$, when $p > q$, just given (42),

$$p = t^2 \pm qu^2 \text{ give } (q/p)_4 = \vartheta_1 . (r/q)_2 = \vartheta_1 . (t/q)_2 \quad [\vartheta_1 \text{ as in } (41a,\ b)], \tag{45a}$$

$$4p = t^2 \pm qu^2 \text{ give } (q/p)_4 = \vartheta_2 . (r/q)_2 = -\vartheta_2 . (t/q)_2 \quad [\vartheta_2 \text{ as in } (41c)], \tag{45b}$$

$$m^2p = t^2 \pm qu^2 \text{ give } (q/p)_4 = \vartheta_m . (r/q)_2 = \vartheta_1 . (t/q)_2 \quad [\vartheta_1 \text{ as in } (41a,\ b)], \tag{45c}$$

and the values $(\vartheta_1, \vartheta_1', \vartheta_1'')$ of ϑ may be taken from the schemes of Art. 8, or from the formulæ (41*a*–*c*). These reduced results are *independent of m* [which occurs in the form of $(m/q)_2$ in ϑ_m, $(p/q)_4$, $(r/q)_2$, but disappears in their product]; they are, in fact, only variants of the previous results (25), (28).

11. *Canonical* 4-*ic Criterion* ($q = 4\kappa+1$).—The 4-ic criterion in the case of $q = 4\kappa+1$ (omitted in Art. 7) may now be expressed in what may be styled the *canonical* form, [*cf.* (36)],

$$\left.\begin{aligned} (q/p)_4 &= (-1)^{\mu} \\ &= +1 \text{ when } \mu = \epsilon; \quad = -1 \text{ when } \mu = \omega \end{aligned}\right\}, \tag{46}$$

and the value of μ, derived from the values of ϑ, ϑ', ϑ'' of Art. 8 is as shown in following scheme :—

FORM.	$(t/q)_2$	T	μ in $p = t^2 \pm qu^2$	μ in $4p = t^2 \pm qu^2$	
$t^2 + qu^2$	$+1$	any	$\frac{1}{4}(q-1)u$		(46*a*)
	-1	any	$\frac{1}{4}(q-1)u+1$		(46*b*)
$t^2 - qu^2$	$+1$	$4t'+1$	$\frac{1}{4}(q-1)u+\frac{1}{4}(p-1)$	$\frac{1}{4}(q-1)u+1$	(46*c*)
		$4t'-1$	$\frac{1}{4}(q-1)u+\frac{1}{4}(p-1)+1$	$\frac{1}{4}(q-1)u$	(46*d*)
	-1	$4t'+1$	$\frac{1}{4}(q-1)u+\frac{1}{4}(p-1)+1$	$\frac{1}{4}(q-1)u$	(46*e*)
		$4t'-1$	$\frac{1}{4}(q-1)u+\frac{1}{4}(p-1)$	$\frac{1}{4}(q-1)u+1$	(46*f*)

12. *Extension of Results* (45), (46).—The results (45*a*, *b*, *c*), (46) obtained for the case of $q =$ a *prime* of form $4\kappa+1$ only (by aid of the law of 4-ic reciprocity) will now be established in a way which shall include all cases of $q = 4\kappa \pm 1$ (any *odd* number).

Firstly, $(q/T)_2 = (T/q)_2, \quad \text{if } q = 4\kappa+1.$ (47*a*)

Next, $(q/T)_2 = (-1)^{\frac{1}{2}(T-1)}.(T/q)_2, \text{ if } q = 4\kappa-1.$

Therefore

$$(\bar{q}/T)_2 = (\bar{1}/T)_2.(q/T)_2 = (\bar{1})^{\frac{1}{2}(T-1)}.(q/T)_2 = (T/q)_2, \text{ when } q = 4\kappa-1. \quad (47b)$$

Hence results (27*a*, *b*, *c*, *d*) may be modified (when q is *odd*) thus :—

$$\left.\begin{array}{l} (T/q)_2 \text{ may be substituted for } (q/T)_2 \text{ or } (\bar{q}/T)_2, \text{ viz.,} \\ \text{for } (q/T)_2 \text{ in results } (27a, c), \text{ when } q = 4\kappa+1; \\ \text{for } (\bar{q}/T)_2 \text{ in results } (27b, d), \text{ when } q = 4\kappa-1. \end{array}\right\} \quad (48)$$

Again, $(t/q)_2 = (2^\tau/q)_2.(T/q)_2$; so that $(T/q)_2 = (2^\tau/q)_2.(t/q)_2$.

Hence, writing for shortness, as in (39*a*, *b*), (39), (40),

$$\vartheta_1 = (\bar{1}/p)_4.(2^{\tau+\nu}/p)_2.(2^\tau/q)_2, \quad \vartheta_1' = (\bar{1}/T)_2.(2^{\tau+\nu}/p)_2.(2^\tau/q)_2. \quad (49)$$

Results (27*a*, *b*, *c*, *d*) will be found to reduce to

$$m^2p = t^2+qu^2 \text{ gives } (q/p)_4 = \vartheta_1.(t/q)_2, \text{ when } q = 4\kappa+1; \quad (50a)$$

$$m^2p = t^2-qu^2 \text{ gives } (\bar{q}/p)_4 = \vartheta_1.(t/q)_2, \text{ when } q = 4\kappa-1; \quad (50b)$$

$$m^2p = t^2-qu^2 \text{ gives } (q/p)_4 = \vartheta_1'.(t/q)_2, \text{ when } q = 4\kappa+1; \quad (50c)$$

$$m^2p = t^2+qu^2 \text{ gives } (\bar{q}/p)_4 = \vartheta_1'.(t/a)_2, \text{ when } q = 4\kappa-1. \quad (50d)$$

Results (50*a*, *c*), confirm (45*c*) for the case of $q = 4\kappa+1$; results (50*b*, *d*) are the analogues for the case of $q = 4\kappa-1$; taken together they give another proof of (27*e*).

13. *Value of* $(q/p)_4$ [q composite].

Let $q = (a_1a_2a_3\ldots).(b_1b_2b_3\ldots)$, n the number of b's, (52)

where every $(a/p)_4 = +1$, and every $(b/p)_4 = -1$. (52*a*)

Then $(q/p)_4 = \Pi\{(a/p)_4\}.\Pi\{(b/p)_4)\} = \pm 1.$ (53)

Hence $(q/p)_4$ follows the same laws of composition as obtain with $(q/p)_2$. Briefly

$$(a_1/p)_4.(a_2/p)_4 = (a_1a_2/p)_4 = +1, \quad (b_1/p)_4.(b_2/p)_4 = (b_1b_2/p)_4 = +1, \quad (53a)$$

$$(a/p)_4.(b/p)_4 = (ab/p)_4 = -1, \quad (53b)$$

$$\Pi\{(a/p)_4\} = (\Pi\{a\}/p)_4 = +1, \quad (53c)$$

$$\Pi\{(b/p)_4\} = (\Pi\{b\}/p)_4 = (-1)^n, \quad (53d)$$

$$\Pi\{(a/p)_4\}.\Pi\{(b/p)_4\} = (\Pi\{ab\}/p)_4 = (-1)^n. \quad (53e)$$

These formulæ enable $(q/p)_4$ to be calculated when the 2-ic forms

$$m_r^2 p = t_r^2 \pm a_r u_r^2, \quad m_r'^2 p = t_r'^2 \pm b_r u_r'^2$$

are known for *each* of the factors a_r, b_r (not necessarily prime). Of course it can always be calculated *directly* from the single partition $m^2 p = t^2 \pm qu^2$ when known; in fact this is one of the great advantages of the processes developed in this paper that (in most cases) q may be *composite.*

14. *Generalized Symbol* $(q/P)_4$.—The use of the symbol $(q/p)_4$ may now be *extended* so as to include the case of a *composite* modulus (P) in a way quite similar to Jacobi's extension of the symbol $(q/p)_2$.

In what follows hereon, the primes p_r, $p'_{r'}$ composing P, P' are supposed to be of the *same special forms.*

$$\text{Every } p_r = 4\varpi_r + 1 = \frac{1}{m_r^2}(t_r^2 \pm qu_r)^2, \quad \text{and} \quad (q/p_r) = +1. \tag{54}$$

Let

$$P = p_1 p_2 p_3, \ldots, \quad P' = p'_1 p'_2 p'_3 \ldots \quad (p_1 \text{ may } = p_2, \text{ \&c.}; \; p'_1 \text{ may } = p'_2, \text{ \&c.}), \tag{55}$$

and let the symbol $(q/P)_4$ be *defined* by the equation

$$(q/P)_4 = (q/p_1)_4 . (q/p_2)_4 . (q/p_3)_4 \ldots \quad (q \text{ prime to } P) \tag{56}$$

or

$$(q/\Pi p_r)_4 = \Pi\{(q/p_r)_4\}. \tag{56a}$$

The properties of this generalized symbol are expressed in the following theorems, analogous to those of Jacobi's symbol $(q/P)_2$.

[For facility of comparison these theorems are here stated in the same order and bear the same numbers 1–5 as the corresponding theorems on $(q/P)_2$ given in Mathews, Art. 42, *q.v.*; as the proofs are quite similar to those given for $(q/P)_2$, a bare statement of results will generally suffice here.]

1. If q is prime to P, P', where $P = \Pi(p_r)$, $P' = \Pi(p'_{r'})$, then

$$(q/PP')_4 = (q/P)_4 . (q/P')_4. \tag{57a}$$

2. If a, b, c, ... are prime to P, then

$$(abc\ldots/P)_4 = (a/P)_4 . (b/P)_4 . (c/P)_4 \ldots. \tag{57b}$$

3. If $q' \equiv q \pmod{P}$, and q *or* q' be prime to P, then

$$q, q' \text{ are both prime to } P, \text{ and } (q'/P)_4 = (q/P)_4. \tag{57c}$$

4. If $P = \Pi(p_r)$, then $\frac{1}{4}(P-1) = \Sigma\frac{1}{4}(p_r - 1)$

and

$$(-1/P)_4 = (-1)^{\frac{1}{4}(P-1)}. \tag{57d}$$

4*a*. The value of $(2/P)_4$ will be worked out in Art. 15.

5. If $P = \Pi(p_r)$, and $P' = \Pi(p'_{r'})$, and P prime to P', and if also every prime p_r, $p'_{r'}$ is of form $(4\varpi+1)$, and also satisfies (54), then

$$(P/P')_4 = \Pi\{(p_r/p'_{r'})_4\}, \quad (P'/P)_4 = \Pi\{(p'_{r'}/p_r)_4\} \qquad (57e)$$

when the products Π extend over *all* the combinations of a factor p_r with a factor $p'_{r'}$. Therefore

$$\begin{aligned}(P/P')_4 . (P'/P)_4 &= \Pi\{(p_r/p'_{r'})_4 . (p'_{r'}/p_r)_4\} \\ &= \Pi\{\vartheta_{r,r'}\} \qquad (57f)\end{aligned}$$

where, as in Art. 8, $\vartheta_{r,r'} = (p_r/p'_{r'})_4 . (p'_{r'}/p_r)_4$. (57*g*)

This result (57*f*) is the generalized law of 4-ic reciprocity.

15. *Reduction of* $(q/P)_4$.—The form of P laid down in Art. 14,

$$P = \Pi(p_r) = \Pi\left\{\frac{1}{m_r^2}(t_r^2 \pm q . u_r^2)\right\} \quad \text{[all the } p_r \text{ of \textit{same} form]}, \qquad (58)$$

is reducible by conformal multiplication of the forms to

$$P = \frac{1}{m_0^2}(t_0^2 \pm q . u_0^2), \quad \text{where} \quad m_0 = \Pi(m_r). \qquad (59)$$

It will now be shown that the (± 1) value of $(q/P)_4$ can be* derived from the form (59) of P in precisely the *same way* as each $(q/p_r)_4$ is derived from its own form (54) [for facility of comparison the results for P are numbered below with the same numbering (but with accents) as the similar results above found for p]. Noting that here

$$t_0^2 \pm q . u_0^2 \equiv 0 \pmod{P, \textit{i.e.}, \bmod \textit{every } p_r},$$

hence, as in (19) of Art. 4, $(\mp q/p_r)_4 = (t_0 u_0/p_r)_2$, for *every* p_r.

Multiplying all such results together gives

$$(\mp q/P)_4 = (t_0 u_0/P)_2. \qquad (19a', b')$$

Next, writing $t_0 = 2^{\tau_0} T_0$, $u_0 = 2^{\nu_0} U_0$, (21′)

the reasoning of Art. 5 leads to

$$(\mp q/P)_4 = (2^{\tau_0+\nu_0}/P)_2 . (\pm q/T_0)_2 \qquad (24a', b')$$

and

$$(q/P)_4 = \theta_0 . (q/T_0)_2. \qquad (25')$$

Again, writing, as in (32),

$$T_0 = t_0 \text{ (when \textit{odd})} = 2q . \mu_0 \pm r_0 \text{ or } r'_0, \qquad (32')$$

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* The possibility of this generalization was pointed out to the author by Mr. A. E. Western.

the reasoning of Art. 7 leads to the *canonical forms*

$$P = t_0^2+q\,.\,u_0^2 \text{ gives } (q/P)_4 = (q/T)_2 = (-1)^{\mu_0}, \qquad (31a', 36')$$

$$4P = t_0^2+q\,.\,u_0^2 \text{ gives } (\bar{q}/P)_4 = (q/T)_2 = (-1)^{\mu_0}. \qquad (31b', 36')$$

Similarly, modifying ϑ_1, ϑ_1' of (49) by writing τ_0, v_0, T_0, P for τ, v, T, p, so as to obtain the *new* ϑ_1, ϑ_1' suited to P, the results (50*a*, *b*, *c*, *d*) yield

$$m^2p = t^2 \pm qu^2 \text{ gives } (\pm q/P)_4 = \vartheta_1\,.\,(t_0/q)_2, \text{ when } q = 4\kappa \pm 1, \quad (50a', b')$$

$$m^2p = t^2 \mp qu^2 \text{ gives } (\pm q/P)_4 = \vartheta_1'\,.\,(t_0/q)_2, \text{ when } q = 4\kappa \pm 1. \quad (50c', d')$$

The results of this article enable $(q/P)_4$ to be reduced *direct* from its 2-ic form (59) without computing the separate quantities $(q/P_r)_4$ each from its own form (54); so that the theorem of this article is of considerable importance.

Ex.—Given $P = C^2+2D^2$, where $P = \Pi(p_r)$ and each $p_r = c^2+2d^2$. Then $C = 4\Gamma \pm 1$, and each $c_r = 4\gamma_r \pm 1$.

By (32) and (32') $C = T_0$, $\mu_0 = \Gamma$, $c_r = T_r$, $\mu_r = \gamma_r$.

Hence, by (36) and (36'), $(2/P)_4 = (-1)^{\Gamma}$, $(2/p_r)_4{}^* = (-1)^{\gamma_r}$, (60)

and it is seen that the results for $(2/P)_4$, $(2/p_r)_4$ are of *similar form*.

[Compare the corresponding results in the 2-ic theory; these are also of *same form*

$$(2/P)_2 = (-1)^{\frac{1}{8}(P^2-1)} \text{ and } (2/p_r)_2 = (-1)^{\frac{1}{8}(p_r^2-1)}.]$$

17. *Tables.*—Here follow tables of data and results for testing the formulæ of this paper. The tables give the quadratic partitions (where possible) of all primes $p < 250$ as follows:—

Table I. $p = t^2 \pm qu^2$ for $q = 5, 13, 17;\ 3, 7, 11;\ 2, 6, 10.$

Table II. $4p = t^2 - qu^2$ for $q = 5$; $4p = t^2 + qu^2$ for $q = 3, 11.$

[The Ω at head of third column denotes the largest *odd* factor in $(p-1)$, so that $(p-1) + \Omega$ denotes the power of 2 contained in $(p-1)$.]

The tables are divided into *two* parts by a thick vertical rule: the left half-table contains the data of primes $p < 250$ for which $(q/p)_4 = +1$; the right half-table contains those for which $(q/p)_4 = -1$. The ($\pm$) signs of $(q/T)_2$, $(t/q)_2$, $(\bar{q}/p)_4$ are shown with each prime: this last result $(\bar{q}/p)_4$ affords an easy test of the results (27*b*, *d*), (27*e*).

[When the values of t, u in the formulæ $p = \frac{1}{4}(t^2 \pm qu^2)$ are *both even*, the results $(q/T)_2$, $(t/q)_2$, $(\bar{q}/p)_4$ are *not given*, because [see (29*c*)] all the results are then more easily obtained from the form $p = (\frac{1}{2}t)^2 \pm q(\frac{1}{2}u)^2$.]

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[*] This is Gauss's criterion for $(2/p)_4$.

q	$(q/p)_4 = +1$									$(q/p)_4 = -1$								
	p	$\frac{p-1}{\Omega}$	(t^2+qu^2)			(t^2-qu^2)			$(\bar{q}/p)_4$	p	$\frac{p-1}{\Omega}$	(t^2+qu^2)			(t^2-qu^2)			$(\bar{q}/p)_4$
			t, u	(q/T)	(t/q)	t, u	(q/T)	(t/q)				t, u	(q/T)	(t/q)	t, u	(q/T)	(t/q)	
5	101	4	9, 2	+,	+	11, 2	+,	+	−	41	8	6, 1	−,	+	11, 4	+,	+	−
	181	4	1, 6	+,	+	19, 6	+,	+	−	61	4	4, 3	+,	+	9, 2	+,	+	+
										241	16	14, 3	−,	+	31, 12	+,	+	−
	109	4	8, 3	+,	−	17, 6	−,	−	−	29	4	3, 2	−,	−	7, 2	−,	−	+
	149	4	12, 1	−,	−	13, 2	−,	−	−	89	8	3, 4	−,	−	13, 4	−,	−	−
										229	4	7, 6	−,	−	27, 10	−,	−	+
13	17	16	2, 1	+.	−	15, 4	−,	−	+	29	4	4, 1	+,	+	9, 2	+,	+	+
	53	4	1, 2	+,	+	51, 14	+,	+	−	101	4	7, 2	−,	−	123, 34	−,	−	+
	61	4	3, 2	+,	+	23, 6	+,	+	−	113	16	10, 1	−,	+	159, 44	+,	+	−
	181	4	8, 3	+,	−	109, 30	−,	−	−	157	4	12, 1	+,	+	25, 6	+,	+	+
										173	4	11, 2	−,	−	15, 2	−,	−	+
										233	8	5, 4	−,	−	21, 4	−,	−	−
17	149	4	9, 2	+,	+	43, 10	+,	+	−	53	4	6, 1	−,	−	11, 2	−,	−	+
	157	4	2, 3	+,	+	15, 2	+,	+	−	89	8	., .	.,	.	19, 4	+,	+	−
	229	.	., .	.,	.	29, 6	−,	−	−	101	4	., .	.,	.	13, 2	+,	+	+
										137	8	., .	.,	.	35, 8	+,	+	−
3	13	4	1, 2	+,	+	4, 1	+,	+	−	37	4	5, 2	−,	−	7, 2	−,	+	+
	109	4	1, 6	+,	+	11, 2	+,	−	−	61	4	7, 2	−,	+	8, 1	+,	−	+
	181	4	13, 2	+,	+	16, 5	+,	+	−	73	8	5, 4	−,	−	10, 3	−,	+	−
	193	64	1, 8	+,	+	14, 1	−,	−	+	97	32	7, 4	−,	+	10, 1	−,	+	−
	229	4	11, 6	+,	−	16, 3	+,	+	−	157	4	7, 6	−,	+	13, 2	+,	+	+
										241	16	7, 8	−,	+	17, 4	−,	−	−
7	29	4	1, 2	+,	−	6, 1	+,	−	−	53	4	5, 2	−,	−	9, 2	+,	+	+
	37	4	3, 2	+,	−	10, 3	−,	−	−	137	8	5, 4	−,	−	12, 1	+,	−	−
	109	4	9, 2	+,	+	19, 6	+,	−	−	149	4	11, 2	−,	+	18, 5	+,	+	+
	113	16	1, 4	+,	+	15, 4	−,	+	+	197	4	13, 2	−,	−	15, 2	−,	+	+
	193	64	9, 4	+,	+	16, 3	+,	+	+	233	8	11, 4	−,	+	24, 7	+,	−	−
11	.	.	., .	.,	.	., .	.,	.	.	53	4	3, 2	−,	+	8, 1	+,	−	+
2	73	8	1, 6	+,	.	9, 2	+,	.	+	17	16	3, 2	−,	.	5, 2	−,	.	−
	89	8	9, 2	+,	.	11, 4	−,	.	+	41	8	3, 4	−,	.	7, 2	+,	.	−
	113	16	9, 4	+,	.	11, 2	−,	.	+	97	32	5, 6	−,	.	13, 6	−,	.	−
	233	8	15, 2	+,	.	19, 8	−,	.	+	137	8	3, 8	−,	.	13, 4	−,	.	−
										193	64	11, 6	−,	.	15, 4	+,	.	−
										241	8	13, 6	−,	.	21, 10	−,	.	−
6	97	32	1, 4	+,	.	11, 2	−,	.	+	73	8	7, 2	−,	.	13, 4	−,	.	−
	241	16	5, 6	+,	.	25, 8	+,	.	+									
10	41	8	1, 2	+,	.	9, 2	+,	.	+	89	8	7, 2	−,	.	27, 8	+,	.	−
	241	16	9, 4	+,	.	41, 12	+,	.	+									

q	$(q/p)_4 = +1$						
	p	$\frac{p-1}{\Omega}$	$\frac{1}{4}(t^2+qu^2)$		$\frac{1}{4}(t^2-qu^2)$		$(\overline{q/p})_4$
			t, u	(q/T) (t/q)	t, u	(q/T) (t/q)	
5	101	4	., .	., .	22, 4		
	181	4	., .	., .	27, 1	−, −	−
	109	4	., .	., .	21, 1	+, +	−
	149	4	., .	., .	26, 4	., .	.
3	13	4	5, 3	−, −	., .	., .	−
	109	4	2, 12	., .	., .	., .	.
	181	4	7, 15	−, +	., .	., .	−
	193	64	23, 9	+, −	., .	., .	+
	229	4	22, 12	., .	., .	., .	.
11	89	8	9, 5	+, +	., .	., .	+
	157	4	23, 3	−, +	., .	., .	−

q	$(q/p)_4 = -1$						
	p	$\frac{p-1}{\Omega}$	$\frac{1}{4}(t^2+qu^2)$		$\frac{1}{4}(t^2-qu^2)$		$(\overline{q/p})_4$
			t, u	(q/T) (t/q)	t, u	(q/T) (t/q)	
5	41	8	., .	., .	13, 1	−, −	−
	61	4	., .	., .	17, 3	−, −	+
	241	16	., .	., .	33, 5	−, −	−
	29	4	., .	., .	11, 1	+, +	+
	89	8	., .	., .	19, 1	+, +	−
	229	4	., .	., .	31, 3	+, +	+
3	37	4	11, 3	+, −	., .	., .	+
	61	4	1, 9	+, +	., .	., .	+
	73	8	7, 9	−, +	., .	., .	−
	97	32	19, 3	−, +	., .	., .	−
	157	4	14, 12	., .	., .	., .	.
	241	16	17, 15	−, −	., .	., .	−
11	37	4	7, 3	+, −	., .	., .	+
	53	4	6, 4	., .	., .	., .	.
	113	16	21, 1	−, −	., .	., .	−
	137	8	3, 7	−, +	., .	., .	−
	181	4	25, 3	+, +	., .	., .	+
	229	4	5, 9	+, +	., .	., .	+

EXTENSION OF TWO THEOREMS ON COVARIANTS

By J. H. GRACE.

[Communicated March 12th, 1903.—Received March 18th, 1903.]

1. Two of the best known elementary results in invariant algebra are the following:—

(i.) The Jacobian of a Jacobian of two binary forms with a third form is reducible.

(ii.) The product of two Jacobians can be expressed as an aggregate of products each containing three factors.

Involved in each statement is the condition that all the forms considered are of order 2 at least; accordingly, in extending the theorems I shall deal with perpetuants, and here content myself with the remark that the corresponding limitation for the extended theorems is easily discovered.

The first theorem has been extended by Jordan,* but, perpetuants not being considered by him, the extended theorem is not in its simplest form.

2. Consider now the first of the two results. Suppressing factors of the type a_x, it is equivalent to saying that a symbolical product of the form $(ab)(ac)$ is reducible, or, in other words, that, if an irreducible covariant contain three symbols, it must have at least three factors of the type (ab).

The extension is now clear, for it is known that a product containing ι symbols is reducible unless it contains at least $2^{\iota-1}-1$ factors of the determinantal type.

But the Jacobian theorem may be stated in a more general form, viz., if a product contain any number of symbols divided into three sets, and the sum of the exponents of factors containing two symbols belonging to different sets be less than three, then the form is reducible.

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* See *Liouville*, 1879. Jordan's results in this connection are correct for forms of order not greater than 12, but after this point his upper limit to the order of an irreducible form is too large. I do not know of a case in which the highest order of an irreducible covariant of a system of binary forms is not given by choosing the greatest of the integers n, $2n-2$, $3n-6$, $4n-14$, $5n-30$, $6n-62$, ... where n is the greatest order of forms in the system. I have examined the cases where $n \not> 30$, and Mr. A. P. Thompson has, I believe, gone considerably further.

More generally, if there be ι sets of symbols, and the number of factors containing two symbols belonging to different sets be less than $2^{\iota-1}-1$, then the product is a reducible one.

3. To make this point clear it is convenient to introduce the idea of generalized transvectants. The r-th transvectant of $f = a_x^m$, $\phi = b_x^n$ is $(ab)^r a_x^{m-r} b_x^{n-r}$, that is to say, a numerical multiple of

$$\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}\right)^r f_x \phi_y,$$

y being replaced by x after the operations are performed.

The latter form of definition is applicable when f and ϕ are symbolical products, and we have the fundamental theorem that any term in a transvectant can be expressed as an aggregate of transvectants, of index equal to or less than that of the original transvectant, of forms derived from the original forms by convolution. To extend this idea we note that the covariant $(bc)^\lambda (ca)^\mu (ab)^\nu a_x^{l-\mu-\nu} b_x^{m-\nu-\lambda} c_x^{n-\lambda-\mu}$ of a_x^l, b_x^m, c_x^n differs only numerically from

$$\left(\frac{\partial^2}{\partial y_1 \partial z_2} - \frac{\partial^2}{\partial z_1 \partial y_2}\right)^\lambda \left(\frac{\partial^2}{\partial z_1 \partial x_2} - \frac{\partial^2}{\partial x_1 \partial z_2}\right)^\mu \left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y}\right)^\nu a_x^l b_y^m c_z^n,$$

y and z being replaced by x in the result.

This latter I call the generalized transvectant of a_x^l, b_x^m, c_x^n, having indices λ, μ, ν; the definition is applicable even when the forms are symbolical products.

In this way we can define a generalized transvectant of any number of forms, and, following the lines of the proof of the ordinary transvectant theorem quoted above, it is easy to show that any term in a transvectant can be expressed as an aggregate of transvectants, having indices equal to or less than the corresponding indices of the original transvectant, of forms derived from the original form by convolution.

4. Now suppose that P is a symbolical product containing ι sets of symbols, that P_1 is the product of all the factors containing only letters of the first set, P_2 the product of those containing only letters of the second set, &c., and let factors of the type a_x be suppressed.

Then, if the number of factors containing letters belonging to different sets be w, it is clear that P is a term in a generalized transvectant of $P_1, P_2, \ldots, P_\iota$, the sum of the indices being w.

Hence P can be expressed as an aggregate of transvectants of $\overline{P}_1, \overline{P}_2, \ldots, P_r, \ldots, \overline{P}_\iota$, where $\overline{P}_r$ is derived by convolution from P_r, and in each such transvectant the sum of the indices is equal to or less than w.

But, if $w < 2^{\iota-1}-1$, each of these transvectants is reducible and not conventionally so, but actually expressible in terms of forms of lower degree.

It follows at once that P is reducible. For example, consider the product $(ab)(bc)^9(cd)^3(de)^2$; it contains 15 factors, and, as 15 is the least possible number for an irreducible product containing five letters, this is at first sight a stable product, and, although in the complete system we should express it in terms of $(ab)^8(bc)^4(cd)^2(de)$ and reducible forms, this is only a conventional reduction. On the other hand, if we divide the letters into the four sets a; b, c; d; e, the sum of the exponents of factors containing letters of different sets is only 6, which is less than the minimum for irreducibility; hence the form in question can be completely expressed in terms of forms of lower degree.

5. We now proceed to the extension of the theorem relating to the product of two Jacobians.

Consider the product of $(ab)^2(bc)^2$ and (de), both of which are irreducible: it is

$$(ab)^2(bc)^2(de) = (ab)^2(bc)^2(ce)-(ab)^2(bc)^2(cd).$$

Now, in the product $(ab)^2(bc)^2(ce)$, d no longer explicitly occurs; therefore the corresponding form is a factor, and further, since there are only five factors and four letters, it follows that $(ab)^2(bc)^2(ce)$ can be expressed in terms of forms of lower degree.

Hence the product of the irreducible forms $(ab)^2(bc)^2$ and (de) can be expressed as an aggregate of products each containing three factors, and one of these factors is the form d or the form e.

Generally consider the product of P and $(a\beta)$, when P contains ι letters and w factors, so that, if P be irreducible, we must have

$$w \geqslant 2^{\iota-1}-1.$$

Using just the same argument as above, it follows that, if $w < 2^{\iota}-1$, the product is expressible as an aggregate of products each containing either the form a or the form β and two other factors.

This is the extension contemplated; it is easy to construct examples, but rather tedious to calculate the right-hand side of the resulting syzygy.

[*Added October* 11*th*, 1903.—The formula for the maximum order of an irreducible covariant given on p. 153 is rigorously established in a paper of Mr. A. Young's to be published shortly in these *Proceedings*.]

THE FIELD OF FORCE DUE TO A MOVING ELECTRON

By A. W. CONWAY.

[Received January 30th, 1903.—Read February 12th, 1903.—Received in revised form May 6th, 1903.]

THE following paper treats of the field of force due to the motion of an electron. No assumption is made as to the nature or shape of an electron except to regard it as a simple singularity of the differential equations which is changing its position in three-dimensional space in any continuous manner whatever. The results will, however, be correct as long as the distance of the point at which the field of force is calculated is great compared with the dimensions of the electron. In § 1 the case of an electron moving in a general manner with a speed less than that of radiation is considered. In § 2 the results are extended to moving distributions of electric charge. Without making an hypothesis concerning the distribution of the charge of the electrical nucleus or electron it it is impossible to obtain the total amount of energy of the system, but the amount of energy radiated, or "wasted," is calculated in § 3. In § 4 the form which the results of § 1 assume when the speed of the singularity is greater than that of radiation is considered.

1. *The Electromagnetic Field due to an Electron moving with a Velocity less than that of Radiation.*

The differential equations which hold in any space free from singularities are, the origin being fixed in the æther,

$$K\frac{\partial}{\partial t}(X, Y, Z) = \operatorname{curl}(\alpha, \beta, \gamma), \qquad -\mu\frac{\partial}{\partial t}(\alpha, \beta, \gamma) = \operatorname{curl}(X, Y, Z)$$

where (X, Y, Z) is the electric force in electromagnetic units, (α, β, γ) is the magnetic force, K is the specific inductive capacity, and μ is the magnetic permeability.

From the above equations it follows that (X, Y, Z) is circuital and that X, Y, and Z are solutions of $\nabla^2 U = V^{-2}\partial^2 U/\partial t^2$ where $V^{-2} = K\mu$.

Let $f_1(u)$, $f_2(u)$, $f_3(u)$ be any functions of u which are real and uniform where u is real, and which are finite and continuous for all finite values of u real or complex. Further, let us suppose that for all real positive values of u less than a positive quantity t the condition

$$[f_1'(u)]^2+[f_2'(u)]^2+[f_3'(u)]^2 < V^2$$

is satisfied where $f_1'(u)$ denotes $\partial f_1(u)/\partial u$, ... This condition, as we shall see later, makes the velocity of the electron less than the velocity of radiation.

Consider first points which lie inside the sphere

$$[x-f_1(0)]^2+[y-f_2(0)]^2+[z-f_3(0)]^2 = V^2t^2.$$

The function

$$\phi(u) = [x-f_1(u)]^2+[y-f_2(u)]^2+[z-f_3(u)]^2 - V^2(t-u)^2$$

has for its first derived

$$\phi'(u) = -2f_1'(u)[x-f_1(u)]-2f_2'(u)[y-f_2(u)]-2f_3'(u)[z-f_3(u)]+2V^2(t-u).$$

Within the range $t > u > 0$ of real values of u, $\phi'(u)$ may have no zero or it may have one or more zeroes. If it has no zero, then, since $\phi(t)$ is positive (being a sum of real squares) and $\phi(0)$ is negative $[(x, y, z)$ lying inside the sphere $\phi(0) = 0]$, it follows that $\phi(u)$ has one, and only one, zero in the range considered. If $\phi'(u)$ has one or more zeroes, let a be one of them, so that $\phi'(a) = 0$. Making use of this, we can write $\phi(a)$ in the form

$$[x-f_1(a)]^2+[y-f_2(a)]^2+[z-f_3(a)]^2$$
$$-V^{-2}\{[x-f_1(a)]f_1'(a)+[y-f_2(a)]f_2'(a)+[z-f_3(a)]f_3'(a)\}^2.$$

If we put

$$\xi = l_1[x-f_1(a)]+m_1[y-f_2(a)]+n_1[z-f_3(a)],$$
$$\eta = l_2[x-f_1(a)]+m_2[y-f_2(a)]+n_2[z-f_3(a)],$$
$$\zeta = l_3[x-f_1(a)]+m_3[y-f_2(a)]+n_3[z-f_3(a)],$$

where the transformation is real and orthogonal and

$$l_1 : m_1 : n_1 = f_1'(a) : f_2'(a) : f_3'(a),$$

then $$\phi(a) = \xi^2(1-V^{-2}\{[f_1'(a)]^2+[f_2'(a)]^2+[f_3'(a)]^2\})+\eta^2+\zeta^2;$$

so that $\phi(a)$ is positive, unless $\xi = 0$, $\eta = 0$, $\zeta = 0$. As before, it follows that $\phi(u)$ has only one zero in the range $t > u > 0$. It also is evident that, if $\phi(a) = 0$ and $\phi'(a) = 0$, we must have $\xi = 0$, $\eta = 0$, $\zeta = 0$. These conditions lead to $x = f_1(a)$, $y = f_2(a)$, $z = f_3(a)$; and, since $\phi(a) = 0$, we get in this case $t = a$. Summing up then, we see that $\phi(u)$ has one, and only one, zero, T (say), between 0 and t, and that $\phi'(T)$ is not zero unless $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$. In this case T is a simple zero of $\phi'(T)$ and a double zero of $\phi(T)$.

The equation $\nabla^2 U = V^{-2}\partial^2 U/\partial t^2$ is satisfied by

$$\{[x-f_1(u)]^2+[y-f_2(u)]^2+[z-f_3(u)]^2-V^2(t-u)^2\}^{-1}$$

where u is independent of x, y, z, and t. It is also satisfied by any number of such solutions provided that the path of summation is independent of x, y, z, and t. Hence $\int\{\phi(u)\}^{-1}du$ is a solution where u is a complex variable, and the path of integration is taken to be a closed contour in the plane of u [to be replaced by the appropriate Riemann surface if $\phi(u)$ is non-uniform in this plane] enclosing no zero of $\phi(u)$ except T. In the same way $\int\frac{f_1'(u)\,du}{\phi(u)}$, $\int\frac{f_2'(u)\,du}{\phi(u)}$, $\int\frac{f_3'(u)\,du}{\phi(u)}$, or functions derived from them by differentiations, are all solutions of $\nabla^2 U = V^{-2}\partial^2 U/\partial t^2$. We may also notice that

$$\frac{\partial}{\partial t}\int\frac{du}{\phi(u)} = \int\frac{\partial}{\partial t}\frac{1}{\phi(u)}du = -\frac{\partial}{\partial x}\int\frac{f_1'(u)}{\phi(u)}du-\frac{\partial}{\partial y}\int\frac{f_2'(u)}{\phi(u)}du-\frac{\partial}{\partial z}\int\frac{f_3'(u)}{\phi(u)}du.$$

Hence

$$\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\int\frac{du}{\phi(u)}+V^{-2}\frac{\partial}{\partial t}\int\frac{f_1'(u)}{\phi(u)}du\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\int\frac{du}{\phi(u)}+V^{-2}\frac{\partial}{\partial t}\int\frac{f_2'(u)}{\phi(u)}du\right\}$$
$$+\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z}\int\frac{du}{\phi(u)}+V^{-2}\frac{\partial}{\partial t}\int\frac{f_3'(u)}{\phi(u)}du\right\} = 0. \tag{1}$$

We shall now consider the electric force given by

$$\left.\begin{aligned} KX &= \frac{\partial}{\partial x}\int\frac{ieV\,du}{\pi\phi(u)}+V^{-1}\frac{\partial}{\partial t}\int\frac{ief_1'(u)\,du}{\pi\phi(u)},\\ KY &= \frac{\partial}{\partial y}\int\frac{ieV\,du}{\pi\phi(u)}+V^{-1}\frac{\partial}{\partial t}\int\frac{ief_2'(u)\,du}{\pi\phi(u)},\\ KZ &= \frac{\partial}{\partial z}\int\frac{ieV\,du}{\pi\phi(u)}+V^{-1}\frac{\partial}{\partial t}\int\frac{ief_3'(u)\,du}{\pi\phi(u)},\end{aligned}\right\} \tag{2}$$

where e is a constant and $i = \sqrt{-1}$.

From what has been proved above they are all solutions of

$$\nabla^2 U = V^{-2}\frac{\partial^2 U}{\partial t^2},$$

and, by (1), they satisfy $\partial X/\partial x+\partial Y/\partial y+\partial Z/\partial z = 0$.

If $t > T > 0$, the electric force at any point of the sphere

$$[x-f_1(T)]^2+[y-f_2(T)]^2+[z-f_3(T)]^2 = V^2(t-T)^2$$

is given by

$$\left.\begin{aligned} KX &= -2eV\frac{\partial}{\partial x}\frac{1}{\phi'(T)} - 2eV^{-1}\frac{\partial}{\partial t}\frac{f_1'(T)}{\phi'(T)}, \\ KY &= -2eV\frac{\partial}{\partial y}\frac{1}{\phi'(T)} - 2eV^{-1}\frac{\partial}{\partial t}\frac{f_2'(T)}{\phi'(T)}, \\ KZ &= -2eV\frac{\partial}{\partial z}\frac{1}{\phi'(T)} - 2eV^{-1}\frac{\partial}{\partial t}\frac{f_3'(T)}{\phi'(T)}. \end{aligned}\right\} \tag{3}$$

X, Y, and Z become infinite only when $\phi'(T) = 0$, which, as we have seen, can take place only when $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$. Then they become infinite to the same order as the components of force due to a point charge at rest. Hence the above expressions represent the field due to a simple singularity moving through the æther in a general manner, its position at a time t being given by the co-ordinates $f_1(t)$, $f_2(t)$, $f_3(t)$. It remains to be shown that the strength of the charge associated with the electron remains constant. This we shall prove by evaluating the surface integral of the normal induction over the sphere $\phi(T) = 0$. Remembering that T is a function of x, y, z, and t, we get

$$\begin{aligned} &\iint K(lX+mY+nZ)\,dS \\ &= 4e[V^2 - \{f_1'(T)\}^2 - \{f_2'(T)\}^2 - \{f_3'(T)\}^2]\iint\frac{dS}{[\phi'(T)]^2} \\ &= 2\pi e[V^2 - \{f_1'(T)\}^2 - \{f_2'(T)\}^2 - \{f_3'(T)\}^2] \\ &\qquad\times\int_{-V(t-T)}^{V(t-T)}[\xi\sqrt{\{f_1'(T)\}^2+\{f_2'(T)\}^2+\{f_3'(T)\}^2} - V^2(t-T)]^{-2}\,d\xi = 4\pi e. \end{aligned}$$

Hence the strength of the electron is constant and equal to e.

From a physical point of view the above analysis shows that the state of the medium at any point depends only on the position and motion of the electron at a time $(t-T)$ previously where T is the real root lying between 0 and t of the equation

$$[x-f_1(T)]^2+[y-f_2(T)]^2+[z-f_3(T)]^2 = V^2(t-T)^2.$$

The electron takes a time T to get from its initial position to the point $f_1(T)$, $f_2(T)$, $f_3(T)$, and it is then producing an æthereal disturbance which after a further time $(t-T)$ reaches every point on the sphere $\phi(T) = 0$. The fact that this sphere is of radius $V(t-T)$ is a direct proof of the fact that these disturbances travel with velocity V.

The condition $\{f_1'(t)\}^2+\{f_2'(t)\}^2+\{f_3'(t)\}^2 < V^2$ means, of course, that the velocity of the electron is less than the velocity of radiation. The theorem proved above that T was a unique zero of $\phi(u) = 0$ shows that there is only one position of the electron which is responsible for the disturbance at any point at any time. Our results also, from their method of investigation, can be applied only to points inside the sphere $\phi(0) = 0$. What goes on outside this can, of course, be calculated from a knowledge of the values of the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ for negative values of t.

The components of electric force can be written in the form

$$KX = -\frac{\partial\psi}{\partial x} - \frac{\partial F}{\partial t}, \quad KY = -\frac{\partial\psi}{\partial y} - \frac{\partial G}{\partial t}, \quad KZ = -\frac{\partial\psi}{\partial z} - \frac{\partial H}{\partial t}, \tag{4}$$

where $$\psi = \frac{2eV}{\phi'(T)}, \quad F = \frac{2e}{V}\frac{f_1'(T)}{\phi'(T)}, \quad \ldots.$$

We can, hence, easily deduce the components of the magnetic force thus:—

$$a = V^2\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right) = 2eV\left(\frac{\partial}{\partial y}\frac{f_3'(T)}{\phi'(T)} - \frac{\partial}{\partial z}\frac{f_2'(T)}{\phi'(T)}\right). \tag{5}$$

It may be noticed that the expressions given above depend on both the velocity and the acceleration of the electron. Hence the expressions become discontinuous over a certain sphere $\phi(t) = 0$ if the acceleration at the time t becomes discontinuous: for instance, if the electron which has been moving with a uniform velocity begins to move with an acceleration, the sphere $\phi(t) = 0$, every point of which is moving outwards with velocity V, is then a surface of discontinuity. The necessary conditions which must be satisfied at such a surface, if it be free from electrical charge, are (*a*) that the normal electric force must be continuous, and (*b*) that the difference between the tangential components of the electrical force should be equal to the vector product of the difference of the tangential components of the magnetic induction and the normal velocity of the surface.* For the components given above it will be found that both these conditions are fulfilled. In fact the electric force given by (3) has the component of the electric force normal to the sphere $\phi(T) = 0$ independent of the acceleration, whilst the tangential component of the electric force is at right angles to the tangential component of the magnetic induction and equal to it multiplied by V, the directions being related to one another in the manner required by condition (*b*).

The equations (2), (3), and (5) give then a complete solution for the field due to a singularity moving in any continuous manner. In the form (3) and (5) they are suitable for numerical calculation; in any given case it is only necessary to calculate a root of a given equation. For the explicit algebraical expression of our results it is necessary to be able to solve the equation $\phi(T) = 0$. We can do this completely if the electron is at rest or moving with uniform velocity. We can also solve the case of an electron moving with a uniform acceleration. To do so we should require the solution of a biquadratic equation, and the results would be too complicated to be of any advantage. There are a few other cases in which the field of force can be obtained at particular points. We now proceed to consider these examples.

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* Heaviside, *Electromagnetic Theory*, Vol. I., p. 56.

For the case of an electron at rest at the point (x', y', z'), the equation $\phi(T) = 0$ becomes $(x-x')^2+(y-y')^2+(z-z')^2 = V^2(t-T)^2$; so that

$$T = t - V^{-1}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

and $$\phi'(T) = 2V^2(t-T) = 2V\sqrt{(x-x')^2+(y-y')^2+(z-z')^2};$$

so that, by (4), $\psi = e/\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$; $F = G = H = 0$.

For the case of a point charge moving along the axis of z with uniform velocity v, the equation $\phi(T) = 0$ becomes $x^2+y^2+(z-vT)^2 = V^2(t-T)^2$; so that

$$T = \frac{-vz+V^2t-\sqrt{(V^2-v^2)(x^2+y^2)+V^2(z-vt)^2}}{V^2-v^2},$$

giving $$\psi = \frac{eV}{\{(V^2-v^2)(x^2+y^2)+V^2(z-vt)^2\}^{\frac{1}{2}}}$$

and $$F = G = 0; \quad H = \frac{veV}{\{(V^2-v^2)(x^2+y^2)+V^2(z-vt)^2\}^{\frac{1}{2}}},$$

which lead to the well known solutions for this case.*

If we have an electron moving in a circle, we can find the forces at any point of the axis. For let the position of the electron be denoted by the coordinates $f_1(t)$, $f_2(t)$, 0 where $\{f_1(t)\}^2+\{f_2(t)\}^2 = R^2$, a constant; then for any point $(0, 0, z)$ on the axis of z we have

$$\phi(T) \equiv \{f_1(T)\}^2+\{f_2(T)\}^2+z^2-V^2(t-T)^2 = 0$$

or $R^2+z^2-V^2(t-T)^2 = 0$, so that $T = t-\dfrac{\sqrt{R^2+z^2}}{V}$, giving

$$KX = -\frac{ef_1(T)}{(R^2+z^2)^{\frac{3}{2}}} - \frac{e}{V^2(R^2+z^2)}\left[\sqrt{R^2+z^2}f_1''(T)+Vf_1'(T)\right],$$

$$KY = -\frac{ef_2(T)}{(R^2+z^2)^{\frac{3}{2}}} - \frac{e}{V^2(R^2+z^2)}\left[\sqrt{R^2+z^2}f_2''(T)+Vf_2'(T)\right],$$

$$KZ = \frac{ez}{(R^2+z^2)^{\frac{3}{2}}},$$

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* J. J. Thomson, *Recent Researches*, p. 18.

$$\alpha = \frac{ez}{V(R^2+z^2)^{\frac{3}{2}}}\left[\sqrt{R^2+z^2}\,f_2''(T)+Vf_2'(T)\right],$$

$$\beta = \frac{ez}{V(R^2+z^2)^{\frac{3}{2}}}\left[\sqrt{R^2+z^2}\,f_1''(T)+Vf_1'(T)\right],$$

$$\gamma = \frac{e}{V(R^2+z^2)^{\frac{3}{2}}}\left[R\{f_1(T)f_2''(T)-f_2(T)f_1''(T)\}\right.$$
$$\left.+V\{f_1(T)f_2'(T)-f_2(T)f_1'(T)\}\right].$$

More generally we can find the force at any point of the axis of a cylinder if an electron is moving on its surface, the velocity resolved along the axis being constant. A similar investigation gives the forces at the centre of a sphere due to an electron moving in any manner on its surface.

If we consider an electron of strength $\frac{1}{2}e$ at the point $[f_1(t), f_2(t), f_3(t)]$ in conjunction with another of strength $-\frac{1}{2}e$ at the point $[-f_1(t), -f_2(t), -f_3(t)]$, we find, on the supposition that the squares and products of $f_1(t)$, $f_1'(t)$, ..., may be neglected,

$$KX = ex\left\{\frac{xf_1''(T)+yf_2''(T)+zf_3''(T)}{V^2r^3}-\frac{3}{V}\,\frac{xf_1'(T)+yf_2'(T)+zf_3'(T)}{r^4}\right.$$
$$\left.+3\,\frac{xf_1(T)+yf_2(T)+zf_3(T)}{r^5}\right\}-\frac{ef_1(T)}{r^3}+\frac{ef_1'(T)}{Vr^2}-\frac{ef_1''(T)}{V^2r},$$

where
$$T = t-\frac{\sqrt{x^2+y^2+z^2}}{V},$$

with similar values for Y and Z. This gives us the solution for various kinds of vibrators of the Hertzian type by taking the moment to be represented by the vector $ef_1(T)$, $ef_2(T)$, $ef_3(T)$.*

In all the above analysis the method consisted in making a contour integration about a certain zero of the function $\phi(u)$, so that the solutions were singular only at one point, and the medium contained the past history of the motion of the electron, the disturbance being propagated in diverging waves. It is interesting to observe that, if by an extension of our analysis we had selected in place of the zero T the real zero which is greater than t, we should then have obtained solutions infinite only at one point, which is the point of *convergence* of the disturbances, and the medium now contains the *future* history of the motion of the electron. For the case of an electron at rest or in uniform motion, these solutions, as we might expect, are not different from one another.

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* Cf. Macdonald, *Electric Waves*, p. 176. For the reverse process—passing from the solutions of Hertz to the case of an electron vibrating over a small path—see Larmor, *Æther and Matter*, pp. 221 *et seq.*; Righi, *Archives Néerlandaises*, Ser. 2, Tome v., p. 348.

THE LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT BY THE TREASURER FOR THE YEAR 1902–03 (NOV. 13TH, 1902, TO NOV. 12TH, 1903).

Cr. *GENERAL CASH ACCOUNT OF THE SOCIETY.* Dr.

		£.	s.	d.		£.	s.	d.
Balance from 1901–02 :—					Printing *Proceedings* to end of Vol. 35, &c. ...	294	11	3
General Fund	£125 1 9				Purchase of £104. 0*s*. 7*d*. India 3 per Cent.			

STATEMENT OF CAPITAL, November, 1903.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment
1. Lieut.-Col. Campbell's Fund... ...	499	7	5 ...	... 385	0	0	Guaranteed 4 per Cent. London & North Western Railway Consolidated Stock.
2. Life Composition Fund, from 1901–2	1687	0	0	1716	3	3	New South Wales 3½ per Cent. Inscribed Stock of 1918.
Amount received during 1902–3	10	10	0				
				10	10	0	In Bank awaiting investment.
3. Invested Surplus Fund...	700	0	0 ...	... 700	7	1	India 3 per Cent. Stock.
Trust Funds.							
1. Lord Rayleigh's Fund	1000	0	0 ...	... 200	0	0	Guaranteed 3 per Cent Great Indian Peninsula Railway Stock.
							Terminable Annuity of £50. 8*s*. 8*d*. (Class B, deduction for Sinking Fund) of same Railway Co.
2. De Morgan Medal Fund	102	5	9 ..	... 60	0	0	Great Western Railway 5 per Cent. Preference Stock.

Examined and found correct, all my requirements as Auditor having been complied with.

Dec. 14*th*, 1903.

(Signed) J. G. LEATHEM.

2. *On Moving Distributions of Electricity.*

If we have any distribution of electrons which on the average amount to a continuous volume, surface, or line distribution of electric charge, we can calculate the resulting field of force from a scalar potential

$$\psi = \int \frac{2V dm}{\phi'(T)},$$

where dm is an element of charge, and from a vector potential (F, G, H), where

$$F = \int \frac{2f_1'(T)}{V\phi'(T)}\,dm, \;\ldots,$$

after the manner of (4) and (5).

As a single example we consider the case of a continuous stream of electrons amounting to a constant line density λ travelling uniformly in a circle. We can then take

$$f_1(t) = a\cos(pt+\theta), \qquad f_2(t) = a\sin(pt+\theta),$$
$$f_3(t) = 0, \qquad dm = a\lambda\, d\theta.$$

In evaluating ψ it will be better, instead of using $2/\phi'(T)$, to replace it by the complex integral used in (2), viz., $\int ieV\pi^{-1}\{\phi(u)\}^{-1}du$. In this case

$$\begin{aligned}\phi(u) &\equiv [x-f_1(u)]^2+[y-f_2(u)]^2+[z-f_3(u)]^2-V^2(t-u)^2\\ &\equiv x^2+y^2+z^2+a^2-V^2(t-u)^2-2xa\cos(pu+\theta)-2ya\sin(pu+\theta).\end{aligned}$$

Hence
$$\psi = -\int dm\int\frac{iV}{\pi}\,\frac{du}{\phi(u)}. \tag{6}$$

The complex integral is supposed to be taken over a contour which encloses a special zero T of $\phi(u)$. These special zeros of $\phi(u)$ for all values of θ are, from their nature, confined to the part of the axis of real quantities lying between 0 and t. If, then, we suppose the path of integration to enclose this region, we are able to change the order of the operations on the right-hand side of (6); hence

$$\psi = -\frac{iV}{\pi}\int du\int_0^{2\pi}\frac{a\lambda\, d\theta}{\phi(u)}.$$

If we write $x^2+y^2+z^2 = r^2$, $x^2+y^2 = \varpi^2$, we get

$$\begin{aligned}\int_0^{2\pi}\frac{a\lambda\, d\theta}{\phi(u)} &= \int_0^{2\pi} a\lambda\, d\theta\,\{r^2+a^2-V^2(t-u)^2-2ax\cos(pu+\theta)\\ &\qquad\qquad -2ay\sin(pu+\theta)\}^{-1}\\ &= \pi a\lambda\,\{[r^2+a^2-V^2(t-u)^2]^2-4\varpi^2a^2\}^{-\frac{1}{2}},\end{aligned}$$

so that
$$\psi = -iVa\lambda\int du\,\{[r^2+a^2-V^2(t-u)^2]^2-4\varpi^2a^2\}^{-\frac{1}{2}}$$

or, changing to real variables,*

$$= \int_{\sqrt{(r^2+a^2-2\varpi a)}}^{\sqrt{(r^2+a^2+2\varpi a)}} a\lambda\, d\zeta\, \{[r^2+a^2-\zeta^2]^2-4\varpi^2a^2\}^{-\frac{1}{2}}.$$

Putting now $\zeta^2 = r^2+a^2-2\varpi a\cos\phi$, we get

$$\psi = \int_0^{\pi} \frac{a\lambda\, d\phi}{(r^2+a^2-2\varpi a\cos\phi)^{\frac{1}{2}}},$$

which is the ordinary electrostatic potential of a uniform ring at rest. In a similar manner we can show that the vector potential is the same as that due to a circular current of strength $pa\lambda$.

A solution of the equation $-V^{-2}\partial^2U/\partial t^2+\nabla^2U+4\pi\rho = 0$ appropriate for the time t is $\iiint r'^{-1}\rho'\, d\xi'\, d\eta'\, d\zeta'$, where

$$r'^2 = (x-\xi')^2+(y-\eta')^2+(z-\zeta')^2$$

and ρ' denotes the value of ρ at the point ξ', η', ζ', and the time $t-r'/V$. Starting from this fact, the solutions for any moving distribution of matter have been given by Macdonald.† We shall now show the connection between the solutions given by him and those given above.

The position of any electron is a function of its initial position and of the time, whilst the density at any part is a function only of the initial position of that part. If, then, ρ be the density at the point ξ, η, ζ, which was initially at the point ξ_0, η_0, ζ_0, we may write

$$\xi = f_1(\xi_0, \eta_0, \zeta_0, t), \qquad \eta = f_2(\ldots), \qquad \zeta = f_3(\ldots), \qquad \rho = F(\xi_0, \eta_0, \zeta_0).$$

In the formula $\psi = \int[\phi'(T)]^{-1}\, 2V\, dm$ we also require T. This is defined by the following equations :—

$$\xi' = f_1(\xi_0, \eta_0, \zeta_0, T), \quad \ldots, \quad (x-\xi')^2+(y-\eta')^2+(z-\zeta')^2 = V^2(t-T)^2. \tag{7}$$

Now $dm = \rho\, d\xi_0\, d\eta_0\, d\zeta_0$, and it is required to transform to the coordinates ξ', η', ζ'. By using such formulæ as

$$\frac{\partial\xi'}{\partial\xi_0} = \frac{\partial f_1}{\partial\xi_0} + \frac{\partial f_1}{\partial T}\frac{\partial T}{\partial\xi_0}$$

$$= \frac{\partial f_1}{\partial\xi_0} + V^{-2}(t-T)^{-1}\frac{\partial f_1}{\partial T}\left\{(x-\xi')\frac{\partial\xi'}{\partial\xi_0}+(y-\eta')\frac{\partial\eta'}{\partial\xi_0}+(z-\zeta')\frac{\partial\zeta'}{\partial\xi_0}\right\},$$

it can be verified without difficulty that

$$\partial(\xi_0, \eta_0, \zeta_0)/\partial(\xi', \eta', \zeta') = \tfrac{1}{2}V^{-1}[(x-\xi')^2+(y-\eta')^2+(z-\zeta')^2]^{-\frac{1}{2}}\,\phi'(T),$$

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* Cf. Forsyth, *Theory of Functions*, p. 193.

† *Electric Waves*, Appendix C. For other investigations from the same point of view, see papers by E. Wiechert, C. H. Wind, H. des Coudres, *Archives Néerlandaises*, Série 2, Tome v. ; T. Levi-Civita, *Nuovo Cimento*, Serie 4, Vol. vi. ; M. Abraham, *Annalen der Physik*, January, 1903.

and it follows that

$$\int 2[\phi'(T)]^{-1}V\,dm = \iiint \rho[(x-\xi')^2+(y-\eta')^2+(z-\zeta')^2]^{-\frac{1}{2}}\,d\xi'd\eta'd\zeta',$$

showing that the two solutions are identical. The difference between them is analogous to that between the two methods of establishing the fundamental equations of fluid motion known as the methods of Lagrange and of Euler. In the one method attention is directed to what goes on in volume elements fixed in the æther, and would be the most suitable for any theory which regards electricity as continuous. In the other method the motion of each corpuscle is followed and advantage is gained when we regard electricity as consisting of discrete electrons.

3. *On the Radiation from an Electron.*

If (in the notation of the preceding section) we consider the shell bounded by the spheres $\phi(T)=0$ and $\phi(T-dT)=0$, the energy in it consists of two parts: one becomes negligible when t becomes large (dT remaining the same size); the other remains finite. This second part is the energy radiated, and when divided by dT we get the rate at which energy is being radiated at the time T. We shall proceed to find this rate of waste for the case of an electron moving in any manner. The thickness of the shell at any point is

$$\tfrac{1}{2}V^{-1}\phi'(T)\,dT\,;$$

so that the required quantity is

$$\frac{1}{16\pi}\iint KV^{-1}[K(X^2+Y^2+Z^2)+\mu(\alpha^2+\beta^2+\gamma^2)]\,\phi'(T)\,dS,$$

the integration being taken over the sphere

$$[x-f_1(T)]^2+[y-f_2(T)]^2+[z-f_3(T)]^2 = V^2(t-T)^2$$

and t being made infinitely great. To simplify the integration we can take the point $[f_1(T), f_2(T), f_3(T)]$ to be origin and the axis of z the direction $[f_1'(T), f_2'(T), f_3'(T)]$. Then, putting

$$v^2=[f_1'(T)]^2+[f_2'(T)]^2+[f_3'(T)]^2, \qquad a^2=[f_1''(T)]^2+[f_2''(T)]^2+[f_3''(T)]^2,$$

α = the angle between v and a,

$$dS = V^2(t-T)^2\sin\theta\,d\theta\,d\phi,$$

θ being the angle made with the direction of the velocity and ϕ the azimuthal angle.

On integrating with respect to ϕ, we get

$$\frac{e^2a^2}{4K}\int_0^\pi \{-(V^2-v^2)(\sin^2 a \sin^2\theta+2\cos^2\theta\cos^2 a)+4v\cos^2 a\cos\theta(V-v\cos\theta)$$
$$+2(V-v\cos\theta)^2\}\,\frac{\sin\theta\,d\theta}{(V-v\cos\theta)^5},$$

which is equal to $$\tfrac{2}{3}\frac{Ve^2a^2}{K}\,\frac{V^2-v^2\sin^2 a}{(V^2-v^2)^3}.$$

If v^2 is negligible, this expression becomes $\frac{2}{3}e^2a^2/KV^3$ (or if $\mu = 1$) $= \frac{2}{3}e^2a^2/V$, which agrees with the rate of radiation given by Larmor* for an electron moving slowly. When v^2 cannot be neglected the result agrees with that given by Heaviside† on writing $Q^2/4\pi$ instead of e^2.

4. *On the Solutions of the Equations when the Velocity is greater than or equal to that of Radiation.*

In this section we consider briefly what the solutions (2) of § 1 become when the velocity of the singularity is not less than V. The integration is supposed to extend around a contour enclosing any real zeros of $\phi(u)$ between 0 and t. We have then to commence by discussing the real zeros of $\phi(u)$ between these limits. If a is a real zero of $\phi'(u) = 0$, then $\phi(a)$ will be negative, zero, or positive, according as the point (x, y, z) is inside, on, or outside the surface which is got by eliminating a between $\phi'(a) = 0$ and $\phi(a) = 0$. This surface, which is the envelope of the sphere $\phi(u) = 0$, was imaginary in § 1 with the exception of one point—the point at which the electron was. In this case it consists of a conical surface, the apex being the electron, and its end is closed by the sphere $\phi(0) = 0$. Leaving out of consideration the case in which this surface meets itself or the sphere again, we see that space is divided into three parts: (A) inside the sphere whose equation is $\phi(0) = 0$, (B) outside the sphere but inside the envelope, (C) outside both. In (A) $\phi(0)$ is negative, $\phi(t)$ positive, $\phi(a)$ negative. Hence there is *one* real zero of $\phi(u) = 0$. In (B) $\phi(0)$ is positive, $\phi(t)$ is positive, and $\phi(a)$ is negative; hence there are *two* real zeros of $\phi(u) = 0$. In (C) there is no real zero. That is to say, in (A) the state of the medium at any point is due to one position of the electron, but in (B) there are two positions. For example, if we take a singularity moving with uniform velocity v along the axis of z, the envelope will be a right circular cone of semi-angle $\sin^{-1} V/v$. In the region (B), *i.e.*, in the cone, the integrals vanish; for they enclose all the zeros of $\phi(u)$. Hence in this region there is no electric or magnetic force. Inside (A) the solutions are of the same form as when $v = V$, and on the boundary the solutions will be infinite.

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* *Æther and Matter*, p. 227.

† *Nature*, November 6th, 1902.

When the velocity $= V$, the result of substituting a zero a of $\phi'(a)$ in $\phi(a)$ is in general positive, but it will be zero if

$$\frac{x-f_1(a)}{f_1'(a)} = \frac{y-f_2(a)}{f_2'(a)} = \frac{z-f_3(a)}{f_3'(a)};$$

so that on any sphere $\phi(u) = 0$ there is one point at which $\phi'(u) = 0$. It is the point of the sphere to which the velocity at the time u is directed. With the exceptions of these points (which lie on an involute of the curve along which the singularity moves), the solution (2) holds inside the sphere $\phi(0) = 0$.

As to the singularities which occur in these solutions they must be regarded in the same way as those which are on a sphere due to the uniform motion in a straight line of an electron moving with the velocity of radiation,* *i.e.*, they are to be regarded as places in which the forces become infinitely great without giving any convergence.

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* Heaviside, *Electromagnetic Theory*.

ON SEQUENCES WHICH DETERMINE THE n^{TH} ROOT OF A RATIONAL NUMBER

By S. M. JACOB (*communicated by* PROF. M. J. M. HILL).

[Received and Read March 12th, 1903.—Received in revised form May 9th, 1903.]

1. Dedekind in his *Tract on Continuity and Irrational Numbers* gives a relation from which two sequences of rational numbers may be found, one sequence consisting of numbers whose square is less than a certain rational number D, and the other of numbers whose square is greater than D. The first sequence has no greatest number, and the second sequence has no least number, provided there is no rational number which has its square equal to D.

The two sequences together, or, as will afterwards be shown, either of them separately, define the irrational number which we call the square root of D.

Dedekind's formula for getting one member of a sequence from the preceding one is

$$y = \frac{x^3+3Dx}{3x^2+D}, \tag{I.}$$

which gives $y-x = \dfrac{2x(D-x^2)}{3x^2+D}$ and $y^2-D = \dfrac{(x^2-D)^3}{(3x^2+D)^2}$;

whence it follows that, if we take a member x of the lower sequence, such that $x^2 < D$, then we get another rational number y, such that $y^2 < D$ and $y > x$.

Similarly, if $x^2 > D$, it follows that $y^2 > D$ and $y < x$. Prof. M. J. M. Hill has observed that Dedekind's result (I.) can be obtained by writing out the binomial expansion for $(y-x)^3$, namely, $y^3-3y^2x+3yx^2-x^3$, substituting D for y^2 wherever it occurs, and equating to zero.

We then get $yD-3Dx+3yx^2-x^3 = 0$, which is identical with (I.). The first section of this paper deals with a set of formulæ which effect nothing more than Dedekind's formula (I.), but include it as a particular case. They were suggested by the above mentioned remark of Prof. Hill. The second section of the paper deals with a generalization of the formulæ when n-th powers are considered instead of squares.

SECTION I.

2. The results of this part of the paper were originally proved by a rather long process, which is now replaced by short proofs kindly placed at my disposal by Mr. A. E. Western. These proofs are given below. If we write down the binomial expansion for $(x-y)^n$, replace all even powers of y (y^{2r}) by powers of D (D^r) and all odd powers of y (y^{2r+1}) by the product of y and powers of D (yD^r), the equation $(x-y)^n=0$ then becomes a simple equation for y, viz.,

$$x^n-{}_nC_1x^{n-1}y+{}_nC_2x^{n-2}D+\ldots+{}_nC_{2r}x^{n-2r}D^r-{}_nC_{2r+1}x^{n-2r-1}D^ry+\ldots$$
$$+\begin{pmatrix}D^{\frac{1}{2}n} \text{ if } n \text{ be even} \\ yD^{\frac{1}{2}(n-1)} \text{ if } n \text{ be odd}\end{pmatrix}=0.$$

Putting $\quad y_1=x^n+{}_nC_2x^{n-2}D+\ldots+{}_nC_{2r}x^{n-2r}D^r+\ldots,$

$$y_2={}_nC_1x^{n-1}+{}_nC_3x^{n-3}D+\ldots+{}_nC_{2r+1}x^{n-2r+1}D^r+\ldots,$$

the equation becomes $y=y_1/y_2$. Now

$$y_1+y_2D^{\frac{1}{2}}=(x+D^{\frac{1}{2}})^n, \qquad y_1-y_2D^{\frac{1}{2}}=(x-D^{\frac{1}{2}})^n;$$

therefore $\quad y_1^2-y_2^2D=(x^2-D)^n$

or $$y^2-D=\frac{(x^2-D)^n}{y_2^2}. \qquad \text{(II.)}.$$

Again, we can write

$$2y_1=(x+D^{\frac{1}{2}})^n+(x-D^{\frac{1}{2}})^n, \qquad 2y_2D^{\frac{1}{2}}=(x+D^{\frac{1}{2}})^n-(x-D^{\frac{1}{2}})^n;$$

therefore $\quad 2D^{\frac{1}{2}}(y_1-y_2x)=(D-x^2)\left[(x+D^{\frac{1}{2}})^{n-1}-(x-D^{\frac{1}{2}})^{n-1}\right],$

which becomes

$$y-x=\frac{(D-x^2)\left\{{}_{n-1}C_1x^{n-2}+\ldots+{}_{n-1}C_{2r+1}x^{n-2r-2}D^r+\ldots\right\}}{y_2}. \qquad \text{(III.)}$$

Both the denominator and the co-factor of $D-x^2$ in the numerator are essentially positive.

Now, in order to get sequences of the kind required, it is necessary that (i.), when $x^2<D$, $y>x$, and, when $x^2>D$, $y<x$. The relation (III.) always satisfies these conditions. Also (ii.), when $x^2<D$, $y^2<D$, and $x^2>D$, $y^2>D$. The relation (II.) satisfies both of these conditions only if n be odd; it satisfies the latter condition only if n be either even or odd.

Hence the generalized form of Dedekind's expression will determine sequences defining an irrational square root, provided n be an odd integer.

SECTION II.

3. We will now investigate an expression for defining an irrational n-th root of a rational number.

The conditions to be satisfied may be put in the following form:—

$$x^n - D = (x-y)\psi, \qquad y^n - D = (x-y)\phi, \tag{IV.}$$

where ϕ and ψ are positive rational functions of x, D. Then we see that, if

$$\left.\begin{array}{lll} \text{(i.) } x^n < D, & \text{then } y > x, & \text{and therefore } y^n < D \\ \text{(ii.) } x^n > D, & \text{then } y < x, & \text{and therefore } y^n > D \end{array}\right\}. \tag{A}$$

We have to make the two equations (IV.) consistent.

This will be the case if $x^n - y^n = (x-y)(\psi - \phi)$,

$$\psi - \phi = x^{n-1} + x^{n-2}y + \ldots + x^{n-r-1}y^r + \ldots + y^{n-1};$$

therefore $\phi = \psi - (x^{n-1} + x^{n-2}y + \ldots + x^{n-r-1}y^r + \ldots + y^{n-1})$

$$= \psi - \frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x}.$$

Now we must make ϕ positive.

Consider first the case $x^n < D$, $y > x$, $y^n < D$; then

$$\frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x} < \frac{ny^n}{x} < \frac{nD}{x}.$$

Secondly, when $x^n > D$, $y < x$, then

$$\frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x} < \frac{nx^n}{x}.$$

Hence in either case

$$\frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x} < n\,\frac{x^n + D}{x}.$$

If we put $\psi = n\dfrac{x^n + D}{x}$, then ψ itself will be a rational positive function of x, D, and ϕ will also be rational and positive.

Also the two equations (IV.) will be consistent when

$$\phi = \frac{x^n + D}{x}n - \frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x}.$$

Substituting for ψ in (IV.),

$$x^n - D = (x-y)\frac{n(x^n + D)}{x}, \tag{V.}$$

$$x - y = \frac{(x^n - D)x}{n(x^n + D)},$$

$$y = x \left\{ \frac{D-x^n+nx^n+nD}{n(x^n+D)} \right\} = x\frac{(n-1)x^n+(n+1)D}{n(x^n+D)}$$

$$= \frac{(n-1)x^{n+1}+(n+1)Dx}{n(x^n+D)}. \qquad \text{(VI.)}$$

This will be the required expression.

There are two equations marked (IV.). The values selected for ϕ and ψ are such as to make these two equations the same, viz.,

$$x^n-D = (x-y)\frac{n(x^n+D)}{x}.$$

It would appear at first sight, then, that this equation could be transformed into both the forms of (IV.); but, on attempting to put it into the form $y^n-D = (x-y)$ (positive rational function of x, D), we see that we get, on multiplication by $x^{n-1}+x^{n-2}y+\ldots+y^{n-1}$,

$$(x^n-D)(x^{n-1}+x^{n-2}y+\ldots+y^{n-1}) = (x^n-y^n)\frac{n(x^n+D)}{x};$$

therefore $$y^n-x^n = \frac{D-x^n}{D+x^n}\,\frac{x^n+x^{n-1}+\ldots+xy^{n-1}}{n};$$

therefore $$y^n-D = \frac{(x^n-D)x}{n(x^n+D)}\left\{\frac{x^n+D}{x}n-\frac{x^n+x^{n-1}y+\ldots+xy^{n-1}}{x}\right\}. \qquad \text{(VII.)}$$

Call the part in $\{\ \}$ L.

Then in the case $x^n < D$ the first equation of (IV.) shows that $y > x$, and the relation (VII.) shows that $y^n < D$, provided L be positive.

But L has been shown to be positive only if we first assume that $y^n < D$. Hence it requires some other method to show that $y^n < D$ when $x^n < D$.

If, however, $x^n > D$, then, as before, $y < x$, and L is now positive whether $y^n >$ or $< D$; hence the relation (VII.) shows that $y^n > D$.

It is not necessary, therefore, in this latter case to verify that the conditions are satisfied by the relation (VI.), provided n be integral. An independent mode of proof has been adopted.

4. Before proceeding to this we will give the verification of the result in the cases when $n = 2, 3, 4$.

(a) When $n = 2$.—

$$y = \frac{x^3+3Dx}{2(x^2+D)}, \qquad y-x = \frac{(D-x^2)x}{2(x^2+D)},$$

$$y^2-D = \frac{(x^2-D)(x^4+3x^2D+4D^2)}{4(x^2+D)^2}.$$

(β) When $n = 3$.—

$$y = \frac{2x^4+4Dx}{3(x^3+D)}, \qquad y-x = \frac{D-x^3}{3(x^3+D)}\,x,$$

$$y^3-D = \frac{(x^3-D)(8x^9+29x^6D+44x^3D^2+27D^3)}{27(x^3+D)^3}.$$

(γ) When $n = 4$.—

$$y = \frac{3x^5+5Dx}{4(x^4+D)}, \qquad y-x = \frac{D-x^4}{4(x^4+D)}\,x,$$

$$y^4-D = \frac{(x^4-D)(81x^{16}+365x^{12}D+691x^8D^2+655x^4D^3+256D^4)}{256(D+x^4)^4}.$$

In all three cases the conditions (A) are satisfied.

5. Proceeding to the proof of the general case, we have

$$y = x\left[1-\frac{x^n-D}{n(x^n+D)}\right];$$

therefore
$$y^n-D = x^n\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n-D.$$

Now we wish to show that (i.), when $x^n < D$, $y^n < D$; (ii.), when $x^n > D$, $y^n > D$.

Case (i.). When $x^n < D$.—We must show that

$$x^n\left[1+\frac{D-x^n}{n(x^n+D)}\right]^n-D < 0.$$

Put $\dfrac{D-x^n}{D+x^n} = a$, where $0 < a < 1$. Then $\dfrac{D}{x^n} = \dfrac{1+a}{1-a}$. Hence we must show that
$$\left(1+\frac{a}{n}\right)^n-\frac{1+a}{1-a} < 0 \quad\text{or}\quad \frac{1+a}{1-a} > \left(1+\frac{a}{n}\right)^n.$$

Now we have, if a and b be positive and m a positive quantity > 1,

$$ma^{m-1}(a-b) > a^m-b^m > mb^{m-1}(a-b). \qquad \text{(VIII.)}$$

(See Chrystal's *Algebra*, chapter xxiv., § 7.)

Putting $a = 1+\dfrac{a}{y}$, $b = 1$, $m = \dfrac{y}{z}$, where $y > z$,

$$\left(1+\frac{a}{y}\right)^{y/z}-1 > \frac{y}{z}\,\frac{a}{y};$$

therefore $\left(1+\dfrac{a}{y}\right)^{y/z} > 1+\dfrac{a}{z}$, $\left(1+\dfrac{a}{y}\right)^y > \left(1+\dfrac{a}{z}\right)^z$, where $y > z$.

Hence the expression $(1+a/n)^n$ continually increases as we increase n. Also
$$\underset{n=\infty}{\mathrm{L}}\left(1+\frac{a}{n}\right)^n = e^a.$$
Hence, dealing only with finite values of n,
$$\begin{aligned}\left(1+\frac{a}{n}\right)^n &< \underset{n=\infty}{\mathrm{L}}\left(1+\frac{a}{n}\right)^n \\ &< e^a \\ &< 1+a+\frac{a^2}{2!}+\ldots+\frac{a^r}{r!}+\ldots,\end{aligned}$$
which is an absolutely convergent series $< 1+a+a^2+\ldots+a^r+\ldots$, which converges when $|a| < 1$. Hence $\left(1+\frac{a}{n}\right)^n < 1-a$, and, since $1+a > 1$, therefore
$$\left(1+\frac{a}{n}\right)^n < \frac{1+a}{1-a}.$$
Hence we have proved what was required.

Case (ii.). When $x^n > D$.—We must show that
$$x^n\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n - D > 0.$$
Putting $\frac{x^n-D}{x^n+D} = a$, so that $0 < a < 1$, $\frac{D}{x^n} = \frac{1-a}{1+a}$. Now, from the formula (VIII.), putting $a = 1$, $b = 1-a/y$, $m = y/z$, where $y > z$,
$$\frac{y}{z}\,\frac{a}{y} > 1-\left(1-\frac{a}{y}\right)^{y/z};$$
therefore $\left(1-\frac{a}{y}\right)^{y/z} > \left(1-\frac{a}{z}\right)$, $\left(1-\frac{a}{y}\right)^y > \left(1-\frac{a}{z}\right)^z$, where $y > z$.

Hence the function $(1-a/n)^n$ continually increases as we increase n. Hence
$$(1-a/n)^n > 1-a, \quad \text{where} \quad n > 1;$$
and $1+a > 1$, therefore
$$1-a > \frac{1-a}{1+a};$$
therefore
$$\left(1-\frac{a}{n}\right)^n > \frac{1-a}{1+a}, \qquad \left(1-\frac{a}{n}\right)^n - \frac{1-a}{1+a} > 0,$$
which is the required result.

We have, therefore, shown that, if
$$y = \frac{(n-1)\,x^{n+1}+(n+1)\,Dx}{n(x^n+D)},$$

then (i.), if $x^n < D$, $y > x$ and $y^n < D$; (ii.), if $x^n > D$, $y < x$ and $y^n > D$. This has been proved for all cases in which n is any quantity greater than unity.

6. A more general expression for y may be given. For, if

$$\phi = \psi - \frac{x^n + x^{n-1}y + \ldots + x^{n-r}y^r + \ldots + xy^{n-1}}{x}$$

and ϕ is positive when $\psi = n\dfrac{x^n+D}{x}$, then clearly ϕ will be positive if $\psi = n\dfrac{x^n+D}{x}$ + any rational positive function of x, D. Call this function F; then $\psi = n\dfrac{x^n+D}{x} + F$.

Substituting in the first equation (IV.),

$$x^n - D = (x-y)\left[\frac{n(x^n+D)}{x} + F\right], \qquad y - x = \frac{(D-x^n)x}{n(x^n+D)+xF},$$

$$\therefore y = \frac{(n-1)x^{n+1} + (n+1)Dx + x^2F}{n(x^n+D)+xF}. \qquad \text{(IX.)}$$

It may readily be shown, by the help of the results proved, that this value for y will satisfy the given conditions.

In the first place, if $x^n < D$, $y > x$, and, if $x^n > D$, $y < x$. Again,

$$y = x\left[1 + \frac{D-x^n}{n(x^n+D)+xF}\right].$$

Case (i.) When $x^n < D$.—We wish to show that $y^n < D$. Now

$$1 + \frac{D-x^n}{n(x^n+D)+xF} < 1 + \frac{D-x^n}{n(x^n+D)};$$

therefore $$\left[1 + \frac{D-x^n}{n(x^n+D)+xF}\right]^n < \left[1 + \frac{D-x^n}{n(x^n+D)}\right]^n.$$

But we have shown that

$$x^n\left[1 + \frac{D-x^n}{n(x^n+D)}\right]^n - D < 0;$$

therefore $$x^n\left[1 + \frac{D-x^n}{n(x^n+D)+xF}\right]^n - D < 0;$$

therefore $y^n - D < 0$, $y^n < D$, which we wished to prove.

Case (ii.). $x^n > D$.—We wish to show that $y^n > D$.

$$y = x\left[1 - \frac{x^n - D}{n(x^n+D)+xF}\right].$$

Now $$1-\frac{x^n-D}{n(x^n+D)+xF} > 1-\frac{x^n-D}{n(x^n+D)};$$

therefore $$x^n\left[1-\frac{x^n-D}{n(x^n+D)+xF}\right]^n > \left[1-\frac{x^n-D}{n(x^n+D)}\right]^n, \text{ if } n>1.$$

But we have shown that
$$x^n\left[1-\frac{x^n-D}{n(x^n+D)}\right]^n-D>0;$$

therefore $$x^n\left[1-\frac{x^n-D}{n(x^n+D)+xF}\right]^n-D>0;$$

therefore $y^n-D>0$, $y^n>D$, which we wished to prove.

7. Two sequences of rational numbers are therefore determined by the relation (VI.), one in ascending and the other in descending order of magnitude.

Let the ascending sequence be $x_1, x_2, \ldots, x_m, \ldots$, such that, if $0<x_1^n<D$, $x_1<x_2<\ldots<x_m$ and $x_m^n<D$; therefore $(x_1^n, x_2^n, \ldots)$ converges, and its limit $\leqslant D$. Similarly, let the descending sequence be $x_1', x_2', \ldots, x_m', \ldots$, where $(x_1'^n, x_2'^n, \ldots)$ converges and its limit $\geqslant D$. It will be shown that both sequences converge to the same limit.

If we start with x_1, we can find another number x_2 of the ascending sequence from the relation
$$x_2=\frac{(n-1)x_1^{n+1}+(n+1)Dx_1}{n(x_1^n+D)};$$

then $$x_3=\frac{(n-1)x_2^{n+1}+(n+1)Dx_2}{n(x_2^n+D)},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots,$$
$$x_{m+1}=\frac{(n-1)x_m^{n+1}+(n+1)Dx_m}{n(x_m^n+D)}.$$

Thus $$x_{m+1}=x_m\left\{1+\frac{D-x_m^n}{n(x_m^n+D)}\right\};$$

therefore $$x_{m+1}-x_m=\frac{D-x_m^n}{n(x_m^n+D)}x_m.$$

Let $x_{m+1}-x_m=\delta_m$. Since $x_{m+1}=x_1+\delta_1+\delta_2+\ldots+\delta_m$ and $x_{m+1}^n \ngtr D$, therefore $x_1+\delta_1+\delta_2+\ldots+\delta_m+\ldots$ ad inf. is a series which converges to some limit E, where, at present, all we know of the limit is that $E^n\leqslant D$.

Now it is a necessary condition for the convergence of the series $\sum_{m=1}^{\infty} \delta_m$ that after a certain term δ_m, where $m \geqslant \mu$, $|\delta_m| < \epsilon$, an arbitrarily small positive quantity. Hence we must have, after a certain term,

$$\left| \frac{D-x_m^n}{n(x_m^n+D)} x_m \right| < \epsilon,$$

$$|D-x_m^n| < \frac{|n(x_m^n+D)|}{|x_m|} \epsilon < \frac{2nD}{x_1} \epsilon;$$

therefore $|D-x_m^n| < \eta$, where η is arbitrarily small. Hence, provided we do not start with x_1 zero, x_m^n converges to the limit D as m is increased. Therefore the ascending sequence $x_1, x_2, \ldots, x_m, \ldots$ converges to the limit E, where $E^n = D$.

In the same way it may be shown that the descending sequence $x'_1, x'_2 \ldots, x'_m, \ldots$ converges to this same limit E. We call E the n-th root of the rational number D. Hence the irrational number is completely defined by either of the two sequences.

8. I am indebted to Prof. Hill for a formula not included in any of the above, for the case $n = 3$, $y-x = \dfrac{x(D-x^3)}{2x^3+D}$. This gives

$$y^3-D = \frac{x^3-D)^3(x^3+D)}{(2x^3+D)^3},$$

which gives, if $x^3 < D$, $y > x$ and $y^3 < D$; but, if $x^3 > D$, $y < x$ and $y^3 > D$. And probably there are a great many other formulæ of a similar nature.

ON FERMAT'S NUMBERS.

By LT.-COL. ALLAN CUNNINGHAM, R.E., *and* MR. A. E. WESTERN.

[Communicated May 14th, 1903.]

Abstract.

FERMAT'S numbers are of the form $2^{2^n}+1 = F_n$, say. The following results have been obtained in regard to the factorisation of these numbers :—

n	Factors of F_n.	Discoverer.	Year.
0–4	$F_0 \dots F_4$, all prime.		
5	$2^7 . 5+1 = 641$ $2^7 . 52347+1$	L. Euler...	1732
6	$2^8 . 9 . 7 . 17+1$	Landry	1880
	$2^8 . 5 . 52562829149+1$...	Landry and Le Lasseur ...	1880
9	$2^{16} . 37+1$	A. E. Western	1903
11	$2^{13} . 3 . 13+1$ $2^{13} . 7 . 17+1$	A. Cunningham	1899
12	$2^{14} . 7+1$	E. Lucas and P. Pervouchine	1878
	$2^{16} . 397+1$ $2^{16} . 7 . 139+1$	A. E. Western	1903
18	$2^{20} . 13+1$	A. E. Western	1903
23	$2^{25} . 5+1$	P. Pervouchine	1878
36	$2^{39} . 5+1$	Seelhoff	1886
38	$2^{41} . 3+1$	J. Cullen, A. Cunningham, A. E. and F. J. Western	1903

Abstract of Negative Results.

No more factors (f) of F_n, as follows :—

1. None $< 10^6$; (certified by each of the authors).
2. No $f = 2^a \pm 2^x+1 < 10^8$.
3. No $f = 2^\nu . Q+1 < 9 . 10^6$; $[Q = 1$ to $1220]$.
4. No $f < 1\frac{1}{4}, 2\frac{1}{2}, 5, 9$ million of F_8, F_9, F_{10}, F_n $(n > 10)$.
5. No $f = 2^{16} . Q+1 < 10^8$.
6. No $f < 10^8$ of F_n, $(n \nless 14)$.

Now it is a necessary condition for the convergence of the series $\sum_{m=1}^{\infty} \delta_m$ that after a certain term δ_m, where $m \geqslant \mu$, $|\delta_m| < \epsilon$, an arbitrarily small positive quantity. Hence we must have, after a certain term,

$$\left| \frac{D-x_m^n}{n(x_m^n+D)} x_m \right| < \epsilon,$$

$$|D-x_m^n| < \frac{|n(x_m^n+D)|}{|x_m|} \epsilon < \frac{2nD}{x_1} \epsilon;$$

therefore $|D-x_m^n| < \eta$, where η is arbitrarily small. Hence, provided we do not start with x_1 zero, x_m^n converges to the limit D as m is increased. Therefore the ascending sequence $x_1, x_2, \ldots, x_m, \ldots$ converges to the limit E, where $E^n = D$.

In the same way it may be shown that the descending sequen $x'_1, x'_2 \ldots, x'_m, \ldots$ converges to this same limit E. We call E the n-th r of the rational number D. Hence the irrational number is compl defined by either of the two sequences.

8. I am indebted to Prof. Hill for a formula not included of the above, for the case $n = 3$, $y-x = \frac{x(D-x^3)}{2x^3+D}$. This gives

$$y^3-D = \frac{x^3-D)^3(x^3+D)}{(2x^3+D)^3},$$

which gives, if $x^3 < D$, $y > x$ and $y^3 < D$; but, if $x^3 >$ $y^3 > D$. And probably there are a great many other formul nature.

It may be useful to remark that the theorem of § 4 can be easily applied to other double-limit problems of elementary analysis; but the results, so far as I have gone, do not seem sufficiently new to justify publication. However, some of the known theorems can be proved very simply by applying this method: for example, Schwarz's theorem on the inversion of two partial differentiations.*

1. *Definition and Properties of a Convergent Double Sequence.*

The double sequence $x_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, \ldots)$ is said *to be convergent*† (or *to converge*) to the limiting value a if for every positive fraction ϵ, however small, integers m, n can be found such that

(1) $$|x_{\mu\nu}-a| \leqslant \epsilon, \quad \text{for} \quad \mu \geqslant m, \quad \nu \geqslant n;$$

or, what comes to the same thing, if an integer p can be found such that

(1*a*) $$|x_{\mu\nu}-a| \leqslant \epsilon, \quad \text{for} \quad \mu, \nu \geqslant p.$$

Clearly p may be taken to be the larger of the two integers m, n.

We express the foregoing facts briefly by the equation $\lim\limits_{\mu, \nu=\infty} x_{\mu\nu} = a$.

The *necessary and sufficient condition for the convergence of the double sequence* $x_{\mu\nu}$ is that an integer q can be found such that

(2) $$|x_{\mu\nu}-x_{qq}| \leqslant \epsilon, \quad \text{for} \quad \mu, \nu \geqslant q,$$

however small the positive fraction ϵ may be.

To avoid complicating the descriptions further on, I now introduce the definitions of *rows* and *columns* of the double sequence. Let the terms be written in the form of an infinite determinant

$$\begin{matrix} x_{00}, & x_{10}, & x_{20}, & x_{30}, & \ldots, \\ x_{01}, & x_{11}, & x_{21}, & x_{31}, & \ldots, \\ x_{02}, & x_{12}, & x_{22}, & x_{32}, & \ldots, \\ \ldots & & \ldots & & \ldots; \end{matrix}$$

then the horizontal lines are called *rows*, and the vertical *columns*. Thus $x_{\mu\nu}$ is placed at the intersection of the $(\mu+1)$-th column with the $(\nu+1)$-th row.

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* Schwarz, *Ges. Math. Abhandlungen*, Bd. II., p. 275; Stolz, *Grundzüge*, Theil 1, p. 146.

† As already stated, the results in §§ 1, 2 are taken from the papers by Pringsheim and London, quoted before. This summary is given with the object of making the rest of the paper more intelligible.

It is to be observed that *the convergence of the double sequence by no means implies the convergence of the single sequences given by the separate rows and columns.* That is to say, we cannot conclude the existence of the two limits

$$(3) \qquad \lim_{\nu=\infty} x_{\mu\nu}, \quad \lim_{\mu=\infty} x_{\mu\nu}$$

(for *finite* values of μ, ν respectively) from the knowledge that $\lim\limits_{\mu,\,\nu=\infty} x_{\mu\nu} = a$. In case the two limits (3) do exist, we shall write

$$(4) \qquad y_\mu = \lim_{\nu=\infty} x_{\mu\nu}, \quad z_\nu = \lim_{\mu=\infty} x_{\mu\nu}.$$

If y_μ and z_ν do not exist, the convergence of $x_{\mu\nu}$ leads to the following equations :—

$$(5) \qquad \lim_{\mu=\infty} (\overline{\lim_{\nu=\infty}} x_{\mu\nu}) = \lim_{\nu=\infty} (\overline{\underline{\lim}}_{\mu=\infty} x_{\mu\nu}) = a,$$

where we use the symbols

$$\overline{\lim_{\nu=\infty}} x_{\mu\nu}, \quad \underline{\lim}_{\nu=\infty} x_{\mu\nu}, \quad \overline{\underline{\lim}}_{\nu=\infty} x_{\mu\nu},$$

in the senses given by Pringsheim.* That is to say, the first is *the greatest limiting value*, the second is *the least limiting value*† of the single sequence $x_{\mu\nu}$ (μ being kept constant); while the third implies that either the first or the second may be used, at pleasure.

To emphasize the theorems just given, we may glance at an instructive example. Let

$$x_{\mu\nu} = (-1)^{\mu+\nu}(1/\mu + 1/\nu) \qquad (\mu,\ \nu = 1,\ 2,\ 3,\ \ldots).$$

Then y_μ and z_ν do not exist, but we have

$$\left.\begin{array}{ll} \overline{\lim\limits_{\nu=\infty}} x_{\mu\nu} = +1/\mu, & \underline{\lim}_{\nu=\infty} x_{\mu\nu} = -1/\mu \\ \overline{\lim\limits_{\mu=\infty}} x_{\mu\nu} = +1/\nu, & \underline{\lim}_{\mu=\infty} x_{\mu\nu} = -1/\nu \end{array}\right\} \lim_{\mu,\,\nu=\infty} x_{\mu\nu} = 0;$$

so that we have

$$\lim_{\mu=\infty} (\overline{\lim_{\nu=\infty}} x_{\mu\nu}) = 0 = \lim_{\nu=\infty} (\overline{\underline{\lim}}_{\mu=\infty} x_{\mu\nu}) = \lim_{\mu,\,\nu=\infty} x_{\mu\nu},$$

in agreement with the general results stated in (5).

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* *Münch. Sitz.-Ber.*, Bd. XXVIII., 1898, p. 62 ; *Encyklopädie Math. Wiss.*, I. A 3, 15.

† *Oberer Limes, Unterer Limes ;* Borel and Goursat use, as the French equivalent for the former, the phrase (introduced by Cauchy) *la plus grande des limites*, which suggests the terms used here. The more literal translations *upper limit* and *lower limit* have been accepted as the English equivalents for the terms *Obere Grenze, Untere Grenze*, which are essentially different terms ; of course it may happen (and often does happen) that the *Grenze* and *Limes* are the same number, but the two conceptions are really distinct ideas.

If, however, the limits y_μ, z_ν have been found to exist, we have in place of (5) the simpler equation

$$\lim_{\mu=\infty} y_\mu = a = \lim_{\nu=\infty} z_\nu, \tag{6}$$

a case which is illustrated by the simple example

$$x_{\mu\nu} = 1/\mu + 1/\nu \qquad (\mu, \nu = 1, 2, 3, \ldots).$$

An obvious deduction from equation (6) is :—*A necessary condition for the convergence of a double sequence, whose rows and columns separately converge, is that the two limits* $\lim\limits_{\mu=\infty} y_\mu$, $\lim\limits_{\nu=\infty} z_\nu$ *must both exist, and be equal. But this necessary condition is by no means sufficient ;* a fact which is at once recognized by examining the special case

$$x_{\mu\nu} = \mu\nu / (\mu^2 + \nu^2) \qquad (\mu, \nu = 1, 2, 3, \ldots).$$

This gives $y_\mu = 0$, $z_\nu = 0$; so that $y_\mu = z_\nu$ for all values of μ, ν, and in particular

$$\lim_{\mu=\infty} y_\mu = \lim_{\nu=\infty} z_\nu = 0.$$

But, nevertheless, the double sequence is not convergent ; for, if $\nu/\mu = t$, $x_{\mu\nu} = t/(1+t^2)$, whatever may be the values of μ, ν.

If, however, the sequence $x_{\mu\nu}$ *is monotonic,** *then the existence of any one of the three limits* $\lim\limits_{\mu=\infty} y_\mu$, $\lim\limits_{\nu=\infty} z_\nu$, $\lim\limits_{\mu,\nu=\infty} x_{\mu\nu}$ *implies the existence of the other two ; and all three are then equal.*

2. *Uniform Convergence of the Rows (or Columns) of a Double Sequence.*

In what follows statements are made with reference to *rows* only ; but exactly corresponding theorems are true if we change to *columns* (making the changes in the letters which are thereby implied, such as putting y_μ in place of z_ν). These statements are taken from Pringsheim's paper ; † London's account, which was formulated independently of Pringsheim's, is essentially the same. ‡

Suppose every *row* of the double sequence to converge, so that

$$\lim_{\mu=\infty} x_{\mu\nu} = z_\nu \qquad (\nu = 0, 1, 2, 3, \ldots).$$

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* That is, we are to have *either* $x_{\mu\nu} \geqslant x_{mn}$, *or* $x_{\mu\nu} \leqslant x_{mn}$, for $\mu \geqslant m$, $\nu \geqslant n$.

† *Math. Ann.*, Bd. LIII., 1900, p. 306 (§ 3, 3 of the paper).

‡ *Ibid.*, pp. 336–338. A similar definition had suggested itself to me, in the interval between reading Pringsheim's first paper and his second.

This equation implies that, corresponding to every positive fraction ϵ, and to every integer ν, an integer m_ν can be found such that

$$(1) \qquad |x_{\mu\nu}-z_\nu| \leqslant \epsilon, \quad \text{for} \quad \mu \geqslant m_\nu,$$

and, in order to make m_ν precise, we suppose that it is the *least* integer which satisfies the condition in (1). It is clear that the numbers m_ν must in general vary both with ν and ϵ; and they may increase without limit when ν does so. Similarly the numbers $|z_\nu|$ may increase indefinitely with ν. If either of these two cases does occur, *the rows are said to converge non-uniformly.*

On the other hand, *if the numbers m_ν have a definite finite maximum*[*] *m, we shall have*

$$(2) \qquad |x_{\mu\nu}-z_\nu| \leqslant \epsilon, \quad \text{for} \quad \mu \geqslant m, \ \nu = 0, 1, 2,$$

where m does not depend on ν, but only on ϵ. The rows are then said to be uniformly convergent, provided that $|z_\nu|$ is always less than a fixed number g.

A simple example of uniform convergence of the rows is provided by

$$x_{\mu\nu} = (-1)^\nu(1/\mu+1/\nu) \qquad (\mu, \nu = 1, 2, 3, \ldots).$$

Then $z_\nu = (-1)^\nu/\nu$ and $|x_{\mu\nu}-z_\nu| = 1/\mu$; so that the condition of uniform convergence is satisfied by taking $m \geqslant 1/\epsilon$.

From this example it is clear that *the uniform convergence of the rows does not ensure the convergence of the columns;* for here we have

$$\overline{\lim_{\nu=\infty}}\, x_{\mu\nu} = +1/\mu, \qquad \underline{\lim_{\nu=\infty}}\, x_{\mu\nu} = -1/\mu;$$

so that $\lim\limits_{\nu=\infty} x_{\mu\nu}$ does not exist.

We sum up the principal properties of a double sequence (in reference to the uniform convergence of the rows) as follows[†] :—

For the convergence of a double sequence (whose rows are known to converge[‡]*), it is necessary and sufficient that the rows should be uniformly convergent and that the limit* $\lim\limits_{\nu=\infty} z_\nu$ *should exist.*

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[*] Since the numbers m_ν are *integers*, the value m must be actually *attained; i.e.*, it is a *maximum* and not an *upper limit.*

[†] Specialized from theorem IV*a*. (*l.c.*, p. 321) of Pringsheim's paper; see also London's paper (*l.c.*, p. 338).

[‡] Of course this convergence is not necessary for the convergence of the double sequence; a fact which follows from § 1.

We have then (and only then)

$$\lim_{\nu=\infty} z_\nu = \lim_{\mu,\,\nu=\infty} x_{\mu\nu} = \lim_{\mu=\infty} (\overline{\lim_{\nu=\infty}}\, x_{\mu\nu}); \tag{3}$$

but we have no guarantee that the columns* converge at all; a fact which is obvious from the example considered on the preceding page.

It is useful to observe that we may, if convenient, exclude a *finite* number of the rows from the condition of being uniformly convergent.

3. *A Simple Test for the Existence of the Infinite Double Integral* $\int_a^\infty \int_b^\infty f(x, y)\, dx\, dy.$

It has been shown by Mr. G. H. Hardy† that the definition adopted by Jordan (followed by Stolz) for the infinite double integral is unnecessarily narrow; essentially, the point is that these authors only recognize *absolutely* convergent double integrals.‡ But, as Hardy remarks, we may have *conditionally* convergent double integrals, just as well as a conditionally convergent double series or as a conditionally convergent single integral.

Following Hardy, I define the infinite double integral as

$$\lim_{H,\,K=\infty} \int_a^H \int_b^K f(x, y)\, dx\, dy, \tag{1}$$

where the double integral is taken over the rectangle bounded by the lines $x = a$, $x = H$, $y = b$, $y = K$.

Using the notation given in (2), § 5, this is the same thing as

$$\lim_{\mu,\,\nu=\infty} I_{\mu\nu}, \tag{2}$$

where I have written

$$I_{\mu\nu} = \int_a^{x_\mu} \int_b^{y_\nu} f(x, y)\, dx\, dy. \tag{3}$$

Applying now the results of § 2, we see that *the following conditions*§ *are*

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[*] However, they are *uniformly bounded* (*gleichmässig begrenzt*). See p. 310 and theorem IV*a*. of Pringsheim's paper; London uses *gleichmässig oscillirend*, p. 339.

[†] *Messenger of Mathematics*, Vol. XXXII., October, 1902, Note X.

[‡] This requires the existence of $\int_a^\infty \int_b^x |f(x, y)|\, dx\, dy$; which, in virtue of the last theorem of § 1, is ensured by the existence of *either* $\int_a^x dx \int_b^\infty |f(x, y)|\, dy$ or $\int_b^\infty dy \int_a^\infty |f(x, y)|\, dx$. Compare § 4, end.

[§] These conditions are deduced from those which give the uniform convergence of the rows of $I_{\mu\nu}$; a corresponding set can be found from the columns, or by interchanging x and y, &c.

sufficient for the existence of the general double limit, i.e., for the existence of the infinite double integral:—

(i.) $\int_a^\infty dx \int_b^y f(x, y)\, dy$ *is to converge for all values of* $y \geqslant b$, *and is to be always less, in absolute value, than a fixed positive number* g.

(ii.) $\lim_{y=\infty} \int_a^\infty dx \int_b^y f(x, y)\, dy$ *is to exist; call this limit* L.

(iii.) ϵ *being given* (*a positive proper fraction*), *it is to be possible to determine* ξ, *such that*

$$\left| \int_x^\infty dx \int_b^y f(x, y)\, dy \right| \leqslant \epsilon \quad (x \geqslant \xi,\ y \geqslant b).$$

Then the infinite double integral exists and is equal to L.

It is to be observed that *condition* (i.) *is not necessary for the existence of the infinite double integral; but that, if condition* (i.) *is known to be verified, then* (ii.) *and* (iii.) *are necessary for its existence.* This is at once clear from the theorems which have been stated in § 2.

Two simple illustrations of cases to which the tests apply are afforded by the first and second examples of § 5 (pp. 189, 192); both of these give an infinite double integral, in Hardy's sense; but the first is only conditionally convergent, a fact which is clear from Stolz's work (*Grundzüge*, Theil 3, p. 156). The third example of § 5 does not give a definite value for the corresponding infinite double integral: it oscillates between 0 and $\frac{1}{2}$.

To illustrate the fact that condition (i.) is not necessary for the existence of an infinite double integral, we may take the simple case

$$\int_1^\infty \int_1^\infty \frac{\partial^2 V}{\partial x\, \partial y}\, dx\, dy,$$

with $$V = \sin \pi x \sin \pi y\, (1/x + 1/y).$$

Here we find that

$$\int_1^H \int_1^K \frac{\partial^2 V}{\partial x\, \partial y}\, dx\, dy = \sin \pi H \sin \pi K\, (1/H + 1/K),$$

and so $$\lim_{H, K=\infty} \int_1^H \int_1^K \frac{\partial^2 V}{\partial x\, \partial y}\, dx\, dy = 0.$$

But yet the limit $\lim_{H=\infty} \int_1^H dx \int_1^K \frac{\partial^2 V}{\partial x\, \partial y}\, dy$ does not exist; so that condition (i.) is not here satisfied.

We may note here an additional illustration of the result obtained by de la Vallée Poussin* that the repeated integral $\int_b^\infty dy \int_a^\infty f(x,y)\,dx$ may have no meaning when $\int_a^\infty dx \int_b^\infty f(x,y)\,dy$ is perfectly precise. This is possible, in fact, even if the double integral $\int_a^\infty \int_b^\infty f(x,y)\,dx\,dy$ exists in addition; as is shown by the simple case

$$f(x,y) = \frac{\partial^2 V}{\partial x\,\partial y}, \qquad V = \sin(\pi x)\left(\frac{1}{x}+\frac{1}{y}\right)\left(1-\frac{1}{y}\right).$$

We find $$\int_1^H dx \int_1^K f(x,y)\,dy = \sin(\pi H)\left(\frac{1}{H}+\frac{1}{K}\right)\left(1-\frac{1}{K}\right);$$

so $$\int_1^\infty \int_1^\infty f(x,y)\,dx\,dy = 0.$$

Further $$\int_1^\infty dx \left[\int_1^\infty f(x,y)\,dy\right] = \int_1^\infty \frac{d}{dx}\left(\frac{\sin(\pi x)}{x}\right) dx = 0.$$

But yet $\int_1^\infty f(x,y)\,dx$ has no meaning, and so $\int_1^\infty dy \int_1^\infty f(x,y)\,dx$ is also meaningless.

Another simple example is suggested by one given by Pringsheim,† for a *finite* area of integration.

4. *Necessary and sufficient Conditions for the Inversion of Two Successive Limits, say* $\lim\limits_{\nu=\infty}(\lim\limits_{\mu=\infty} x_{\mu\nu})$, $\lim\limits_{\mu=\infty}(\lim\limits_{\nu=\infty} x_{\mu\nu})$.

If these two limits are to be equal, it is clearly implied that the rows and columns all converge‡ (except, possibly, a *finite* number of them); so that we can write, as in (4), § 1,

$$(1) \qquad y_\mu = \lim_{\nu=\infty} x_{\mu\nu}, \qquad z_\nu = \lim_{\mu=\infty} x_{\mu\nu}.$$

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* *Annales de la Soc. Sci. de Bruxelles*, t. XVI., 1891–2, p. 179. This result and reference I take from Brunel's article "Bestimmte Integrale," *Encyklopädie Math. Wiss.*, II. A 8, 8.

† *Münchener Sitz.-Ber.*, Bd. XXVIII., 1898, p. 71; the example occurs in the course of his article on double integrals.

‡ If we choose to understand our problem as the determination of conditions in order that

$$\lim_{\nu=\infty}(\underline{\lim}_{\mu=\infty} x_{\mu\nu}) = \lim_{\mu=\infty}(\underline{\lim}_{\nu=\infty} x_{\mu\nu}),$$

then the existence of y_μ, z_ν is by no means certain (see § 1). But this generalization is not needed here.

We shall now prove that *the following conditions are both necessary and sufficient to give*

$$(2) \qquad \lim_{\mu=\infty} y_\mu = \lim_{\nu=\infty} z_\nu,$$

(i.) $\lim\limits_{\mu=\infty} y_\mu$ *is to exist and to be equal, say, to* a.

(ii.) *Corresponding to every positive fraction* ϵ, *and to every prescribed integer* m_0, *it must be possible to find integers* m ($\geqslant m_0$), n *such that*

$$(3) \qquad | z_\nu - x_{m\nu} | \leqslant \epsilon, \quad \textit{for all values of } \nu \geqslant n,$$

where one value of m *only need be considered.*

First, the conditions are *sufficient*; for, assuming them to be verified, we have

$$z_\nu - a = (z_\nu - x_{m\nu}) + (x_{m\nu} - y_m) + (y_m - a).$$

Choose first m_0 to give

$$| y_\mu - a | \leqslant \epsilon, \quad \text{for} \quad \mu \geqslant m_0,$$

which is possible as a consequence of condition (i.). Next, fix m, n as in condition (ii.), so that

$$| z_\nu - x_{m\nu} | \leqslant \epsilon, \quad \text{for} \quad \nu \geqslant n.$$

Finally, m having been fixed, determine n' to give

$$| x_{m\nu} - y_m | \leqslant \epsilon, \quad \text{for} \quad \nu \geqslant n'.$$

Then we have $| z_\nu - x_{m\nu} | \leqslant \epsilon, \quad | x_{m\nu} - y_m | \leqslant \epsilon, \quad \text{for} \quad \nu \geqslant n_1,$

where n_1 is the greater of the integers n, n'. Hence, as

$$| z_\nu - a | \leqslant | z_\nu - x_{m\nu} | + | x_{m\nu} - y_m | + | y_m - a |,$$

we find

$$| z_\nu - a | \leqslant 3\epsilon, \quad \text{for} \quad \nu \geqslant n_1,$$

or

$$\lim_{\nu=\infty} z_\nu = a = \lim_{\mu=\infty} y_\mu.$$

Second, the conditions are *necessary*; for condition (i.) is clearly necessary if we are to attach any definite meaning to equation (2). Now

$$z_\nu - x_{m\nu} = (z_\nu - a) + (a - y_m) + (y_m - x_{m\nu}).$$

We first determine m to give $| y_m - a | \leqslant \frac{1}{3}\epsilon$, which is possible, since $\lim\limits_{\mu=\infty} y_\mu = a$. Having fixed m, we can determine n, so that

$$| z_\nu - a | \leqslant \tfrac{1}{3}\epsilon, \quad | y_m - x_{m\nu} | \leqslant \tfrac{1}{3}\epsilon, \quad \text{for} \quad \nu \geqslant n.$$

Hence $| z_\nu - x_{m\nu} | \leqslant | z_\nu - a | + | a - y_m | + | y_m - x_{m\nu} | \leqslant \epsilon, \text{ for } \nu \geqslant n,$

which is inequality (3) given in the condition (ii.), as enunciated.

Condition (ii.) of the foregoing may also be replaced by the following :—

(ii.) *Corresponding to every positive fraction* ϵ, *it must be possible to determine an integer* m_0 *such that, if any integer* μ *is taken* ($\mu \geqslant m_0$), *then another integer* n_μ *can be found to give*

(4) $$|z_\nu - x_{\mu\nu}| \leqslant \epsilon, \quad \textit{for all values of } \nu \geqslant n_\mu.$$

It appears, at first sight, that this condition must imply further restrictions on the sequence $x_{\mu\nu}$; and accordingly we need only verify that it is necessary, as well as the first form.

Now, if $$\lim_{\mu=\infty} y_\mu = a = \lim_{\nu=\infty} z_\nu,$$

we can certainly choose m_0 to give

$$|y_\mu - a| \leqslant \tfrac{1}{3}\epsilon, \quad \text{for} \quad \mu \geqslant m_0,$$

and then, picking out any special value of μ ($\geqslant m_0$), we can find n_μ to give

$$|z_\nu - a| \leqslant \tfrac{1}{3}\epsilon, \quad |y_\mu - x_{\mu\nu}| \leqslant \tfrac{1}{3}\epsilon, \quad \text{for} \quad \nu \geqslant n_\mu.$$

Hence $$|z_\nu - x_{\mu\nu}| \leqslant \epsilon, \quad \text{for} \quad \nu \geqslant n_\mu,$$

and this is the inequality (4).

To illustrate the application of these conditions, let us take the simple case

$$x_{\mu\nu} = \frac{\mu^2 + a\mu\nu + \beta\nu^2}{\mu^2 + \nu^2} \qquad (\mu, \nu = 1, 2, 3, \ldots).$$

Here $y_\mu = \beta$, $z_\nu = 1$; so that the first condition is satisfied. Next

$$z_\nu - x_{m\nu} = \frac{(1-\beta)\,\nu^2 - am\nu}{m^2 + \nu^2},$$

from which it is clear that the inequality (3) cannot be satisfied unless $1-\beta = 0$; and, if $\beta = 1$, this inequality will be satisfied by taking $n \geqslant m\,|a|/\epsilon$.

In like manner, to satisfy inequality (4), we have to take $\beta = 1$, and $n_\mu \geqslant \mu\,|a|/\epsilon$. Thus, in our present case, the necessity and sufficiency of the conditions is verified; for the equation $\beta = 1$ is exactly equivalent to

$$\lim_{\mu=\infty} y_\mu = \lim_{\nu=\infty} z_\nu.$$

We can treat similarly the example

$$x_{\mu\nu} = \frac{a_0\mu^p + a_1\mu^{p-1}\nu + a_2\mu^{p-2}\nu^2 + \ldots + a_p\nu^p}{b_0\mu^p + b_1\mu^{p-1}\nu + b_2\mu^{p-2}\nu^2 + \ldots + b_p\nu^p} \qquad (p > 2),$$

in which we may take $b_0, b_1, b_2, \ldots, b_p > 0$, in order to avoid the possible vanishing of the denominator.

It is now clear that, in general, the equation

$$\lim_{\nu=\infty}(\lim_{\mu=\infty} x_{\mu\nu}) = \lim_{\mu=\infty}(\lim_{\nu=\infty} x_{\mu\nu})$$

by no means implies the convergence of the double sequence $x_{\mu\nu}$; and, consequently, it does not involve the uniform convergence of either rows or columns of that sequence (§ 2).

But in a special case which is sometimes of interest the convergence of the double sequence does follow from the equation just written down :—

If every row[*] *gives a single sequence which never decreases,*[†] *then the equation*

$$\lim_{\mu=\infty}(\lim_{\nu=\infty} x_{\mu\nu}) = \lim_{\nu=\infty}(\lim_{\mu=\infty} x_{\mu\nu}) = a$$

leads to the convergence of the double sequence, provided that $|x_{\mu\nu}|$ *is always less than a fixed number* g.[‡]

Under these circumstances, we have

$$z_\nu = \lim_{\mu=\infty} x_{\mu\nu} \geqslant x_{\mu\nu};$$

so that $$z_\nu - x_{\mu\nu} = |z_\nu - x_{\mu\nu}|.$$

Now choose an integer m to give

$$|y_m - a| \leqslant \tfrac{1}{4}\epsilon,$$

and, afterwards, n to give

$$|z_\nu - a| \leqslant \tfrac{1}{4}\epsilon, \quad |x_{m\nu} - y_m| \leqslant \tfrac{1}{4}\epsilon, \quad \text{for} \quad \nu \geqslant n.$$

Then $$|z_\nu - x_{m\nu}| \leqslant |z_\nu - a| + |a - y_m| + |y_m - x_{m\nu}|$$
$$\leqslant \tfrac{3}{4}\epsilon, \quad \text{for} \quad \nu \geqslant n,$$

i.e., $$z_\nu - x_{m\nu} \leqslant \tfrac{3}{4}\epsilon, \quad \text{for} \quad \nu \geqslant n,$$

But $$x_{\mu\nu} \geqslant x_{m\nu}, \quad \text{for} \quad \mu \geqslant m,$$

and so[§] $$z_\nu - x_{\mu\nu} \leqslant \tfrac{3}{4}\epsilon, \quad \text{for} \quad \mu \geqslant m, \quad \nu \geqslant n.$$

Hence $$|x_{\mu\nu} - a| \leqslant |z_\nu - a| + |z_\nu - x_{\mu\nu}|$$
$$\leqslant \epsilon, \quad \text{for} \quad \mu \geqslant m, \quad \nu \geqslant n,$$

i.e., $$\lim_{\mu,\,\nu=\infty} x_{\mu\nu} = a.$$

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[*] Of course "row" may be replaced by "column."

[†] Again "never decreases" may be replaced by "never increases."

[‡] This theorem should be compared with Pringsheim's on monotonic double sequences, given at the end of § 1.

[§] This is the condition for uniform convergence of the rows of the double sequence; and so we could infer the existence of the double limit by appealing to § 2, but the direct method seems just as easy.

Two simple examples of this last test are given by the following :—

$$x_{\mu\nu} = \frac{\mu\nu}{(\mu+1)(\nu+1)}.$$

Here the numbers in every row never decrease, and

$$\lim_{\mu=\infty}(\lim_{\nu=\infty} x_{\mu\nu}) = 1 = \lim_{\nu=\infty}(\lim_{\mu=\infty} x_{\mu\nu});$$

so that $\lim_{\mu,\nu=\infty} x_{\mu\nu} = 1,$

which is at once verified.

$$x_{\mu\nu} = \frac{\mu\nu}{(\mu+1)^2+(\nu+1)^2}.$$

Here in every row the numbers first increase and afterwards decrease ; so that, although

$$\lim_{\mu=\infty}(\lim_{\nu=\infty} x_{\mu\nu}) = 0 = \lim_{\nu=\infty}(\lim_{\mu=\infty} x_{\mu\nu}),$$

yet the general limit $\lim_{\mu,\nu=\infty} x_{\mu\nu}$ is not determinate.

Another obvious conclusion, which is suggested by comparing §§ 2, 4, is the following :—

The double-limit problems which are commonly encountered in elementary analysis depend usually on the inversion of two successive limits and not on the existence of a general double limit. In all such problems a test which involves *uniform convergence* must be *sufficient only* and *not necessary*. Uniform convergence is only necessary when the general double limit is required to exist,

This remark has been made in a number of special cases, but not in any general form, so far as I know. For example, it is now well known that uniformity of convergence of an infinite series is sufficient (but not necessary) both for the continuity of its sum and for the term-by-term integration of the series; but no attempt seems to have been made to attribute the two facts to a common origin.

5. *Inversion of a Repeated Infinite Integral.*

Conditions are now to be found under which it will be permissible to invert the order of integration in a repeated integral which has ∞ as the upper limit in both integrations ; that is, we wish to investigate the truth of the equation

$$(1) \qquad \int_a^\infty dx \int_b^\infty f(x, y)\,dy = \int_b^\infty dy \int_a^\infty f(x, y)\,dx.$$

For the sake of simplifying the argument we consider only the case* in which $f(x, y)$ has no singularity for any *finite* values of x, y such that $x \geqslant a$, $y \geqslant b$.

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* This case is the one most commonly used in the ordinary applications of the theorem ; but, if it should happen that singularities occur at a finite number of points (within the limits of integration), each can be discussed separately by a method analogous to the one used in what follows. However, if singularities occur at all points of certain *curves*, the problem is more difficult ; and other methods become necessary.

Consider the double sequence

$$(2) \qquad I_{\mu\nu} = \int_a^{x_\mu} dx \int_b^{y_\nu} f(x, y)\, dy = \int_b^{y_\nu} dy \int_a^{x_\mu} f(x, y)\, dx,$$

in which x_μ, y_ν represent two independent sequences each having the limiting value $+\infty$; we shall now see how to connect our problem with the one discussed in § 4, which gave the conditions for the truth of the equation

$$(3) \qquad \lim_{\mu=\infty} (\lim_{\nu=\infty} I_{\mu\nu}) = \lim_{\nu=\infty} (\lim_{\mu=\infty} I_{\mu\nu}).$$

For we may write

$$(4) \qquad \lim_{\nu=\infty} I_{\mu\nu} = \lim_{\nu=\infty} \int_a^{x_\mu} dx \int_b^{y_\nu} f\, dy = \int_a^{x_\mu} dx \int_b^{\infty} f\, dy,$$

provided that $$\lim_{\nu=\infty} \int_a^{x_\mu} dx \int_{y_\nu}^{\infty} f\, dy = 0.$$

The necessary and sufficient conditions for the truth of the last equation are :—

(i.) *That* $\int_b^\infty f\, dy$ *converges for all finite values of* $x\,(\geqslant a)$.

(ii.) *That when* ξ *is fixed we can determine* η_0 *corresponding to any assigned positive fraction* ϵ *such that*

$$\left| \int_a^{\xi} dx \int_\eta^\infty f\, dy \right| \leqslant \epsilon, \quad \textit{for all} \quad \eta \geqslant \eta_0.$$

In like manner we may write

$$(5) \qquad \lim_{\mu=\infty} I_{\mu\nu} = \int_b^{y_\nu} dy \int_a^\infty f\, dx,$$

provided that

(i.) (*cont.*) $\int_a^\infty f\, dx$ *converges for all finite values of* $y\,(\geqslant b)$.

(iii.) *When* η *is fixed we can determine* ξ_0 *corresponding to any assigned positive fraction* ϵ *such that*

$$\left| \int_b^{\eta} dy \int_\xi^\infty f\, dy \right| \leqslant \epsilon, \quad \textit{for all} \quad \xi \geqslant \xi_0.$$

When equations (4), (5) are true it is clear that equation (1) is equivalent to equation (3); and, according to § 4, the necessary and sufficient conditions for the truth of (3) are as follows:—

(iv.) $\int_b^\infty dy \int_a^\infty f\,dx$ *must converge.*

(v.) *When* η_0 *is given we must be able to determine* ξ, η *corresponding to any assigned positive fraction* ϵ *such that* $\eta \geqslant \eta_0$, *and*

$$\left| \int_a^x dx \int_\eta^\infty f\,dy \right| \leqslant \epsilon, \quad \textit{for all} \quad x \geqslant \xi.$$

We have thus found a sufficient set of conditions which justify the inversion indicated by equation (1); but it seems unlikely that they are also necessary. For the method does not establish equation (1) directly, but replaces this equation by the set (3)–(5); now, so far as can be seen, *a priori*, there is no reason to suppose that the truth of equation (1) necessarily involves the truth of (3)–(5). That is, it seems quite possible that cases may exist* in which equation (1) is true, while two (or all) of equations (3)–(5) are untrue.

It is sometimes useful to observe that conditions (iv.), (v.) may be replaced by an exactly similar pair, found by interchanging x and y (with corresponding changes in the other letters). This liberty of choice is often of use in the elementary problems of this class; thus it may happen that one (but not both) of the integrals

$$(6) \qquad \int_\eta^\infty f\,dy, \quad \int_\xi^\infty f\,dx$$

can be expressed by the aid of known functions, and then this fact will guide us in selecting the most suitable form of test. If neither of the integrals (6) can be expressed in terms of known functions, the work often becomes very long and tedious.

We shall now apply conditions (i.)–(v.) to some simple cases of repeated integrals which are employed in familiar pieces of analysis.

We take first the case $f(x, y) = [e^{-ay} \sin(xy) \cos x]/x$ $(a > 0)$, with $a = 0$, $b = 0$. There is then no difficulty in verifying conditions (i.) and (iv.); so pass to the other tests. We can conveniently treat conditions (ii.) and (v.) simultaneously; and we find by elementary integrals

$$\int_\eta^\infty f\,dy = \frac{e^{-a\eta} \cos x}{x(a^2+x^2)} \left[a \sin(x\eta) + x \cos(x\eta)\right],$$

---

* I have made some attempts to construct such cases; but, so far, without obtaining any decisive example.

i.e.,
$$\left|\int_\eta^\infty f\,dy\right| < \frac{e^{-a\eta}}{x(a^2+x^2)}\left[a(x\eta)+x\right],$$

and so
$$\left|\int_\eta^\infty f\,dy\right| < \frac{(1+a\eta)e^{-a\eta}}{a^2+x^2}.$$

Hence
$$\left|\int_0^x dx\int_\eta^\infty f\,dy\right| < (1+a\eta)e^{-a\eta}\int_0^x\frac{dx}{a^2+x^2} < \frac{\pi}{2a}e^{-a\eta}(1+a\eta),$$

for
$$\int_0^x\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) < \frac{\pi}{2a}.$$

Thus, by properly selecting η we can make

$$\left|\int_0^x dx\int_\eta^\infty f\,dy\right| \leqslant \epsilon, \quad \text{for } all\ x\,;$$

and this satisfies conditions (ii.) and (v.).

The analysis required to verify condition (iii.) is somewhat longer, and we need the two preliminary results *

$$(7)\qquad \left|\int_\xi^\infty\frac{\sin(kx)}{x}\,dx\right| < \frac{2}{k\xi}$$

$$(8)\qquad \left|\int_{\xi_1}^{\xi_2}\frac{\sin(kx)}{x}\,dx\right| < \pi,$$

where k, ξ, ξ_1, ξ_2 are all supposed to be *positive.*

---

* The first of these is commonly given in books on the integral calculus; the second is perhaps not so well known. To obtain the first, we use the formula

$$\int\frac{\sin(kx)}{x}\,dx = -\frac{\cos(kx)}{kx} + \int\frac{\cos(kx)}{kx^2}\,dx.$$

The second may be found as follows:—

Write $\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{x}\,dx = (-1)^n v_n$, so that $v_n = \int_0^\pi\frac{\sin x}{x+n\pi}\,dx$; then it is clear that

$$v_0 > v_1 > v_2 > \dots > v_n > \dots > 0,$$

since $\sin x$ is positive if x is between 0 and π. Also $v_0 = \int_0^\pi\frac{\sin x}{x}\,dx < \pi$, since $0 < \frac{\sin x}{x} < 1$ in this integral. Now
$$\int_{\xi_2}^{\xi_1}\frac{\sin(kx)}{x}\,dx = \int_{k\xi_2}^{k\xi_1}\frac{\sin x}{x}\,dx;$$

and, if we find integers p, q, such that $p\pi \leqslant k\xi_1 < (p+1)\pi$, $q\pi \leqslant k\xi_2 < (q+1)\pi$, we can put the integral in the form

$$(-1)^p\left[w_p - v_{p+1} + v_{p+2} - \dots + (-1)^{p-q}(-v_{q-1}+w_q)\right]$$

where
$$0 < w_p \leqslant v_p, \quad 0 \leqslant w_q \leqslant v_q.$$

From this expression and the properties of the v's we find that our integral is numerically less than v_p; hence
$$\left|\int_{\xi_2}^{\xi_1}\frac{\sin(kx)}{x}\,dx\right| < v_p < v_0 < \pi.$$

We have now

$$\int_0^y e^{-ay}\,dy \int_\xi^\infty \frac{\sin(xy)\cos x}{x}\,dx$$

$$= \tfrac{1}{2}\int_0^y e^{-ay}\,dy\left[\int_\xi^\infty \frac{\sin(y+1)x}{x}\,dx + \int_\xi^\infty \frac{\sin(y-1)x}{x}\,dx\right] = I_1 + I_2, \text{ say.}$$

Then $$|I_1| < \frac{1}{\xi}\int_0^y \frac{e^{-ay}\,dy}{y+1}, \text{ by using inequality (7),}$$

or $$|I_1| < \frac{1}{\xi}\int_0^\infty e^{-ay}\,dy < 1/a\xi.$$

In order to examine I_2 we divide the integral into three parts, from 0 to $1-\theta$, from $1-\theta$ to $1+\theta$, and from $1+\theta$ to y, where θ is a positive proper fraction but is not yet precisely determined. Then

$$\left|\tfrac{1}{2}\int_0^{1-\theta} e^{-ay}\,dy \int_\xi^\infty \frac{\sin(y-1)x}{x}\,dx\right| < \int_0^{1-\theta}\frac{e^{-ay}}{\xi(1-y)}\,dy < \frac{1}{\xi\theta}$$

by using first the inequality (7) and then the inequality $e^{-ay}/(1-y) < 1/\theta$.

Next $$\left|\tfrac{1}{2}\int_{1-\theta}^{1+\theta} e^{-ay}\,dy \int_\xi^\infty \frac{\sin(y-1)x}{x}\,dx\right| < \frac{\pi}{2}\int_{1-\theta}^{1+\theta} e^{-ay}\,dy < \pi\theta$$

where we use inequality (8); and, last of all,

$$\left|\tfrac{1}{2}\int_{1+\theta}^\infty e^{-ay}\,dy \int_\xi^\infty \frac{\sin(y-1)x}{x}\,dx\right| < \int_{1+\theta}^\infty \frac{e^{-ay}}{\xi(y-1)}\,dy < \frac{1}{\xi\theta}\int_{1+\theta}^\infty e^{-ay}\,dy.$$

Taking the last three results together, we have $|I_2| < 1/\xi\theta(1+1/a)+\pi\theta$,

and so $$|I_1+I_2| < \frac{1}{a\xi} + \frac{1}{\xi\theta}\left(1+\frac{1}{a}\right) + \pi\theta,$$

from which it is clear that we can make $|I_1+I_2| < \epsilon$ by choosing first $\theta < \frac{1}{3}\epsilon/\pi$ and then ξ_0 so as to give both $1/a\xi_0 < \frac{1}{3}\epsilon$, $(1+1/a)/\xi_0\theta < \frac{1}{3}\epsilon$, after which ξ may be any number greater than ξ_0.

Thus condition (iii.) is satisfied; and so the inversion of the order of integration is permissible in this case. Using the results

$$\int_0^\infty f(x, y)\,dx = \tfrac{1}{2}\pi e^{-ay} \quad (y > 1)$$

$$= 0 \quad (y < 1),$$

$$\int_0^\infty f(x, y)\,dy = \frac{\cos x}{a^2+x^2},$$

we deduce that
$$\int_0^\infty \frac{\cos x}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-a};$$
a familiar result which can be proved in other ways.*

As a second example I take one from the investigation which is ordinarily given † of Dirichlet's result
$$\frac{\Gamma'(p)}{\Gamma(p)} = \int_0^\infty \left[e^{-x}-(1+x)^{-p}\right]\frac{dx}{x}.$$
In this work we are concerned with the inversion of the repeated integral
$$\int_0^\infty x^{p-1}e^{-x}dx \int_0^\infty (e^{-y}-e^{-xy})\,\frac{dy}{y}.$$

To deal with this we take conditions (iv.), (v.) in the form which is obtained by interchanging x, y, &c.; there is no difficulty in seeing that conditions (i.) and (iv.) are satisfied. To apply conditions (iii.), (v.) we have to consider
$$\int_0^y dy \int_\xi^\infty f(x, y)\,dx.$$

Now, by the mean value theorem of the differential calculus, we have $e^{-y}-e^{-xy} = (xy-y)e^{-\lambda y}$ where λ is between x and 1; hence, if $x > 1$, we have $|f(x, y)| < x^q e^{-(x+y)}$ where q is the real part of p, so as to include the case when p is complex.

Hence
$$\left|\int_0^y dy \int_\xi^\infty f\,dx\right| < \int_0^y e^{-y}dy \int_\xi^\infty x^q e^{-x}dx \quad \text{if} \quad \xi > 1,$$
and
$$\int_0^y e^{-y}dy < \int_0^\infty e^{-y}dy = 1.$$
Thus
$$\left|\int_0^y dy \int_\xi^\infty f\,dx\right| < \int_\xi^\infty x^q e^{-x}dx,$$
and this integral can be made less than ϵ by properly selecting ξ; *i.e.*, we have
$$\left|\int_0^y dy \int_\xi^\infty f\,dx\right| < \epsilon, \quad \text{for } all\ y,$$
and so conditions (iii.) and (v.) are both satisfied. To verify condition (ii.),

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* In many of the older books on the calculus this method is given without any justification of the step of inversion; even in Harnack's book the argument is inadequate. It seems that de la Vallée-Poussin was the first to justify the step in this case (see his Prize Essay, *Annales de la soc. sci. de Bruxelles*, t. XVI., 1891-2, p. 167). Unfortunately I have not been able to consult this essay; but Stolz (*Grundzüge*, Theil 3, pp. 31-33) gives an investigation which I suppose to be essentially the same as de la Vallée-Poussin's.

† For instance:—Jordan, *Cours d'Analyse*, t. II., §§ 175-178; or Whittaker, *Modern Analysis*, § 103. Jordan's treatment is based on de la Vallée-Poussin's tests (see § 6), while Whittaker omits the proof that the inversion is permissible.

we take $$\left|\int_0^x dx \int_\eta^\infty f\,dy\right| \leqslant \left|\int_0^1 dx \int_\eta^\infty f\,dy\right| + \left|\int_1^x dx \int_\eta^\infty f\,dy\right|.$$

Now $$\left|\int_0^1 dx \int_\eta^\infty f\,dy\right| < \frac{1}{\eta}\int_0^1 x^{q-2}\,dx$$

by using the obvious result

$$\int_\eta^\infty (e^{-xy}-e^{-y})\frac{dy}{y} < \int_\eta^\infty e^{-xy}\frac{dy}{y} \quad (\text{if } x<1) \quad < \frac{1}{\eta}\int_\eta^\infty e^{-xy}\,dy < \frac{1}{x\eta}.$$

Thus $$\left|\int_0^1 dx \int_\eta^\infty f\,dy\right| < \frac{1}{(q-1)\eta},$$

provided that $q>1$. Again, if $x>1$, we have

$$\int_\eta^\infty (e^{-y}-e^{-xy})\frac{dy}{y} < \int_\eta^\infty e^{-y}\frac{dy}{y} < \frac{1}{\eta}\int_\eta^\infty e^{-y}\,dy < \frac{1}{\eta},$$

and so $$\left|\int_1^x dx \int_\eta^\infty f\,dy\right| < \frac{1}{\eta}\int_1^x x^{q-1}e^{-x}\,dx < \frac{1}{\eta}\Gamma(q).$$

Hence $$\left|\int_0^x dx \int_\eta^\infty f\,dy\right| < \frac{1}{\eta}\left[\frac{1}{q-1}+\Gamma(q)\right],$$

and this can, accordingly, be made less than ϵ by properly choosing η, provided that $q>1$; and so condition (ii.) is also satisfied. Thus the inversion is justified when the real part of p is greater than 1; however, this restriction on p does not appear to be necessary for the truth of the final result, as we can see in the following way. We have

$$\int_0^\infty \left[\frac{1}{(1+x)^{p-1}} - \frac{1}{(1+x)^p}\right]\frac{dx}{x} = \int_0^\infty \frac{dx}{(1+x)^p} = \frac{1}{p-1},$$

provided that the real part of p is greater than 1. Hence

$$\frac{\Gamma'(p)}{\Gamma(p)} - \frac{1}{p-1} = \int_0^\infty \left[e^{-x}-(1+x)^{-(p-1)}\right]\frac{dx}{x};$$

for the inversion (on which Dirichlet's equation depends) has been justified if the real part of p is greater than 1.

Now $$\Gamma(p) = (p-1)\Gamma(p-1),$$

and so $$\frac{\Gamma'(p)}{\Gamma(p)} - \frac{1}{p-1} = \frac{\Gamma'(p-1)}{\Gamma(p-1)};$$

thus $$\frac{\Gamma'(p-1)}{\Gamma(p-1)} = \int_0^\infty \left[e^{-x}-(1+x)^{-(p-1)}\right]\frac{dx}{x}.$$

Hence Dirichlet's result holds if the real part of p is *positive*; and it seems likely, therefore, that the inversion is still permissible if the real part of p is a positive proper fraction. This fact tends to confirm the conclusion to which we were led before (p. 189), that conditions (i.)–(v.) are sufficient but not necessary for the inversion; however, the evidence is not complete, as I have not been able to show that the restriction ($q > 1$) is necessary to satisfy condition (ii.).

A number of examples somewhat similar to this one will be found in the analysis which leads to the expression for $\log \Gamma(p)$ as a definite integral in Binet's form.*

As a third example let us take

$$f(x, y) = \frac{\partial^2 V}{\partial x \partial y}, \quad V = \frac{xy}{1+x^2+y^2}, \quad a = 0, \quad b = 0.$$

This example is artificial, in the sense that it has no application such as those which were found for the other two cases, but it illustrates a point which the others do not. For this example enables us to distinguish clearly between the conditions obtained in this paragraph and those given by other writers.

We find at once that conditions (i.), (iv.) are satisfied, and

$$\int_0^x dx \int_\eta^\infty f\, dy = \int_0^x \frac{-\eta(1-x^2+\eta^2)}{(1+x^2+\eta^2)^2}\, dx = -\frac{x\eta}{1+x^2+\eta^2}.$$

Thus we can satisfy condition (ii.) by taking $\eta_0 \geqslant \xi/\epsilon$, and (v.) by taking $\xi \geqslant \eta/\epsilon$.

In like manner we find

$$\int_0^y dy \int_\xi^\infty f\, dx = -\frac{\xi y}{1+\xi^2+y},$$

and so we can satisfy condition (iii.) by taking $\xi_0 \geqslant \eta/\epsilon$.

Hence all the conditions are satisfied and the inversion is permissible. This result is clearly right, for the repeated integral is completely symmetrical in x and y. But, as will be seen in § 6, the conditions given by de la Vallée-Poussin would *not* allow of inversion in this case.

To illustrate the effect of breaking one of the prescribed conditions, we may consider the following case (again an artificially constructed integral):—

$$f(x, y) = \frac{\partial^2 V}{\partial x \partial y} = \frac{1-x^2y^2}{(1+x^2y^2)^2}, \quad V = \tan^{-1}(xy), \quad a = 0, \quad b = 1.$$

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* See § 104 of Whittaker's book, just quoted. I have not met with any justification of the inversions which are there employed. The work involved in the various cases is rather long and no special point seems to occur, so that I shall not stop to reproduce the details.

In this case the inversion is *not* permissible, for we have

$$\int_1^\infty f\,dy = -\frac{1}{1+x^2}, \qquad \int_0^\infty dx \int_1^\infty f\,dy = -\frac{\pi}{2};$$

but
$$\int_0^\infty f\,dx = 0, \qquad \int_1^\infty dy \int_0^\infty f\,dx = 0.$$

To examine the condition which is broken we calculate

$$\int_0^x dx \int_\eta^\infty f\,dy = -\int_0^x \frac{\eta\,dx}{1+x^2\eta^2} = -\tan^{-1}(x\eta),$$

from which it is clear that neither condition (ii.) nor (v.) can be satisfied. However, we find

$$\int_1^y dy \int_\xi^\infty f\,dx = -\int_1^y \frac{\xi\,dy}{1+\xi^2 y^2} = \tan^{-1}(\xi) - \tan^{-1}(\xi y),$$

and so by proper choice of ξ we can satisfy condition (iii.) and, moreover, the form of condition (v.) which is found by interchanging x and y.

Before leaving this paragraph it will be well to point out a misleading kind of "proof" which has sometimes been adopted in justifying the process of inverting a repeated integral. The following example (taken from Harnack's *Calculus*, § 158) illustrates the point at issue.

In order to justify the inversion of the integral $\int_0^\infty \sin y\,dy \int_0^\infty e^{-yx^2}\,dx$ Harnack argues thus (according to the English translation):—"These integrations can be interchanged, for $\sin y e^{-yx^2}$ is everywhere a continuous function, that for $y = \infty$ becomes infinitely small in a higher order than any algebraic function." But this kind of argument would only be valid if the integral were *absolutely* convergent: and it is not. For we have

$$|\sin y\, e^{-yx^2}| \geqslant \sin^2 y\, e^{-yx^2}.$$

Thus

$$\int_0^\infty dy \int_0^\infty \left|\sin y\, e^{-yx^2}\right| dx \geqslant \int_0^\infty \sin^2 y\,dy \int_0^\infty e^{-yx^2}\,dx \geqslant \frac{2}{\sqrt{\pi}} \int_0^\infty y^{-\frac{1}{2}} \sin^2 y\,dy,$$

and the last integral does not converge. Thus the original integral is *not* absolutely convergent; and, accordingly, Harnack's argument is not sufficient to justify the inversion. We proceed to apply the conditions already enunciated, and calculate first an upper limit to

$$\left| \int_0^x dx \int_\eta^\infty \sin y\, e^{-yx^2}\,dy \right|.$$

We find that
$$\left| \int_\eta^\infty \sin y\, e^{-yx^2}\,dy \right| < (1+x^2)\,e^{-x^2\eta},$$

and so $$\left| \int_0^x dx \int_\eta^x \sin y \, e^{-yx^2} dy \right| < \tfrac{1}{2} \sqrt{\frac{\pi}{\eta}} \left(1 + \frac{1}{2\eta}\right).$$

From this it is clear that conditions (ii.) and (v.) can be satisfied by proper choice of η only.

To consider $\int_0^y dy \int_\xi^\infty f\,dx$, we shall find it convenient to observe that

$$\int_\xi^\infty e^{-x^2 y} dx = \int_{\xi\sqrt{y}}^\infty e^{-z^2} dz,$$

and so this integral may be denoted by $F(\xi\sqrt{y})$. From the definition of $F(\xi\sqrt{y})$ it is clear that this function is positive and continually diminishes as y increases (ξ being kept fixed); moreover, the function converges towards zero as y increases to ∞. It follows that we may write*

$$\int_0^y \sin y \, F(\xi\sqrt{y}) \, dy \leqslant \int_0^\pi \sin y \, F(\xi\sqrt{y}) \, dy.$$

Now $$F(\xi\sqrt{y}) = \int_\xi^\infty e^{-x^2 y} dx < \frac{1}{2\xi y} \int_\xi^\infty 2xy \, e^{-x^2 y} dx < \frac{1}{2\xi y}.$$

Hence $$\int_0^y \sin y \int_\xi^\infty e^{-x^2 y} dx < \int_0^\pi \frac{\sin y}{2\xi y} dy < \frac{\pi}{2\xi},$$

the last step following from inequality (8) given above. It is now clear that condition (iii.) can be satisfied by proper choice of ξ.

6. *A Comparison of the Tests in § 5 with those at present known.*

Undoubtedly the best known and most generally used tests are those of de la Vallée-Poussin† :—

A. *If the integral* $\int_a^\infty f(x, y)\,dx$ *is uniformly convergent in an arbitrary interval,* $\int_b^\infty f(x, y)\,dy$ *uniformly convergent (in general) in an arbitrary interval, and* $\int_a^\infty dx \int_b^y f(x, y)\,dy$ *uniformly convergent in an unlimited*

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* This is proved by dividing the range of integration into intervals: 0 to π, π to 2π, 2π to 3π, …, $n\pi$ to y $[y \leqslant (n+1)\pi]$. We then obtain a number of decreasing terms, alternately positive and negative; thus the sum is positive and less than the first term.

† Given first in his prize essay, already quoted, and in *Liouville's Journal de Mathématiques* (4), t. VIII., 1892, p. 421. The conditions, as quoted, are taken from §§ 76, 77 of the second paper; the first paper has unfortunately been inaccessible to me.

interval, then

$$\int_a^\infty dx \int_b^\infty f(x, y)\,dy = \int_b^\infty dy \int_a^\infty f(x, y)\,dx.$$

B. *If the integrals* $\int_a^\infty f(x, y)\,dx$, $\int_b^\infty f(x, y)\,dy$ *are uniformly convergent (in general) in arbitrary intervals; if the integrals* $\int_b^\infty dy \int_a^x f(x, y)\,dx$, $\int_a^\infty dx \int_b^y f(x, y)\,dy$ *are uniformly convergent in unlimited intervals; and if one of the two integrals* $\int_a^\infty dx \int_b^\infty f(x, y)\,dy$, $\int_b^\infty dy \int_a^\infty f(x, y)\,dx$ *has a determinate value, then the other is also determinate and is equal to the first.*

In order to make clear theorems A and B, we must give de la Vallée-Poussin's definitions of the various terms involved (§ 69 of his second paper):—

The integral $\int_a^\infty f(x, y)\,dx$ is *uniformly convergent* in the interval $b \leqslant y \leqslant d$, if, ϵ being given, we can take N large enough to give

$$\left| \int_X^\infty f(x, y)\,dx \right| < \epsilon, \quad \text{for} \quad X \geqslant N,$$

whatever may be the value of y in the interval.*

If, however large d may be, this condition is satisfied, the integral is said *to converge uniformly in an arbitrary interval.*

If the condition is satisfied for all values of $y \geqslant b$, the *convergence is uniform in an unlimited interval.*

If the condition is verified for every part of an interval which does not contain any point of a certain aggregate of the first kind,† *the convergence is, in general, uniform in the interval.*

We shall now show that, if we rely on de la Vallée-Poussin's tests, the third example of § 5 will not allow of an inversion. This example is given by

$$f(x, y) = \frac{\partial^2 V}{\partial x \partial y}, \qquad V = \frac{xy}{1+x^2+y^2}.$$

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* In case $f(x, y)$ becomes infinite within the limits of integration additional conditions have to be satisfied (*l.c.*, § 61), which I omit, for brevity.

† *I.e.*, with a finite number of derived aggregates: the simplest case occurs when the aggregate is finite.

Of the two tests, we choose A (for B is only required in case we wish to make more use of the saving clause "in general"). Then we find

$$\int_X^\infty f(x, y)\,dx = -\frac{X(1+X^2-y^2)}{(1+X^2+y^2)^2};$$

so

$$\left|\int_X^\infty f(x, y)\,dx\right| < \frac{1}{X},$$

and accordingly $\int_0^\infty f(x, y)\,dx$ is uniformly convergent for all values of y; and, in like manner, $\int_0^\infty f(x, y)\,dx$ is uniformly convergent for all values of x. Next we have to take $\int_X^\infty dx\int_0^y f(x, y)\,dy$, which is found to be $-\dfrac{Xy}{1+X^2+y^2}$, and this gives

$$\left|\int_X^\infty dx\int_0^X f(x, y)\,dy\right| = \frac{X^2}{1+2X^2} > \tfrac{1}{3}, \quad \text{if} \quad X > 1;$$

so that the condition of uniform convergence is not satisfied.

7. *Continuity of an Infinite Integral which contains a Variable Parameter.*

We consider here the question of the continuity of the integral

$$\phi(y) = \int_a^\infty f(x, y)\,dx,$$

and we shall simply discuss the question of the continuity of $\phi(y)$ in the neighbourhood of some special value of y, say $y = b$. For this purpose we introduce two sequences x_μ, y_ν; the first is supposed to increase to ∞ with μ, while the second converges to the limit b. Then consider the double sequence

$$I_{\mu\nu} = \int_a^{x_\mu} f(x, y_\nu)\,dx;$$

it is clear that the condition for the continuity of $\phi(y)$ follows from the conditions that we may have

$$\lim_{\mu=\infty}(\lim_{\nu=\infty} I_{\mu\nu}) = \lim_{\nu=\infty}(\lim_{\mu=\infty} I_{\mu\nu})$$

by taking all possible types for the sequences x_μ, y_ν.

Thus, using § 3, we find *the necessary and sufficient conditions for the continuity of* $\phi(y)$ *in the neighbourhood of* $y = b$:—

(i.) $\phi(y)$ *is to exist for values of* y *sufficiently near to* b, *and* $\int_a^x f(x, b)\, dx$ *is to exist for sufficiently large values of* x.

(ii.) $\lim_{x=\infty} \int_a^x f(x, b)\, dx$ *is to converge.*

(iii.) *Corresponding to every positive fraction* ϵ *and to every number* ξ_0 (*however large*), *it must be possible to determine* δ *and* ξ *such that*

$$\left| \int_\xi^\infty f(x, y)\, dx \right| \leqslant \epsilon, \quad for \quad b-\delta \leqslant y \leqslant b+\delta,$$

where $\xi \geqslant \xi_0$, *and* δ *may depend on* ξ.

This test is, essentially, the same as one given by Dini;* however, I have not met with any applications of the test in published work. As a simple illustration, let us take

$$\phi(y) = \int_0^\infty \frac{y(1-ax^2y^2)}{(1+x^2y^2)^2}\, dx ;$$

then it will be found that

$$\phi(y) = \tfrac{1}{2}(1-a) \qquad (y \gtrless 0),$$

$$\phi(0) = 0.$$

Thus $\phi(y)$ is discontinuous at $y = 0$ unless $a = 1$. Now the test given above requires that

$$\left| \int_\xi^\infty \frac{y(1-ax^2y^2)}{(1+x^2y^2)^2}\, dx \right| \leqslant \epsilon, \quad \text{for} \quad |y| \leqslant \delta,$$

i.e.,

$$\left| \frac{\frac{1}{2}(1-a)-\xi y}{1+\xi^2y^2} \right| \leqslant \epsilon, \quad \text{for} \quad |y| \leqslant \delta.$$

But this condition cannot be satisfied unless $a = 1$, and, if $a = 1$, we need only take $\delta < \epsilon/\xi$.

If, however, instead of applying § 3 to $I_{\mu\nu}$, we apply § 2, and make $I_{\mu\nu}$ uniformly convergent (rows or columns), in place of condition (iii.) as given above, we obtain :—

(iii.) *Corresponding to every positive fraction* ϵ, *it must be possible to find numbers* ξ_0, δ *such that*

$$\left| \int_\xi^\infty f(x, y)\, dx \right| \leqslant \epsilon, \quad for \quad \xi \geqslant \xi_0, \quad |y-b| \leqslant \delta.$$

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* *Grundlagen*, § 293 (in the German edition). It may be noticed that another test can be obtained similarly, by interchanging the parts played by μ and ν.

This form of test is equivalent to the one given in § 292 of Dini's book; it is necessary and sufficient for the existence of the *general* double limit $\lim_{x=\infty,\, y=b} \int_a^x f(x, y)\, dx$, but it is by no means necessary as a test of the continuity of $\phi(y)$ in the neighbourhood of $y = b$. For example, in the illustration last discussed an application of this test would fail, even if $a = 1$.

A simple case* which both forms of test will exclude is given by

$$\phi(y) = \int_a^\infty \frac{y\, dx}{(1+x^2y^2)^{\frac{3}{2}}},$$

for which

$$\phi(y) = +1-[ay/(1+a^2y^2)^{\frac{1}{2}}] \quad (y > 0),$$

$$\phi(0) = 0,$$

$$\phi(y) = -1-[ay/(1+a^2y^2)^{\frac{1}{2}}] \quad (y < 0).$$

It may be useful to point out how a special form of test given by Poincaré† can be obtained from either of those found here. Changing the notation used by Poincaré to agree with ours, we find that his conditions are:—

$$f(x, y) = F(x, y)\,.\,G(x), \qquad \phi(y) = \int_0^\infty f(x, y)\, dx,$$

with—

(i.) $\int_0^\infty G(x)\, dx$ *absolutely convergent, so that* $\int_0^\infty |\, G(x)\,|\, dx$ *is finite;*

(ii.) $|\, F(x, y)\,| < A$, *for all finite values of* x, y;

(iii.) $\lim_{y=0} F(x, y) = 1$, *for all finite values of* x;

then we can deduce $\lim_{y=0} \phi(y) = \int_0^\infty G(x)\, dx$.

For we have $|\, f(x, y)\,| < A\, |\, G(x)\,|$, and so

$$\left| \int_\xi^\infty f(x, y) \right| dx < A \int_\xi^\infty |\, G(x)\,|\, dx,$$

which can be made less than ϵ, by taking

$$\int_\xi^\infty |\, G(x)\,|\, dx \leqslant \epsilon/A.$$

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[*] Originally suggested by, but slightly transformed from, the integral which gives the normal attraction of a distribution of surface density over a circular area.

[†] *Potentiel Newtonien*, Art. 43.

This can be done, by proper choice of ξ, without any restriction on y, in virtue of condition (i.).

Hence, both our tests are satisfied and $\phi(y)$ is continuous near $y = 0$;

i.e.,
$$\lim_{y=0} \phi(y) = \phi(0) = \int_0^\infty G(x)\,dx.$$

I may record a set of conditions closely allied to those given by Poincaré, which can be applied if $G(x)$ is not absolutely convergent.

(i.) $\int_0^\infty G(x)\,dx$ *is convergent.*

(ii.) $F(x, y)$ *is positive and never increases as* x *ranges from* 0 *to* ∞, y *being kept constant.*

(iii.) $F(x, y) < A$ *for all values of* x, y.

(iv.) $\lim_{y=0} F(x, y) = 1.$

Then
$$\int_x^\infty f(x, y)\,dx = F(\xi, y)\int_\xi^{\xi'} G(x)\,dx \quad (\xi' \geqslant \xi),$$

by using O. Bonnet's form for the second mean-value theorem of the integral calculus. Hence
$$\left|\int_x^\infty f(x, y)\,dx\right| < A\left|\int_\xi^{\xi'} G(x)\,dx\right|,$$

and this can be made less than ϵ by proper choice of ξ, whatever value y may have, according to the first condition. Thus $\phi(y)$ is continuous near $y = 0$, and accordingly
$$\lim_{y=0} \int_0^\infty f(x, y)\,dx = \int_0^\infty G(x)\,dx.$$

A familiar and very useful case is given by $F(x, y) = e^{-xy}$.

ON COVARIANT TYPES OF BINARY n-ICS*

By A. YOUNG.

[Communicated April 16th, 1903.—Received April 22nd, 1903.—Received in revised form August 3rd, 1903.]

1. The "finiteness" of the complete system of concomitants of a single binary form was first proved by Gordan† in the year 1868. Before that time it was believed that quantics of order higher than 4 did not possess a finite system of concomitants. Since then many different proofs of this central fact in the theory of forms have been given by various writers. Its truth has been established for ternary forms and forms having any greater number of variables; for forms possessing two or more different *sets* of variables, whether cogredient or otherwise; for any simultaneous system of forms; and for types of concomitants when the actual number of base forms becomes infinite, but the order of each is finite. Information, with references, as to the various proofs of the finiteness is to be found in any of the various editions of Meyer's "Bericht über den gegenwärtigen Stand der Invariantentheorie."‡

The majority of the proofs, especially of the later ones, keep the attention fixed on the collective idea of finiteness, and give little or no information as to the actual composition, formation, or extent of the complete system. There are two exceptions, which stand out pre-eminently from the above general rule: these are the proof given by Gordan and that due to Jordan.

Gordan proved the theorem by actually giving a process by means of which the complete system might be established, and by showing that this process yielded only a finite number of irreducible concomitants. In the later editions of his proof the process was considerably modified, but the principle remained the same.

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* The paper as originally written consisted of §§ 2–8. The introduction (§ 1) and the table (§ 9) have been written at the suggestion of the referees.

† *Jour. für Math.* (*Crelle*), Vol. LXIX., pp. 323–354.

‡ The original edition appeared in 1892 in the *Jahresbericht der Deutschen Math.-Vereinigung*, Bd. I.; French revised edition by Fehr, 1897 (Gauthier-Villars, Paris); Italian revised edition by Vivanti, 1899 (Pellerano, Naples); and, finally, the article "Invariantentheorie" by Meyer in the *Encycl. Math. Wiss.*, Bd. I., p. 320 (I. B. 2), 1899.

Jordan's proof was published in 1876.* It is remarkable that it has not attracted more attention. The first half of the paper referred to is devoted to an introduction to the theory of symbolic algebra. The author then proceeded to examine the covariants of degree 3. His discussion was exhaustive and established the following theorem :—

If a_x^p, b_x^q, c_x^r be any three binary forms, the order of each being $\geqslant n$, then the various covariants furnished by the expression

$$(ab)^\lambda \, (bc)^\mu \, (ca)^\nu \, a_x^{p-\nu-\lambda} \, b_x^{q-\lambda-\mu} \, c_x^{r-\mu-\nu} \qquad (\lambda+\mu+\nu = n)$$

may be linearly expressed in terms of the $n+1$ covariants $C_{abc}^\rho \, (\rho = 0, 1, \ldots, n)$, where

$$C_{abc}^\rho = (ab)^{n-\rho} \, (bc)^\rho \, a_x^{p-n+\rho} \, b_x^{q-n} \, c_x^{r-\rho}.$$

They may also be linearly expressed as a function of the covariants C_{abc}^ρ, C_{bca}^ρ, C_{cab}^ρ, where $\rho \ngtr \frac{1}{3}n$. If $n = 3k+2$, these last covariants will be all independent. If $n = 3k+1$, they will be connected by a relation which will permit us to express the sum $C_{abc}^k + C_{bca}^k + C_{cab}^k$ as a linear function of those C's for which $\rho < k$. If $n = 3k$, there will be two relations permitting us to express two of C_{abc}^k, C_{bca}^k, C_{cab}^k as a function of the third and of those covariants C for which ρ is less than k.†

In the next paragraph covariants of a system of quantics, the order of each being at least $2l$, were considered. It was proved that these belonged to three classes : (i.) covariants which have a factor $(ab)^{l+1}$; (ii.) covariants which have a factor $(ab)^l \, (bc)^l \, (ca)^\rho$ $(\rho > 0)$; (iii.) covariants and products of covariants of the form $(ab)^\mu \, (bc)^\nu \, (cd)^{\mu'} \, (de)^{\nu'} \ldots a_x^{p-\mu} \ldots$; the symbols $a, b, c, \ldots$ corresponding to any quantics of the system, and the exponents $\mu, \nu, \mu', \nu', \ldots$ being different from zero and satisfying the inequalities $\nu \leqslant \frac{1}{2}\mu$, $\nu' \leqslant \frac{1}{2}\mu'$, $\nu'' \leqslant \frac{1}{2}\mu''$, ... ; $\mu \leqslant l$, $\mu' \leqslant \mu-\nu-\epsilon$, $\mu'' \leqslant \mu'-\nu'-\epsilon'$, ..., where $\epsilon^{(i)}$ is equal to 0 or 1, according as $\mu^{(i)}$ is odd or even.

In the next paragraph an upper limit to the degree in the coefficients of irreducible covariants of the third class was obtained.

In the last paragraph the finiteness was proved for any simultaneous system, the proof being equally applicable to the finiteness of types of concomitants, for the letters of a symbolical product were regarded as not interchangeable. In addition, an upper limit to the order (in the variables) of an irreducible concomitant of a system of binary forms, the order of each being equal to or less than n, was obtained. This limit, which for any value of n could be calculated by a definite arithmetical process, is the same as that given here for the first eleven values of n, but for $n = 12$, and for higher values of n, Jordan's upper limit is here shown to be too high.

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* *Liouville's Jour. de Math.*, 1876.

† This theorem is fundamental to the method used in the present paper. It is extended in Jordan's second paper, *Liouville's Jour. de Math.*, 1879. It was rediscovered by Stroh, *Math. Ann.*, Bd. XXXI., pp. 444–454, who proved it in a more general form than that given above, using an entirely different method.

In *Liouville's Journal de Mathématiques*, 1879, Jordan wrote a second paper to determine an upper limit for the degree of an irreducible concomitant.

Upper limits for the degree and order of an irreducible concomitant have also been given by Sylvester.* He assigned the limit

$$\Sigma\left(\frac{n^2+1}{2}\right)+\Sigma\left(\frac{\nu^2}{2}\right)-2$$

as an upper limit to the order of a concomitant of a simultaneous system of binary forms of odd orders $n, n', n'', \ldots$, and even orders $\nu, \nu', \nu'', \ldots$; but he gave no proof. The limit is higher than that given by Jordan.

Our knowledge of complete systems of concomitants of binary forms is still limited to a few cases. Complete systems have been calculated for single forms of orders from 1 to 8 inclusive. For two quantics (h, k) the systems are known for the cases (2, 1), (2, 2), (2, 3), (3, 3), (3, 4), (2, 5), (2, 6), (4, 4). Systems of types of concomitants are known for the first four orders separately. The system for any number of quadratics and linear forms has been found. Also rules have been given by which the additional concomitants may be at once obtained when a linear or quadratic form is added to an already known complete system. References in nearly all the cases quoted are to be found in Meyer's *Bericht*.

2. The main object of the present paper is to obtain the complete irreducible system of types of covariants of binary forms of order n whose grade does not exceed $\frac{1}{2}n$.

The theorem to be proved is in reality an extension of the theorem proved for perpetuant types by Grace,† and follows his work closely both in statement and in demonstration.

The determinant factors of the symbolical products used here are alone written down, for, the orders of the various quantics considered being known, the remaining factors can at once be supplied.

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* *Proc. Roy. Soc.*, Vol. XXVII., pp. 11, 12 (1878); *Comptes Rendus*, Vol. LXXXVI., pp. 1437-1441, 1491-1492, 1519-1522 (1878).

† *Proc. London Math. Soc.*, Vol. XXXV., pp. 107-111.

3. Any covariant which is of unit degree in the coefficients of each of the quantics $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, …, $a_{\delta_x}^{n_\delta}$ can be expressed linearly in terms of (i.) covariants of the form $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, …, $\lambda_{\delta-1} \nless 1$, and the arrangement of the letters $a_1, a_2, \dots, a_\delta$ is fixed; (ii.) covariants which have a factor of the form $(a_h a_k)^\lambda (a_k a_l)^{n_k-\lambda}$; (iii.) products of covariants of lower total degree.

Assuming that this theorem is true when the number of quantics concerned is less than δ, we shall first prove its truth when the number is equal to δ.

Any covariant of the kind under discussion may be expressed in terms of transvectants of the form $(a_{1_x}^{n_1}, C_{\delta-1})^\mu$, where $C_{\delta-1}$ is a covariant of unit degree in the coefficients of each of the quantics $a_{2_x}^{n_2}$, $a_{3_x}^{n_3}$, …, $a_{\delta_x}^{n_\delta}$. Hence, on the above assumption, the transvectant written down can be expressed linearly in terms of transvectants

$$[a_{1_x}^{n_1},\ (a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}\ \dots\ (a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}]^\mu,$$

and of covariants belonging to the second and third classes.*

Now all possible transvectants of this form are under consideration; hence each may be replaced by one of its terms.

Unless $\lambda_2+\mu < n_2$, the transvectant contains a term belonging to the second class. Hence all the covariants considered can be expressed in terms of covariants of the form $(a_1a_2)^\mu(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$, and of covariants belonging to the second and third classes.

4. Let

$$(a_4a_5)^{\lambda_4}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}\,a_{4_x}^{n_4-\lambda_4}a_{5_y}^{n_5-\lambda_4-\lambda_5}a_{6_y}^{n_6-\lambda_5-\lambda_6}\dots a_{\delta_y}^{n_\delta-\lambda_{\delta-1}} \equiv a_4^{n_4-\lambda_4}q_y^\rho.$$

Then $(a_1a_2)^\mu(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$ differs from any other term of

$$[(a_1a_2)^\mu(a_2a_3)^{\lambda_2},\quad a_{4_x}^{n_4-\lambda_4}q_y^\rho]_{y=x}^{\lambda_3} \qquad \text{(I.)}$$

by covariants in which the number of factors involving a_1, a_2, a_3 only is greater than $\lambda_2+\mu$.

Consider the transvectant $[(ab)^\lambda(bc)^\kappa,\ a_{4_x}^{n_4-\lambda_4}q_y^\rho]_{y=x}^{\lambda_3}$, where $\kappa < 2^{\delta-3}$ and a, b, c are the letters a_1, a_2, a_3 in some order.

The systems of covariants $(ab)^\lambda(bc)^\kappa$ and $[(ab)^\lambda, c_x^{m_3}]^\kappa$, where $\lambda+\kappa$ has a constant value and κ remains less than $2^{\delta-3}$, are equivalent—in the sense that any member of one system can be expressed linearly in terms of members of the other.

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* It is easy to verify that, if $C_{\delta-1}$ is reducible, the transvectant $(a_{1_x}^{n_1}, C_{\delta-1})^\mu$ belongs, on the assumption made, to the second or third class.

Let us introduce a new symbol a where $a_x^{m_1+m_2-2\lambda}$ is the covariant $(ab)^\lambda$; m_1, m_2 being the orders of a and b. Then it will be sufficient to consider the transvectants $\left[(ac)^\kappa,\ a_{4_x}^{n_4-\lambda_4}q_y^\rho\right]_{y=x}^{\lambda_3}$, where $\kappa < 2^{\delta-3}$.

But on the assumption made, since this is a covariant of the $\delta-1$ quantics $a_x^{m_1+m_2-2\lambda}$, $c_x^{m_3}$, $a_{4_x}^{n_4}$, …, $a_{\delta_x}^{n_\delta}$, it can be expressed in terms of (i.) covariants $(ac)^{\mu_2}(ca_4)^{\mu_3}(a_4a_5)^{\mu_4}\ldots(a_{\delta-1}a_\delta)^{\mu_{\delta-1}}$, where $\mu_2 \not< 2^{\delta-3}$, $\mu_3 \not< 2^{\delta-4}$, …, $\mu_{\delta-1} \not< 1$; (ii.) covariants which have a factor of the form $(a_ha_k)^\nu(a_ka_l)^{n_k-\nu}$; (iii.) products of covariants of lower total degree.

Amongst the covariants here in the second class, there are included those which have a factor $(a_ha)^\nu(aa_k)^{m_1+m_2-2\lambda-\nu}$; but we may suppose that $m_1 \not> m_2$ and $\nu \not< m_1+m_2-2\lambda-\nu$, and hence that $\nu \not< m_1-\lambda$. Then every such covariant can be expressed in terms of such as have a factor of the form $(a_ha)^{m_1-\lambda}(ab)^\lambda$, and is therefore properly to be included in the second class.

5. It follows from what we have just proved that, if either μ or λ_2 is less than $2^{\delta-3}$, the transvectant (I.) can be expressed in terms of covariants which contain a greater number of factors involving a_1, a_2, a_3 only. We may then suppose that $\lambda_2+\mu \not< 2.2^{\delta-3}$. Again, the covariant $(a_1a_2)^\mu(a_2a_3)^{\lambda_2}$ belongs to the second class unless n_1, n_2, n_3 are each greater than $\lambda_2+\mu$. For, if $n_1 \leqslant \lambda_2+\mu$,

$$\begin{aligned}(a_1a_2)^\mu(a_2a_3)^{\lambda_2} &= (a_1a_2)^\mu\left[(a_1a_3)-(a_1a_2)\right]^{n_1-\mu}(a_2a_3)^{\lambda_2+\mu-n_1}\\ &= \Sigma A_r(a_1a_3)^r(a_1a_2)^{n_1-r}(a_2a_3)^{\lambda_2+\mu-n_1}.\end{aligned}$$

In this case the transvectant (I.), and every one of its terms, belongs to the second class.

When n_1, n_2, n_3 are all greater than $\lambda_2+\mu$, and $\lambda_2+\mu$ is not less than $2^{\delta-2}$, we may express the covariant $(a_1a_2)^\mu(a_2a_3)^{\lambda_2}$ linearly in terms of the covariants*

$$\begin{array}{llll}(a_2a_3)^{\lambda_2+\mu}, & (a_2a_3)^{\lambda_2+\mu-1}(a_3a_1), & \ldots, & (a_2a_3)^{\lambda_2+\mu-2^{\delta-3}+1}(a_3a_1)^{2^{\delta-3}-1};\\ (a_3a_1)^{\lambda_2+\mu}, & (a_3a_1)^{\lambda_2+\mu-1}(a_1a_2), & \ldots, & (a_3a_1)^{\lambda_2+\mu-2^{\delta-3}+1}(a_1a_2)^{2^{\delta-3}-1};\\ (a_1a_2)^{\lambda_2+\mu}, & (a_1a_2)^{\lambda_2+\mu-1}(a_2a_3), & \ldots, & (a_1a_2)^{2^{\delta-2}}(a_2a_3)^{\lambda_2+\mu-2^{\delta-2}}.\end{array}$$

Now, by § 3, if C be any one of the covariants in the first two rows just written down, the number of factors involving a_1, a_2, a_3 only in the transvectant $\left(C,\ a_{4_x}^{n_4-\lambda_4}q_y^\rho\right)_{y=x}^{\lambda_3}$ can be increased.

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[*] Stroh, *Math. Ann.*, Bd. XXXI., pp. 444–454; Jordan, *Liouville's Jour. de Math.*, 1876, 1879.

Hence these covariants can ultimately be expressed in terms of covariants of the second and third classes, and of covariants which contain the factor $(a_1 a_2)^{\lambda_1}$, where $\lambda_1 \nless 2^{\delta-2}$.

If, now, we write $a_x^{n_1+n_2-2\lambda_1} \equiv (a_1 a_2)^{\lambda_1}$, and use the assumption for covariants of total degree $\delta-1$, we see, as in § 3, that all covariants can be expressed linearly in terms of covariants of the second and third classes, and of covariants of the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_\delta-1}$, where

$$\lambda_1 \nless 2^{\delta-2}, \quad \lambda_2 \nless 2^{\delta-3}, \quad \dots, \quad \lambda_\delta \nless 1.$$

Thus the theorem is true when there are δ quantics involved, provided that it is true when any $\delta-1$ quantics are involved.

6. When only three quantics are involved we have to consider covariants of the form $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} (a_3 a_1)^{\nu}$.

If $\lambda+\mu = n_2$, this covariant belongs to the second class.

If $\lambda+\mu < n_2$, the index of $(a_3 a_1)$ can be diminished by means of the identity

$$(a_3 a_1) = -(a_1 a_2)-(a_2 a_3).$$

Hence we have only to consider covariants $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu}$, where $\lambda+\mu < n_2$.

If $\mu = n_3$, this covariant belongs to the second class.

If $\mu < n_3$ and $\lambda = 1$, we can, by means of the relation

$$\begin{aligned}(a_1 a_3)^{\mu+1} &= \{(a_1 a_2)+(a_2 a_3)\}^{\mu+1} \\ &= (a_2 a_3)^{\mu+1} + \binom{\mu+1}{1}(a_1 a_2)(a_2 a_3)^{\mu} + \binom{\mu+1}{2}(a_1 a_2)^2 (a_2 a_3)^{\mu-2} + \dots,\end{aligned}$$

express the covariant in terms of members of the first and third classes.

Hence the theorem is true when three quantics are involved; it is therefore true in general.

7. Let us suppose that all the quantics considered in the theorem just proved are of the same order n. Then the covariants of the second class all contain a factor of the form $(ab)^{\lambda}(bc)^{n-\lambda}$. These can be expressed in terms of transvectants, such as $[(ab)^{\lambda}(bc)^{n-\lambda}(ca)^{r}, C]^{\varpi}$, where r is positive or zero.

Now Jordan* has proved that the covariant $(ab)^{\lambda}(bc)^{n-\lambda}(ca)^{r}$ can be expressed in terms of covariants $(ab)^{\mu}(bc)^{n-\mu}(ca)^{r}$, where $\mu \geqslant \frac{1}{2}n$, and $\mu-r \geqslant 2(n-\mu-r)$, *i.e.*, $r \nless 2n-3\mu$.

Hence all covariant types of binary forms of order n can be expressed in terms of (i.) covariant types of the form $(a_1 a_2)^{\lambda_1}(a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_\delta-1}$, where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, ..., $\lambda_{\delta-1} \nless 1$; (ii.) covariant types which have a factor $(ab)^{\lambda}(bc)^{n-\lambda}(ca)^{r}$, where $\lambda \nless \frac{1}{2}n$, $r \nless 2n-3\lambda$; (3) products of covariants of lower degree. This theorem at once gives a system of types in terms of which all irreducible types of grade not exceeding $\frac{1}{2}n$ may be expressed.

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* *Loc cit.*

8. It appears practically certain that the theorem for perpetuant types is exact, *i.e.*, that the covariant types for binary forms of infinite order $(a_1 a_2)^{\lambda_1}(a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda}{}_{\delta}$, where $\lambda_1 \nless 2^{\delta-2}$, $\lambda_2 \nless 2^{\delta-3}$, ..., $\lambda_\delta \nless 1$, and the order of the letters is fixed beforehand, are both independent and irreducible (this system being equivalent to that used by Grace). If this is the case, covariants of this form must be independent and irreducible for quantics of finite order. It does not follow, however, that they cannot be expressed in terms of covariants of higher grade, or else in terms of covariants belonging to the second class. In fact, as it is easy to verify, if $\lambda_1+\lambda_r \geqslant n$, such a covariant type of a system of binary n-ics can be expressed in terms of members of the second class, and of covariants $(a_1 a_2)^{\lambda'_1}(a_2 a_3)^{\lambda'_2} \dots (a_{\delta-1} a_\delta)^{\lambda'_{\delta-1}}$, where one or more of the differences $\lambda'_1-\lambda_1$, $\lambda'_2-\lambda_2$, ..., $\lambda'_{r-1}-\lambda_{r-1}$ is positive, and the rest are zero.

NOTE ON THE FOREGOING PAPER.

By J. H. GRACE.

Consider any number of forms of order not exceeding n.

Following Jordan, we add to the system every covariant of the second degree whose order does not exceed n.

Thus, if a_x^p and b_x^q be two forms, we add the covariant $(ab)^{\lambda} a_x^{p-\lambda} b_x^{q-\lambda}$ to the system whenever $p+q-2\lambda \leqslant n$, and therefore it is always added to the original system when $\lambda \geqslant \frac{1}{2}p$ or $\frac{1}{2}q$.

Hence by the elementary theory of transvectants, if $\lambda \geqslant \frac{1}{2}n$, a covariant involving the factor $(ab)^{\lambda}$ can be expressed as an aggregate of covariants in each of which the letters a and b are replaced by a single symbol also belonging to a form of order not greater than n.

It follows at once that in seeking for the covariant of highest order such a covariant can be neglected because the same order would occur for a lower degree.

The covariants of Class II. in the preceding paper can therefore be neglected because (see § 1) either λ or $n_k-\lambda$ must be at least equal to $\frac{1}{2}n_k$.

Hence a covariant of the highest order will appear in the first class.

Now the highest order for degree ι is $n\iota-2^{\iota}+2$, corresponding to the form $(a_1 a_2)^{2^{\iota-2}}(a_2 a_3)^{2^{\iota-3}} \dots (a_{\iota-1} a_\iota)$.

We therefore have the following rule for finding the maximum order:—

Choose the greatest of the integers n, $2n-2$, $3n-6$, $4n-14$, $5n-30$, $6n-62$, ...*

Since the form from which this order arises is the fundamental perpetuant of degree ι, and this certainly irreducible (although not rigorously proved to be so), it follows that the order given above is a true maximum. It is actually attained by the covariants of a single quantic of order n, and it is not surpassed by the covariants of any number of forms whose orders do not exceed n. The value of ι which makes $n\iota-2^\iota+2$ a maximum is the integer next greater than $\log_2 n$.

9. It will be seen that, if $2^{j+1} > n > 2^j$, the maximum order of a covariant of a quantic or quantics of order n is $(j+1)n-2^{j+1}+2$. Thus for values of n from 1 to 100 we have the values of the maximum order as follows:—

n	Max. order.	n	Max. order.	n	Max. order.	n	Max. order.	n	Max. order.
1	1	21	75	41	184	61	304	81	441
2	2	22	80	42	190	62	310	82	448
3	4	23	85	43	196	63	316	83	455
4	6	24	90	44	202	64	322	84	462
5	9	25	95	45	208	65	329	85	469
6	12	26	100	46	214	66	336	86	476
7	15	27	105	47	220	67	343	87	483
8	18	28	110	48	226	68	350	88	490
9	22	29	115	49	232	69	357	89	497
10	26	30	120	50	238	70	364	90	504
11	30	31	125	51	244	71	371	91	511
12	34	32	130	52	250	72	378	92	518
13	38	33	136	53	256	73	385	93	525
14	42	34	142	54	262	74	392	94	532
15	46	35	148	55	268	75	399	95	539
16	50	36	154	56	274	76	406	96	546
17	55	37	160	57	280	77	413	97	553
18	60	38	166	58	286	78	420	98	560
19	65	39	172	59	292	79	427	99	567
20	70	40	178	60	298	80	434	100	574

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* See a note to a preceding paper of my own, *supra*, p. 151.

ON THE APPLICATION OF QUATERNIONS TO THE ORTHOGONAL TRANSFORMATION AND INVARIANT THEORY

By Major P. A. MACMAHON, D.Sc., F.R.S.

[Communicated June 11th, 1903.—Received October 6th, 1903.]

THE purpose of the present communication is, by means of known results in quaternions, to present the formulæ for the orthogonal transformation of three and four variables in a more convenient and symmetrical form than has yet been given, to show that the invariant theory is implied by and involved in the quaternion representation, and to establish, in the case of the ternary quantic, the irreducible orthogonal forms which do not exceed the fourth degree in the coefficients in the case of invariants or the third degree in the case of covariants as far as the ternary quantic inclusive.

1. *Orthogonal Ternary Transformation.*

Suppose the ternary quantic $a_x^n = (a_1x_1+a_2x_2+a_3x_3)^n$ connected with the form $A_x^n = (A_1x_1+A_2x_2+A_3x_3)^n$ by the orthogonal transformation

$$\left.\begin{aligned} x_1 &= p_1X_1+q_1X_2+r_1X_3 \\ x_2 &= p_2X_1+q_2X_2+r_2X_3 \\ x_3 &= p_3X_1+q_3X_2+r_3X_3 \end{aligned}\right\}. \tag{1}$$

The quaternion imaginaries i, j, k, I denote by i_1, i_2, i_3; so that in Aronhold's notation the quaternion vector

$$x_1i+x_2j+x_3k = x_1i_1+x_2i_2+x_3i_3 = x_i.$$

Hamilton, Cayley, and others have shown that the transformation may be represented by the quaternion identity

$$x_i = (a_0+a_i)\,X_i\,(a_0-a_i), \tag{2}$$

wherein the squared tensor of the quaternion a_0+a_ι is unity, viz.,

$$a_0^2+a_1^2+a_2^2+a_3^2 = 1;$$

so that the identity involves only three independent quantities a. Thus the conjugate quaternions a_0+a_ι, a_0-a_ι are such that each is the inverse of the other, $(a_0+a_\iota)(a_0-a_\iota) = 1$. It is easy to verify the quaternion identity, for

$$x_1 i_1+x_2 i_2+x_3 i_3$$
$$= (a_0+a_\iota)\,i_1\,(a_0-a_\iota)\,X_1+(a_0+a_\iota)\,i_2\,(a_0-a_\iota)\,X_2+(a_0+a_\iota)\,i_3\,(a_0-a_\iota)\,X_3, \tag{3}$$

while, from equations (1),

$$x_1 i_1+x_2 i_2+x_3 i_3 = p_\iota X_1+q_\iota X_2+r_\iota X_3, \tag{4}$$

giving

$$\left.\begin{aligned} p_\iota &= (a_0+a_\iota)\,i_1\,(a_0-a_\iota) \\ q_\iota &= (a_0+a_\iota)\,i_2\,(a_0-a_\iota) \\ r_\iota &= (a_0+a_\iota)\,i_3\,(a_0-a_\iota) \end{aligned}\right\}, \tag{5}$$

and, squaring each side,

$$p_\iota^2 = -p_p = -1, \qquad q_\iota^2 = -q_q = -1, \qquad r_\iota^2 = -r_r = -1$$

and

$$p_\iota q_\iota+q_\iota p_\iota = -p_q = 0, \quad q_\iota r_\iota+r_\iota q_\iota = -q_r = 0, \quad r_\iota p_\iota+p_\iota r_\iota = -r_p = 0,$$

the six relations required by orthogonality.

From equations (5) we derive

$$\left.\begin{aligned} (a_0-a_\iota)\,p_\iota\,(a_0+a_\iota) &= i_1 \\ (a_0-a_\iota)\,q_\iota\,(a_0+a_\iota) &= i_2 \\ (a_0-a_\iota)\,r_\iota\,(a_0+a_\iota) &= i_3 \end{aligned}\right\} \tag{6}$$

and also

$$\left.\begin{aligned} p_\iota(a_0+a_\iota) &= (a_0+a_\iota)\,i_1 \\ q_\iota(a_0+a_\iota) &= (a_0+a_\iota)\,i_2 \\ r_\iota(a_0+a_\iota) &= (a_0+a_\iota)\,i_3 \end{aligned}\right\}. \tag{7}$$

From equations (5) are derived

$$\left.\begin{aligned} p_1 &= a_0^2+a_1^2-a_2^2-a_3^2, & p_2 &= 2(a_0a_3+a_1a_2), & p_3 &= 2(-a_0a_2+a_1a_3) \\ q_1 &= 2(-a_0a_3+a_1a_2), & q_2 &= a_0^2-a_1^2+a_2^2-a_3^2, & q_3 &= 2(a_0a_1+a_2a_3) \\ r_1 &= 2(a_0a_2+a_1a_3), & r_2 &= 2(-a_0a_1+a_2a_3), & r_3 &= a_0^2-a_1^2-a_2^2+a_3^2 \end{aligned}\right\} \tag{8}$$

and from equations (7)

$$\left.\begin{aligned}
a_1p_1+a_2p_2+a_3p_3 &= a_1\\
a_0p_1+a_3p_2-a_2p_3 &= a_0\\
-a_3p_1+a_0p_2+a_1p_3 &= a_3\\
a_2p_1-a_1p_2+a_0p_3 &= -a_2\\
a_1q_1+a_2q_2+a_3q_3 &= a_2\\
a_0q_1+a_3q_2-a_2q_3 &= -a_3\\
-a_3q_1+a_0q_2+a_1q_3 &= a_0\\
a_2q_1-a_1q_2+a_0q_3 &= a_1\\
a_1r_1+a_2r_2+a_3r_3 &= a_3\\
a_0r_1+a_3r_2-a_2r_3 &= a_2\\
-a_3r_1+a_0r_2+a_1r_3 &= -a_1\\
a_2r_1-a_1r_2+a_0r_3 &= a_0
\end{aligned}\right\}. \qquad (9)$$

From equations (1) we obtain $a_x = a_pX_1+a_qX_2+a_rX_3$. Hence $A_1 = a_p$, $A_2 = a_q$, $A_3 = a_r$, and, from equation (2), $X_\iota = (a_0-a_\iota)\,x_\iota\,(a_0+a_\iota)$, so that, if

$$X_1 = P_1x_1+Q_1x_2+R_1x_3, \qquad X_2 = P_2x_1+Q_2x_2+R_2x_3,$$
$$X_3 = P_3x_1+Q_3x_2+R_3x_3,$$

$$\begin{aligned}
&X_1i_1+X_2i_2+X_3i_3\\
&\qquad = (a_0-a_\iota)\,i_1\,(a_0+a_\iota)\,x_1+(a_0-a_\iota)\,i_2\,(a_0+a_\iota)\,x_2+(a_0-a_\iota)\,i_3\,(a_0+a_\iota)\,x_3\\
&\qquad = P_\iota x_1+Q_\iota x_2+R_\iota x_3,
\end{aligned}$$

and it is clear that in the formulæ (5), (6), (7), (8), (9) we may change p, q, r into P, Q, R, if we at the same time change a_ι into $-a_\iota$ or a_1, a_2, a_3 into $-a_1$, $-a_2$, $-a_3$. Thus

$$\begin{aligned}
P_1 &= (a_0^2+a_1^2-a_2^2-a_3^2), & Q_1 &= 2\,(a_0a_3+a_1a_2), & R_1 &= 2\,(-a_0a_2+a_1a_3)\,;\\
P_2 &= 2\,(-a_0a_3+a_1a_2), & Q_2 &= (a_0^2-a_1^2+a_2^2-a_3^2), & R_2 &= 2\,(a_0a_1+a_2a_3)\,;\\
P_3 &= 2\,(a_0a_2+a_1a_3), & Q_3 &= 2\,(-a_0a_1+a_2a_3), & R_3 &= (a_0^2-a_1^2-a_2^2+a_3^2).
\end{aligned}$$

Also, from (1) and (10), $x_\alpha = X_\alpha$.

Now, since $A_1 = a_p$, $A_2 = a_q$, $A_3 = a_r$, $A_\iota = a_pi_1+a_qi_2+a_ri_3$; but $X_\iota = x_pi_1+x_qi_2+x_ri_3$; hence A and a are connected in the same way as X and x, and we have

$$a_\iota = (a_0+a_\iota)\,A_\iota\,(a_0-a_\iota), \qquad A_\iota = (a_0-a_\iota)\,a_\iota\,(a_0+a_\iota),$$
$$a_\alpha = A_\alpha.$$

It follows from the above results that the quaternion vectors x_i, a_i may be regarded as pseudo-invariants, and it may be shown that the real invariants may be exhibited as rational integral combinations of pseudo-invariants.

Since
$$a_i x_i = (a_0+a_i)A_i(a_0-a_i)(a_0+a_i)X_i(a_0-a_i)$$
$$= (a_0+a_i)A_i X_i(a_0-a_i),$$

it is seen that every product of powers of vectors a_i, b_i, c_i, ..., x_i, y_i, z_i, ..., and also every linear function of such products is a pseudo-invariant.

If f be a function of pseudo-invariants which is scalar, and F its transform, $f = (a_0+a_i)F(a_0-a_i) = F$, and f is a real invariant.

The problem of the orthogonal ternary invariants is thus reduced to the discovery of those functions of quaternion vectors which are mere scalars: *e.g.*,
$$x_i^2 = -x_1^2-x_2^2-x_3^2 = -x_x,$$
$$a_i^2 = -a_1^2-a_2^2-a_3^2 = -a_a,$$

showing that x_x and a_a are invariants, for the quaternion formula immediately yields $x_x = X_X$, $a_a = A_A$.*

The form a_a or x_x is the only type of invariant symbolic factor which involves one set of letters or umbræ.

When two sets are involved we have
$$a_i b_i + b_i a_i = -2a_b, \qquad a_i x_i + x_i a_i = -2a_x,$$

and we know that no other types exist which involve two sets of letters or umbræ.

The remaining symbolic factors (abc), (abx) are derived from the formulæ
$$a_i b_i c_i - c_i b_i a_i = -2(abc), \quad a_i b_i x_i - x_i b_i a_i = -2(abx),$$

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* The fundamental formula $x_i = (a_0+a_i)X_i(a_0-a_i)$ is one of a set of four, for the formulæ of transformation given by (1) and (8) are unchanged

(i.) by changing the signs of a_2, a_3, x_2, x_3, X_2, X_3;
(ii.) ,, ,, a_3, a_1, x_3, x_1, X_3, X_1;
(iii.) ,, ,, a_1, a_2, x_1, x_2, X_1, X_2;

so that, writing in general
$$p_1 i_1 - p_2 i_2 - p_3 i_3 = p_i', \quad -p_1 i_1 + p_2 i_2 - p_3 i_3 = p_i'', \quad -p_1 i_1 - p_2 i_2 + p_3 i_3 = p_i''',$$
we obtain the quaternion formulæ
$$x_i' = (a_0+a_i')X_i'(a_0-a_i'), \quad x_i'' = (a_0+a_i'')X_i''(a_0-a_i''), \quad x_i''' = (a_0+a_i''')X_i'''(a_0-a_i''').$$

and now we have the complete set of factors a_a, a_b, (abc), a_x, (abx), x_x in terms of which all orthogonal invariants and covariants are expressible.

The result $a_i x_i + x_i a_i = -2a_x$ shows that we may take the fundamental ternary quantic of order n to be $(a_i x_i + x_i a_i)^n$, which on expansion is

$$\begin{aligned} &\{(a_i x_i)^n + (x_i a_i)^n\} \\ &+\binom{n}{1}\{(a_i x_i)^{n-2} + (x_i a_i)^{n-2}\}\, a_a x_x \\ &+\binom{n}{2}\{(a_i x_i)^{n-4} + (x_i a_i)^{n-4}\}\, a_a^2 x_x^2 \\ &+\dots, \end{aligned}$$

yielding a fundamental set of covariants of degree 1 in the coefficients, the covariants being of type $a_a^p\{(a_i x_i)^{n-2p} + (x_i a_i)^{n-2p}\}$ of degree and order 1, $n-2p$, respectively. The number of these is $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ according as n is even or uneven.

A formula of reduction for such forms is

$$(a_i x_i)^m + (x_i a_i)^m = -2a_x\{(a_i x_i)^{m-1} + (x_i a_i)^{m-1}\} - a_a x_x\{(a_i x_i)^{m-2} + (x_i a_i)^{m-2}\},$$

leading to their rapid calculation.

We may continue the series of scalars

$$\begin{gathered} a_i b_i c_i d_i + d_i c_i b_i a_i, \\ a_i b_i c_i d_i e_i - e_i d_i c_i b_i a_i, \\ a_i b_i c_i d_i e_i f_i + f_i e_i d_i c_i b_i a_i, \\ \dots \quad \dots \quad \dots \quad \dots \end{gathered}$$

indefinitely.

I find $\quad a_i b_i c_i d_i + d_i c_i b_i a_i = 2(a_b c_d - a_c b_d + a_d b_c)$,

$$a_i b_i c_i d_i e_i - e_i d_i c_i b_i a_i = 2\{a_b(cde) + b_c(ade) - a_c(bde) + d_e(abc)\}.$$

The ten expressions of form $a_b(cde)$ are not independent, for they are connected by four independent syzygies, viz.,

$$\begin{aligned} a_b(cde) - b_c(ade) + b_d(ace) - b_e(acd) &= 0, \\ a_c(bde) - b_c(ade) + c_d(abe) - c_e(abd) &= 0, \\ a_d(bce) - b_d(ace) + c_d(abe) - d_e(abc) &= 0, \\ a_e(bcd) - b_e(acd) + c_e(abd) - d_e(abc) &= 0. \end{aligned}$$

Any three of these may be derived from the fourth by interchange of letters. As a consequence, in the theory of the invariants, any one of the equations may be regarded as the fundamental syzygy for the reduction of symbolic expressions.

We can derive such useful syzygies as

$$a_x(bcd)-b_x(acd)+c_x(abd)-d_x(abc)=0,$$
$$a_a(bcd)-a_b(acd)+a_c(abd)-a_d(abc)=0,$$
$$x_x(bcd)-b_x(cdx)+c_x(bdx)-d_x(abc)=0,$$
$$a_a(bcx)-a_b(acx)+a_c(abx)-a_x(abc)=0,$$

which will be of great use in the sequel.

In the case of six sets of umbræ

$$\begin{aligned}&a_\iota b_\iota c_\iota d_\iota e_\iota f_\iota+f_\iota e_\iota d_\iota c_\iota b_\iota a_\iota\\ &=2(-a_b c_d e_f+a_b c_e d_f-a_b c_f d_e\\ &\qquad+a_c b_d e_f-a_c b_e d_f+a_c b_f d_e\\ &\qquad-a_d b_c e_f+a_d b_e c_f-a_d b_f c_e\\ &\qquad+a_e b_c d_f-a_e b_d c_f+a_e b_f c_d\\ &\qquad-a_f b_c d_e+a_f b_d c_e-a_f b_e c_d).\end{aligned}$$

There exist the fundamental identity

$$(abc)(def)=\begin{vmatrix}a_d & a_e & a_f\\ b_d & b_e & b_f\\ c_d & c_e & c_f\end{vmatrix}$$

and the fundamental syzygy

$$\begin{vmatrix}a_e & a_f & a_g & a_h\\ b_e & b_f & b_g & b_h\\ c_e & c_f & c_g & c_h\\ d_e & d_f & d_g & d_h\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3 & 0\\ b_1 & b_2 & b_3 & 0\\ c_1 & c_2 & c_3 & 0\\ d_1 & d_2 & d_3 & 0\end{vmatrix}\begin{vmatrix}e_1 & f_1 & g_1 & h_1\\ e_2 & f_2 & g_2 & h_2\\ e_3 & f_3 & g_3 & h_3\\ 0 & 0 & 0 & 0\end{vmatrix}=0.$$

2. I propose now to determine the irreducible orthogonal invariants and covariants of the ternary quantics of lower orders when the degree in the coefficients does not exceed 4.

The symbolic expression of any invariant or covariant involves factors of the forms a_a, a_b, (abc), (abx), a_x, x_x. We may omit x_x from consideration because it is itself a covariant.

The product of any two determinant factors has been shown to be expressible in terms of symbolic factors a_a, a_b, … ; so that we may regard any form as involving at most only one determinant factor. This may be either of the nature (abc) or of the nature (abx).

If the form involves (abc) and be a covariant, it must contain as well a factor of the form e_x, and, since there exists the syzygy

$$e_x(abc) = a_e(bcx) - b_e(acx) + c_e(abx),$$

we can express any such covariant in terms of covariants involving the symbolic factors of nature (abx). In consequence of this fact we are only concerned with the determinant factor (abc) when invariants are in question.

The investigation is thus divisible into four portions—

(i.) Invariants which involve factors a_a, a_b, ... only.

(ii.) Covariants which involve factors a_a, a_b, a_x, ... only.

(iii.) Invariants which involve the single determinant factor (abc).

(iv.) Covariants which involve the single determinant factor (abx).

Observing that every ternary quantic a_x^n has the special orthogonal covariant x_x, we commence by considering the invariants and covariants of degree 1 in the coefficients.

If the order of the quantic be even and equal to $2m$, we have the irreducible system a_a^m, $a_a^{m-1}a_x^2$, $a_a^{m-2}a_x^4$, ..., a_x^{2m}, and, if the order be uneven and equal to $2m+1$, the irreducible system $a_a^m a_x$, $a_a^{m-1}a_x^3$, $a_a^{m-2}a_x^5$, ..., a_x^{2m+1}. The number of invariantive forms of the first degree in the coefficients for the quantic of order n is thus the coefficient of x^n in the expansion of the fraction $\dfrac{1}{(1-x)(1-x^2)}$.

I pass to the forms of the second degree in the coefficients which are without a determinant factor (abx).

Since three letters only appear, viz., a, b, x, there are no syzygies, and the general form to be considered is

$$a_a^{p_{11}} b_b^{p_{22}} a_b^{p_{12}} a_x^{p_1} b_x^{p_2},$$

with the condition $$2p_{11}+p_1 = 2p_{22}+p_2.$$

For irreducibility p_{12} must be greater than zero; also by interchange of letters we can always take p_{11}, p_{22} to be in descending order of magnitude, so that without loss of generality we may assume

$$p_{11} \geqslant p_{22};$$

putting then $$p_{11} = p_{22}+q, \qquad p_2 = p_1+2q,$$

the general irreducible form is found to be

$$(a_a b_b)^{p_{22}} a_b^{p_{12}+1} (a_x b_x)^{p_1} (a_a b_x^2)^q,$$

and the number of forms for the quantic of order n is the coefficient of x^n in the expansion of $\dfrac{x}{(1-x)^2(1-x^2)^2}$.

For the quantics of the first few orders, we have

order 1: a_b;
order 2: a_b^2, $a_b a_x b_x$;
order 3: a_b^3, $a_b^2 a_x b_x$, $a_b a_a b_b$,
$a_b a_a b_x^2$,
$a_b a_x^2 b_x^2$;
order 4: a_b^4, $a_b^3 a_x b_x$, $a_b^2 a_a b_b$, $a_b a_a b_b a_x b_x$,
$a_b^2 a_a b_x^2$, $a_b a_a a_x b_x^3$,
$a_b^2 a_x^2 b_x^2$, $a_b a_x^3 b_x^3$.

We now come to the invariants of the third degree in the coefficients which have no determinant factor. The form is

$$a_a^{p_{11}} b_b^{p_{22}} c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}},$$

with the conditions

$$2p_{11}+p_{13} = 2p_{22}+p_{23}, \qquad 2p_{22}+p_{12} = 2p_{33}+p_{13};$$

we may suppose p_{11}, p_{22}, p_{33} to be in descending order of magnitude, and therefore p_{12}, p_{13}, p_{23} in ascending order; so that, writing

$$p_{11} = p_{33}+p+q, \quad p_{22} = p_{33}+p, \quad p_{13} = p_{12}+2p, \quad p_{23} = p_{12}+2p+2q,$$

we find the form to be

$$(a_a b_b c_c)^{p_{33}} (a_a b_b a_c^2 b_c^2)^p (a_a b_c^2)^q (a_b a_c b_c)^{p_{12}}.$$

For irreducibility p and p_{12} must not vanish together; this shows that the number of irreducible forms is the coefficient of x^n in the expansion of

$$\frac{1}{(1-x^2)^3(1-x^4)} - \frac{1}{(1-x^2)^2} = \frac{x^2+x^4-x^6}{(1-x^2)^3(1-x^4)}$$
$$= x^2+4x^4+9x^6+17x^8+28x^{10}+43x^{12}+\dots.$$

Thus for the quadric there is the form

$$a_b a_c b_c;$$

for the quartic the four forms

$$a_a b_b c_c a_b a_c b_c, \qquad a_a a_b a_c b_c^3,$$
$$a_a b_b a_c^2 b_c^2, \qquad a_b^2 a_c^2 b_c^2;$$

and for the sextic the nine forms

$$a_a^2 b_b^2 c_c^2 a_b a_c b_c, \qquad a_a b_b c_c a_b^2 a_c^2 b_c^2, \qquad a_a^2 a_b a_c b_c^5,$$
$$a_a^2 b_b^2 c_c a_c^2 b_c^2, \qquad a_a^2 b_b a_c^2 b_c^4, \qquad a_a a_b^2 a_c^2 b_c^4,$$
$$a_a^2 b_b c_c a_b a_c b_c^3, \qquad a_a b_b a_b a_c^3 b_c^3, \qquad a_b^3 a_c^3 b_c^3;$$

and the series can be continued without difficulty.

Taking up now the covariants of the third degree in the coefficients which do not possess any determinant factor, we have to deal with four letters a, b, c, x, and we must take account of the syzygy

$$\begin{vmatrix} a_a & a_b & a_c & a_x \\ b_a & b_b & b_c & b_x \\ c_a & c_b & c_c & c_x \\ a_x & b_x & c_x & x_x \end{vmatrix} = 0.$$

The general form of covariant is $a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3}$, wherein p_{11}, p_{22}, p_{33} may be taken to be in descending order of magnitude.

The conditions

$$2p_{11}+p_{13}+p_1 = 2p_{22}+p_{23}+p_2, \quad 2p_{22}+p_{12}+p_2 = 2p_{33}+p_{13}+p_3$$
$$p_{11} \geqslant p_{22} \geqslant p_{33}$$

are easily dealt with, and the result is that we have the twelve ground symbolic products

$$a_b c_x, \quad a_c b_x, \quad b_c a_x, \quad a_x b_x c_x,$$
$$a_a b_c^2, \quad a_a b_b c_c, \quad a_a b_x^2 c_x^2, \quad a_a b_b c_x^2,$$
$$a_a b_c b_x c_x, \quad a_b a_c b_c,$$
$$a_a b_b a_c b_c c_x,$$
$$a_a b_b a_c^2 b_c^2.$$

Four of these appertain to the linear quantic and are visibly reducible.

In the case of the quadric the only form that is not visibly reducible is $a_b a_c b_x c_x$, but, reducing the fundamental syzygy, we find that

$$2a_b a_c b_x c_x - 2c_c a_b a_x b_x + a_a b_b c_x^2 - a_b^2 c_x^2$$

contains x_x as a factor; hence $a_b a_c b_x c_x$ for the quadric is reducible.

For the cubic the irreducible forms may be

$$a_b^2 a_c b_c c_x, \quad a_a b_b a_c b_c c_x, \quad a_b a_c b_c a_x b_x c_x,$$
$$a_a a_b b_c^2 c_x, \quad a_b b_c^2 a_x^2 c_x, \quad a_a a_b b_c b_x c_x^2\,;$$

but, multiplying the fundamental syzygy by

$$a_b c_x$$

and reducing for the cubic, we find that

$$2a_a a_b b_c b_x c_x^2 - a_b a_c b_c a_x b_x c_x - a_b b_c^2 a_x^2 c_x$$

is reducible. We therefore discard

$$a_b a_c b_c a_x b_x c_x,$$

and take the remaining five forms as the fundamental covariants of the cubic of the third degree in the coefficients.

Passing to the quartic, the possible irreducible forms are found to be

Order 2 *in Variables.*

$$\begin{aligned} a_a a_b^2 b_c^2 c_x^2 &= A_{42}, & a_b^3 a_c b_c c_x^2 &= E_{42}, \\ a_a a_b a_c b_c^2 b_x c_x &= B_{42}, & a_b^2 a_c^2 b_c b_x c_x &= F_{42}, \\ a_a a_b b_c^3 a_x c_x &= C_{42}, & a_a b_b a_b a_c b_c c_x^2 &= G_{42}, \\ a_a b_b c_c a_b a_c b_x c_x &= D_{42}, & a_a b_b a_c^2 b_c b_x c_x &= H_{42}. \end{aligned}$$

These reduce to five independent forms, since multiplication of the fundamental syzygy by $a_a b_c^2$, $a_b a_c b_c$, $a_a b_b c_c$ shows that the linear functions

$$A_{42}-B_{42}-2C_{42}+2H_{42}, \quad 2B_{42}+E_{42}-2F_{42}-G_{42}, \quad F_{42}$$

are reducible. We take as fundamental forms A_{42}, B_{42}, C_{42}, D_{42}, E_{42}.

Order 4 *in Variables.*

$$\begin{aligned} a_b^3 a_c b_x c_x^3 &= A_{44}, & a_a b_b c_c a_b a_x b_x c_x^2 &= F_{44}, \\ a_b^2 a_c^2 b_x^2 c_x^2 &= B_{44}, & a_a b_b a_b a_c b_x c_x^3 &= G_{44}, \\ a_b^2 a_c b_c a_x b_x c_x^2 &= C_{44}, & a_a b_b a_c b_c a_x b_x c_x^2 &= H_{44}, \\ a_a b_c^3 a_x^2 b_x c_x &= D_{14}, & a_a a_b^2 b_c b_x c_x^3 &= I_{44}, \\ a_a a_b b_c^2 a_x b_x c_x^2 &= E_{44}, & a_a a_b a_c b_c b_x^2 c_x^2 &= J_{44}, \end{aligned}$$

and now, multiplying the syzygy by $a_a b_c b_x c_x$, $a_a b_b c_x^2$, $a_b^2 c_x^2$, $a_b a_c b_x c_x$, we find that the linear functions

$$2E_{44}-2H_{44}-I_{44}+J_{44}, \quad 2G_{44}+H_{44}, \quad 2A_{44}-B_{44}+C_{44}-2I_{44},$$
$$2A_{44}-2B_{44}-3C_{44}+4E_{44}+2J_{44}$$

are reducible.

We have thus six independent forms of order 4 in the variables which we take to be A_{44}, B_{44}, C_{44}, D_{44}, E_{44}, F_{44}.

Order 6 *in Variables.*

$$\begin{aligned} a_b^2 a_c a_x b_x^2 c_x^3 &= A_{46}, & a_a a_b a_c b_x^3 c_x^3 &= D_{46}, \\ a_b a_c b_c a_x^2 b_x^2 c_x^2 &= B_{46}, & a_a a_b b_c a_x b_x^2 c_x^3 &= E_{46}. \\ a_a b_c^2 a_x^2 b_x^2 c_x^2 &= C_{46}, \end{aligned}$$

Multiplying the syzygy by $a_b a_x b_x c_x^2$, $a_a b_x^2 c_x^2$, we find that the linear functions

$$A_{46}+B_{46}-2E_{46}, \quad D_{46}+2E_{46}$$

are reducible. We therefore take the three independent covariants of order 6 in the variables to be A_{46}, B_{46}, C_{46}.

Order 8 *in Variables.*

$$a_b a_c a_x^2 b_x^3 c_x^3.$$

When the syzygy is multiplied by $a_x^2 b_x^2 c_x^2$, this single form is at once seen to be reducible. This completes the discussion of covariants of degree 3 in the coefficients as far as the ternary quartic inclusive, the symbolic forms being free from a determinant factor.

For the invariants of the fourth degree in the coefficients which do not involve a determinant factor the form is

$$a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} d_d^{p_{44}} a_b^{p_{12}} a_c^{p_{13}} a_d^{p_{14}} b_c^{p_{23}} b_d^{p_{24}} c_d^{p_{34}},$$

with the conditions
$$2p_{11}+p_{13}+p_{14} = 2p_{22}+p_{23}+p_{24},$$
$$2p_{22}+p_{12}+p_{24} = 2p_{33}+p_{13}+p_{34},$$
$$2p_{33}+p_{13}+p_{23} = 2p_{44}+p_{14}+p_{24},$$

and we may take $$p_{11} \geqslant p_{22} \geqslant p_{33} \geqslant p_{44},$$

implying $$p_{24} \geqslant p_{13}, \quad p_{34} \geqslant p_{12}.$$

Put therefore $$p_{33} = p_{44}+q_{33}, \quad p_{22} = p_{44}+q_{22}+q_{33},$$
$$p_{11} = p_{44}+q_{11}+q_{22}+q_{33}, \quad p_{24} = p_{13}+q_{24}, \quad p_{34} = p_{12}+q_{34},$$

and the form becomes

$$(a_a b_b c_c d_d)^{p_{44}} (a_a b_b c_c)^{q_{33}} (a_a b_b)^{q_{22}} a_a^{q_{11}} (a_b c_d)^{p_{12}} (a_c b_d)^{p_{13}} a_d^{p_{14}} b_c^{p_{23}} b_d^{q_{24}} c_d^{q_{34}},$$

where $$q_{24} = q_{11}+q_{33}, \quad q_{34} = q_{11}+2q_{22}+q_{33}.$$

Hence the form is

$$(a_a b_b c_c d_d)^{p_{44}} (a_a b_b c_c b_d c_d)^{q_{33}} (a_a b_b c_d^2)^{q_{22}} (a_a b_d c_d)^{q_{11}} (a_b c_d)^{p_{12}} (a_c b_d)^{p_{13}} a_d^{p_{14}} b_c^{p_{23}},$$

with the single condition $$q_{11}+p_{14} = q_{33}+p_{23}.$$

Hence it is easy to see that the form finally is composed of powers of the symbolic factors

$$a_b c_d, \quad a_c b_d, \quad a_d b_c,$$
$$a_a b_b c_c d_d, \quad a_a b_b c_d^2, \quad a_a b_c b_d c_d,$$
$$a_a b_b c_c a_d b_d c_d,$$
$$a_a^2 b_b c_c b_d^2 c_d^2.$$

There are obviously no irreducible forms for the quantic of the first order.

For the quadric the only possible form is $a_b a_c b_d c_d$, as other forms are visibly reducible.

As regards this form observe that the syzygy

$$\begin{vmatrix} a_a & a_b & a_c & a_d \\ a_b & b_b & b_c & b_d \\ a_c & b_c & c_c & c_d \\ a_d & b_d & c_d & d_d \end{vmatrix} = 0$$

reduces for the quadric to

$$6a_b a_c b_d c_d - 8a_a b_c b_d c_d - 3a_b^2 c_d^2 + 6a_a b_b c_d^2 - a_a b_b c_c d_d = 0,$$

showing that $a_b a_c b_d c_d$ is in fact reducible. Therefore the quadric has no invariant of the fourth degree in the coefficients.

For the cubic the possible irreducible forms are

$$a_b^2 a_c b_d c_d^2, \quad a_b a_c a_d b_c b_d c_d, \quad a_a b_b a_c b_d c_d^2, \quad a_a a_b b_c b_d c_d^2, \quad a_a b_b c_c a_d b_d c_d,$$

and these are in one syzygy, for on multiplying the fundamental syzygy by $a_b c_d$ and reducing we find that

$$2a_b^2 a_c b_d c_d^2 + 2a_b a_c a_d b_c b_d c_d + 4a_a b_b a_c b_d c_d^2 - 8a_a a_b b_c b_d c_d^2 - a_a b_b c_c a_d b_d c_d$$

is reducible. There is no other syzygy; so that there are four fundamental invariants for the cubic, and we will take the first four above written as exemplar forms.

For the quartic the possible irreducible forms are

$a_b^3 a_c b_d c_d^3$ (I.), $a_a b_b a_b a_c b_d c_d^3$ (VI.),
$a_b^2 a_c a_d b_c b_d c_d^2$ (II.), $a_a b_b a_c a_d b_c b_d c_d^2$ (VII.),
$a_b^2 a_c^2 b_d^2 c_d^2$ (III.), $a_a a_b^2 b_c b_d c_d^3$ (VIII.),
$a_a b_b c_c d_d a_b a_c b_d c_d$ (IV.), $a_a a_b a_d b_c^2 b_d c_d^2$ (IX.),
$a_a b_b a_c^2 b_d^2 c_d^2$ (V.), $a_a b_b c_c a_b a_d b_d c_d^2$ (X.),

ten in number; but these are connected by a number of syzygies with reducible forms. These are five in number, found by multiplying the fundamental syzygy by

$$a_a b_b^2 c_c d_d, \quad a_a b_b c_d^2, \quad a_b^2 c_d^2, \quad a_a b_c b_d c_d, \quad a_b a_d b_c c_d.$$

We then find that the following linear functions are reducible, viz.:—

IV.,
V.$-$2VI.$-$VII.$+$2X.,
2I.$+$II.$-$III.$+$2V.$-$4VIII.,
2VII.$+$VIII.$-$2IX.$-$X.,
2I.$-$3II.$-$2III.$-$4VI.$-$2VII.$+$8IX.,

and we may consider I., II., III., V., VI. to be the ground forms.

3. *The Determinant Forms.*

For invariants of degree 3 we take the form

$$a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} (abc).$$

We may take p_{11}, p_{22}, p_{33} and also p_{23}, p_{13}, p_{12} in descending order of magnitude, and that the form may not vanish it is merely necessary that no equalities shall exist between the numbers p_{11}, p_{22}, p_{33}. These conditions are satisfied by writing

$$p_{22} = p_{33}+\kappa+1, \qquad p_{13} = p_{12}+2\kappa+2,$$
$$p_{11} = p_{33}+\kappa+\lambda+2, \qquad p_{23} = p_{12}+2\kappa+2\lambda+4,$$

and then the form becomes

$$(a_a b_b c_c)^{p_{33}} (a_a b_b a_c^2 b_c^2)^{\kappa} (a_a b_c^2)^{\lambda} (a_b a_c b_c)^{D} a_a^2 b_b a_c^2 b_c^4 (abc).$$

Every form contains therefore the factor

$$a_a^2 b_b a_c^2 b_c^4 (abc),$$

and may contain powers of the factors

$$a_a b_b c_c, \quad a_a b_c^2, \quad a_b a_c b_c, \quad a_a b_b a_c^2 b_c^2$$

in addition.

The simplest invariant of the kind is therefore

$$a_a^2 b_b a_c^2 b_c^4 (abc),$$

for the ternary septimic and in general for the ternary n-ic, the number of independent invariants of the kind is given by the coefficients of x^n the expansion of $\dfrac{x^7}{(1-x^2)^3(1-x^4)}$.

Proceeding to covariants which involve the determinant factor (abx), observe that syzygies present themselves for the first time when four letters appear in the forms. Hence the discussion of the form

$$a_a^{p_{11}} b_b^{p_{22}} a_b^{p_{12}} a_x^{p_1} b_x^{p_2} (abx)$$

is very simple.

We may consider $p_{11} > p_{22}$, $p_2 > p_1$ without loss of generality. Put then

$$p_{11} = p_{22} + \kappa + 1, \qquad p_2 = p_1 + 2\kappa + 2,$$

so that the form becomes

$$(a_a b_x^2)^\kappa a_b^{p_{12}} (a_x b_x)^{p_1} a_a (abx) b_x^2.$$

Every form must contain $\quad a_a (abx) b_x^2,$

and in addition may contain the factors

$$a_b, \quad a_x b_x, \quad a_a b_x^2$$

any number of times repeated.

The generating function of irreducible forms is

$$\frac{a_a (abx) b_x^2}{1 - a_b . 1 - a_x b_x . 1 - a_a b_x^2},$$

yielding for the cubic $\quad a_a (abx) b_x^2,$

and for the quartic $\quad a_a a_b (abx) b_x^2, \quad a_a (abx) a_x b_x^3.$

When we come to the covariants of degree 3 in the coefficients which involve the determinant factor (abx) we reach a complicated system of forms and the indeterminate equations are troublesome to handle. Moreover, syzygies have to be considered. I propose, without exhibiting the whole of the work, to show shortly how the irreducible forms appertaining to the quadric, cubic, and quartic have been obtained.

The general form is

$$a_a^{p_{11}} b_b^{p_{22}} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3} (abx),$$

and we may clearly take p_{11}, p_{22} in descending order of magnitude. The form thus becomes

$$(a_a b_b)^{p_{22}} a_a^\kappa c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3} (abx),$$

with the conditions

$$2\kappa + p_{13} + p_1 = p_{23} + p_2, \qquad 2p_{22} + p_{12} + p_2 + 1 = 2p_{33} + p_{13} + p_3;$$

from the second of these conditions it is clear that p_{33}, p_{13}, and p_3 cannot be zero simultaneously, and a short examination shows that every form must contain one of the five combinations

$$c_x (abx), \quad c_c a_x b_x (abx), \quad c_c a_b (abx), \quad a_c b_c (abx), \quad a_a c_c b_c b_x (abx),$$

and with each of these we may connect the form

$$(a_a b_b)^{p_{22}} a_a^{\kappa} c_c^{p_{33}} a_b^{p_{12}} a_c^{p_{13}} b_c^{p_{23}} a_x^{p_1} b_x^{p_2} c_x^{p_3}$$

with the conditions

$$2\kappa+p_{13}+p_1 = p_{23}+p_2, \qquad 2p_{22}+p_{12}+p_2 = 2p_{33}+p_{13}+p_3.$$

The fundamental solutions of these equations are given in the following table:—

p_{22}	κ	p_{33}	p_{12}	p_{13}	p_{23}	p_1	p_2	p_3
			1					1
				1			1	
					1	1		
						1	1	1
	1				2			
1		1						
	1						2	2
1								2
	1				1		1	1
			1	1	1			
1				1	1			1
1				2	2			
		1	1			1	1	
	1	1	1		1		1	
	1	1					2	
		1	2					
		1				2	2	

yielding the seventeen factors:—

$$a_b c_x, \quad a_c b_x, \quad b_c a_x, \quad a_x b_x c_x,$$

$$a_a b_c^2, \quad a_a b_b c_c, \quad a_a b_x^2 c_x^2, \quad a_a b_b c_x^2, \quad a_a b_c b_x c_x,$$

$$a_b a_c b_c, \quad c_c a_b a_x b_x, \quad a_a c_c b_x^2, \quad c_c a_b^2, \quad c_c a_x^2 b_x^2,$$

$$a_a b_b a_c b_c c_x, \quad a_a c_c a_b b_c b_x,$$

$$a_a b_b a_c^2 b_c^2.$$

There is no irreducible form for the quantic of order 1. Taking account of the syzygies, I find, after a laborious process, one irreducible form for the quadric, nine irreducible forms for the cubic, and twenty-five irreducible forms for the quartic, and the whole of the irreducible forms, involving at most four letters, so far as the quartic inclusive are set forth in the subjoined list.

Irreducible Orthogonal Invariants.

Linear Quantic Invariant: a_b.

Covariant: a_x.

Quadric Invariants: a_a, a_b^2, $a_b a_c b_c$.

Covariants: a_x^2, $a_b a_x b_x$, $a_c b_x c_x (abx)$.

Cubic Invariants:

a_b^3, $a_a b_b a_c b_d c_d^2$, $a_a b_b c_c a_d b_d c_d$,

$a_a b_b a_b$, $a_a a_d b_c^2 b_d c_d$, $a_b^2 a_c b_d c_d^2$.

Covariants:

$a_a a_x$, $a_a b_x^2 (abx)$, $a_b b_c^2 a_x^2 c_x$, $a_c^2 b_x^2 c_x (abx)$,

a_x^3, $a_a a_c b_c^2 b_x$, $a_a b_c^2 c_x (abx)$, $a_b a_c b_x c_x^2 (abx)$,

$a_b^2 a_x b_x$, $a_b^2 a_c b_c c_x$, $a_c^2 b_c b_x (abx)$, $a_a b_c b_x c_x^2 (abx)$,

$a_a a_b b_x^2$, $a_a b_b a_c b_c c_x$, $c_c a_b a_c b_x (abx)$, $c_c a_c a_x b_x^2 (abx)$,

$a_b a_x^2 b_x^2$, $a_a a_c b_c b_x^2 c_x$, $a_a c_a b_c b_x (abx)$, $a_c a_x b_x^2 c_x^2 (abx)$.

Quartic Invariants:

a_a^2, $a_b^2 a_c^2 b_b c_c$, $a_b a_c b_c a_a b_b c_c$, $a_b^2 a_c^2 b_d^2 c_d^2$,

a_b^4, $a_b^2 a_c^2 b_c^2$, $a_b^3 a_c b_d c_d^3$, $a_a b_b a_c^2 b_d^2 c_d^2$,

$a_b^2 a_a b_b$, $a_b^3 a_c b \; c_c$, $a_b^2 a_c a_d b_c b_d c_d^2$, $a_a b_b a_b a_c b_d c_d^3$.

Covariants :

$a_a a_x^2$,

a_x^4,

$a_b^3 a_x b_x$,

$a_a a_b^2 b_x^2$,

$a_b^2 a_x^2 b_x^2$,

$a_a b_b a_b a_x b_x$,

$a_a a_b a_x b_x^3$,

$a_b a_x^3 b_x^3$,

$a_a a_b b_x^2 (abx)$,

$a_a a_x b_x^3 (abx)$,

$a_a a_b^2 b_c^2 c_x^2$,

$a_a a_b a_c b_c^2 b_x c_x$,

$a_a a_b b_c^3 a_x c_x$

$a_a b_b c_c a_b a_c b_x c_x$,

$a_b^3 a_c b_c c_x^2$,

$a_b^3 a_c b_x c_x^3$,

$a_b^2 a_c^2 b_x^2 c_x^2$,

$a_b^2 a_c b_c a_x b_x c_x^2$,

$a_a a_b b_c^2 a_x b_x c_x^2$,

$a_a b_c^3 a_x^2 b_x c_x$,

$a_a b_b c_c a_b a_x b_x c_x^2$,

$a_b^2 a_c a_x b_x^2 c_x^3$,

$a_b a_c b_c a_x^2 b_x^2 c_x^2$,

$a_a b_c^2 a_x^2 b_x^2 c_x^2$,

$a_a a_c b_c^3 (abx)$,

$a_a c_c a_b b_c^2 (abx)$,

$c_c a_b^2 b_c a_x c_x (abx)$,

$a_a b_c^3 a_x c_x (abx)$,

$a_a a_c b_c^2 b_x c_x (abx)$,

$a_a b_b c_c a_c b_x c_x (abx)$,

$a_b a_c^2 b_c b_x c_x (abx)$,

$a_a c_c a_b b_c b_x c_x (abx)$,

$a_c^3 b_c b_x^2 (abx)$,

$a_a c_c a_c b_c b_x^2 (abx)$,

$c_c a_b a_c^2 b_x^2 (abx)$,

$a_a a_c b_c^2 a_x b_x (abx)$,

$b_c^3 a_x^3 c_x (abx)$,

$a_c b_c^2 a_x^2 b_x c_x (abx)$,

$a_b b_c^2 a_x^2 c_x^2 (abx)$,

$a_a b_c^2 a_x b_x c_x^2 (abx)$,

$c_c a_b a_c a_x b_x^2 c_x (abx)$,

$a_a c_c a_c b_x^3 c_x (abx)$,

$a_a b_b a_c b_x c_x^3 (abx)$,

$c_c a_c^2 a_x b_x^3 (abx)$,

$b_c^2 a_x^3 b_x c_x^2 (abx)$,

$a_b b_c a_x^2 b_x c_x^3 (abx)$,

$a_a b_c a_x b_x^2 c_x^3 (abx)$,

$c_c a_c a_x^2 b_x^3 c_x (abx)$,

$b_c a_x^3 b_x^2 c_x^3 (abx)$.

4. *Orthogonal Quaternary Transformation.*

Suppose the quantic

$$a_x^n = (a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3)^n$$

to be converted into

$$A_x^n = (A_0 X_0 + A_1 X_1 + A_2 X_2 + A_3 X_3)^n$$

by the orthogonal transformation

$$(x_0, x_1, x_2, x_3) = \left(\begin{array}{cccc} p_0 & q_0 & r_0 & s_0 \\ p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{array}\right)(X_0, X_1, X_2, X_3).$$

It is well known that the transformation may be represented by the quaternion identity

$$x_0+x_i = (\alpha_0+\alpha_i)(X_0+X_i)(\beta_0-\beta_i)$$

or
$$X_0+X_i = (\alpha_0-\alpha_i)(x_0+x_i)(\beta_0+\beta_i),$$

where
$$\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2 = \beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2 = 1\,;$$

so that only six independent constants are involved.

To verify that the identity does, in fact, give an orthogonal transformation, observe that

$$\begin{aligned} x_0+x_1i_1+x_2i_2+x_3i_3 \\ &= (\alpha_0+\alpha_i)(\beta_0-\beta_i)X_0+(\alpha_0+\alpha_i)i_1(\beta_0-\beta_i)X_1 \\ &\quad +(\alpha_0+\alpha_i)i_2(\beta_0-\beta_i)X_2+(\alpha_0+\alpha_i)i_3(\beta_0-\beta_i)X_3 \\ &= (p_0+p_i)X_0+(q_0+q_i)X_1+(r_0+r_i)X_2+(s_0+s_i)X_3. \end{aligned}$$

Hence
$$\left.\begin{aligned} p_0+p_i &= (\alpha_0+\alpha_i)(\beta_0-\beta_i) \\ q_0+q_i &= (\alpha_0+\alpha_i)i_1(\beta_0-\beta_i) \\ r_0+r_i &= (\alpha_0+\alpha_i)i_2(\beta_0-\beta_i) \\ s_0+s_i &= (\alpha_0+\alpha_i)i_3(\beta_0-\beta_i) \end{aligned}\right\}, \qquad \text{(i.)}$$

and, taking the conjugate quaternions of each side of the four identities,

$$\begin{aligned} p_0-p_i &= (\beta_0+\beta_i)(\alpha_0-\alpha_i), \\ q_0-q_i &= -(\beta_0+\beta_i)i_1(\alpha_0-\alpha_i), \\ r_0-r_i &= -(\beta_0+\beta_i)i_2(\alpha_0-\alpha_i), \\ s_0-s_i &= -(\beta_0+\beta_i)i_3(\alpha_0-\alpha_i), \end{aligned}$$

whence, by multiplication,

$$p_0^2-p_i^2 = p_p = 1, \quad q_0^2-q_i^2 = q_q = 1, \quad r_0^2-r_i^2 = r_r = 1, \quad s_0^2-s_i^2 = s_s = 1.$$

Moreover

$$\begin{aligned} (p_0+p_i)(q_0-q_i)+(q_0+q_i)(p_0-p_i) \\ &= 2p_q = -(\alpha_0+\alpha_i)i_1(\alpha_0-\alpha_i)+(\alpha_0+\alpha_i)i_1(\alpha_0-\alpha_i) = 0, \end{aligned}$$

$$\begin{aligned} (q_0+q_i)(r_0-r_i)+(r_0+r_i)(q_0-q_i) \\ &= 2q_r = -(\alpha_0+\alpha_i)i_1i_2(\alpha_0-\alpha_i)-(\alpha_0+\alpha_i)i_2i_1(\alpha_0-\alpha_i) = 0, \end{aligned}$$

and so forth; so that also $p_q = p_r = p_s = q_r = q_s = r_s = 0$, and we have the ten relations between the sixteen coefficients required by orthogonality.

From the relations (i.) we deduce the transformation formulæ

$$\begin{aligned}
p_0 &= \alpha_0\beta_0+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3, & r_0 &= \alpha_0\beta_2+\alpha_1\beta_3-\alpha_2\beta_0-\alpha_3\beta_1,\\
p_1 &= -\alpha_0\beta_1+\alpha_1\beta_0-\alpha_2\beta_3+\alpha_3\beta_2, & r_1 &= -\alpha_0\beta_3+\alpha_1\beta_2+\alpha_2\beta_1-\alpha_3\beta_0,\\
p_2 &= -\alpha_0\beta_2+\alpha_1\beta_3+\alpha_2\beta_0-\alpha_3\beta_1, & r_2 &= \alpha_0\beta_0-\alpha_1\beta_1+\alpha_2\beta_2-\alpha_3\beta_3,\\
p_3 &= -\alpha_0\beta_3-\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_0, & r_3 &= \alpha_0\beta_1+\alpha_1\beta_0+\alpha_2\beta_3+\alpha_3\beta_2,
\end{aligned}$$

$$\begin{aligned}
q_0 &= \alpha_0\beta_1-\alpha_1\beta_0-\alpha_2\beta_3+\alpha_3\beta_2, & s_0 &= \alpha_0\beta_3-\alpha_1\beta_2+\alpha_2\beta_1-\alpha_3\beta_0,\\
q_1 &= \alpha_0\beta_0+\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3, & s_1 &= \alpha_0\beta_2+\alpha_1\beta_3+\alpha_2\beta_0+\alpha_3\beta_1,\\
q_2 &= \alpha_0\beta_3+\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_0, & s_2 &= -\alpha_0\beta_1-\alpha_1\beta_0+\alpha_2\beta_3+\alpha_3\beta_2,\\
q_3 &= -\alpha_0\beta_2+\alpha_1\beta_3-\alpha_2\beta_0+\alpha_3\beta_1, & s_3 &= \alpha_0\beta_0-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3,
\end{aligned}$$

and the corresponding formulæ for expressing X_0, X_1, X_2, X_3 in terms of x_0, x_1, x_2, x_3 by changing the signs of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$.

Note also that, since $(\alpha_0-\alpha_i)(x_0+x_i) = (X_0+X_i)(\beta_0-\beta_i)$, we obtain by equating scalars $x_\alpha = X_\beta$. Also

$$a_0+a_i = (\alpha_0+\alpha_i)(A_0+A_i)(\beta_0-\beta_i),$$

$$a_0-a_i = (\beta+\beta_i)(A_0-A_i)(\alpha_0-\alpha_i),$$

and $$a_\alpha = A_\beta.$$

We can now prove that a_x is an invariant symbol because

$$(a_0+a_i)(x_0-x_i) = (\alpha_0+\alpha_i)(A_0+A_i)(X_0-X_i)(\alpha_0-\alpha_i),$$

and, taking the conjugates of each side,

$$(x_0+x_i)(a_0-a_i) = (\alpha_0+\alpha_i)(X_0+X_i)(A_0-A_i)(\alpha_0-\alpha_i),$$

and, since

$$(a_0+a_i)(x_0-x_i)+(x_0+x_i)(a_0-a_i) = 2a_0x_0-a_ix_i-x_ia_i = 2a_x, \text{ a scalar,}$$

we have $$a_x = A_X,$$

and similarly $\quad a_a = A_A, \quad x_x = X_X, \quad a_b = A_B.$

It is clear that, if f be any quaternion product involving umbræ $a, b, c, \ldots$, or variables $x, y, z, \ldots$, and F its transform, such that

$$f = (\alpha_0+\alpha_i)F(\alpha_0-\alpha_i),$$

and $\bar{f}$, $\bar{F}$ be the quaternions conjugate to f, F respectively,

$$f+\bar{f} = F+\bar{F},$$

and $f+\bar{f}$ or Sf is an invariant combination of symbols.

It can now be shown that the determinant $(abcd)$ is an invariant factor. It is easy to prove that

$$S(a_0-a_i)(b_0+b_i)(c_0-c_i)(d_0+d_i) = \quad (abcd)+a_b c_d-a_c b_d+a\ b_c,$$

$$S(a_0+a_i)(b_0-b_i)(c_0+c_i)(d_0-d_i) = -(abcd)+a_b c_d-a_c b_d+a_d b_c.$$

Hence
$$2(abcd) = S(a_0-a_i)(b_0+b_i)(c_0-c_i)(d_0+d_i) \\ -S(a_0+a_i)(b_0-b_i)(c_0+c_i)(d_0-d_i),$$

and, since

$$(a_0-a_i)(b_0+b_i)(c_0-c_i)(d_0+d_i) \\ = (\beta_0+\beta_i)(A_0-A_i)(B_0+B_i)(C_0-C_i)(D_0+D_i)(\beta_0-\beta_i),$$

$$(a_0+a_i)(b_0-b_i)(c_0+c_i)(d_0-d_i) \\ = (\alpha_0+\alpha_i)(A_0+A_i)(B_0-B_i)(C_0+C_i)(D_0-D_i)(\alpha_0-\alpha_i),$$

it follows that $(abcd) = (ABCD)$. Hence the invariant symbolic factors appertaining to the quaternary quantic are of types a_a, a_b, a_x, x_x, $(abcx)$, $(abcd)$.

ON CLOSED SETS OF POINTS AND CANTOR'S NUMBERS

By W. H. YOUNG.

[Received May 12th, 1903.—Read May 14th, 1903.]

1. In a paper on "Sets of Intervals" (*Proc. London Math. Soc.*, Vol. xxxv., p. 245) we classified the points of a segment (A, B), in which a set of intervals is given, into (1) internal, (2) simple end-points, (3) semi-external, and (4) external points of a set of intervals, and proved certain properties of the external points, and that the semi-external points are at most countably infinite.

The set of external points by themselves is an unclosed set. The theory of unclosed sets is far more incomplete than that of closed sets. In the first part of the present paper I propose to deal exclusively with closed sets. It is an important fact that

By adding the semi-external points to the external points we get a closed set of points. (1)

For, if P is either an internal point or an ordinary end-point of intervals, a neighbourhood of P can be found so small that it contains no semi-external or external points, for such neighbourhood can be taken entirely within either one or two of the intervals; hence P cannot be a limiting point of the set of external and semi-external points; this set is therefore closed.

Similarly, of course,

The set obtained by adding all the end-points of the black intervals to the external points is closed. (2)

The extra points added are isolated points, whose limiting points were included before.*

The set (1) is none other than the first derived set of the set (2).

2. *Closed Sets and their Complementary Intervals.*

The latter theorem is for many purposes the more valuable of the two, since it has an important converse, which is an immediate result of the investigations into sets of intervals.

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* *Loc. cit.*, p. 253.

It may be expressed as follows:—*Any set of non-overlapping intervals* $\delta_1, \delta_2, \ldots$ *on the straight line defines a closed set of points, viz., all those points which are not internal to any of the intervals.* The converse is that any closed set of points, which does not consist merely of the whole segment (A, B) under discussion, defines a set of intervals such that the whole segment (A, B) consists of the internal points of those intervals and the points of the given set. This may be expressed more shortly by saying *a closed set of points is always complementary to a set of intervals*—open intervals, of course. This close connexion between closed sets of points and sets of intervals suggests that the theory of sets of intervals may be used to develop the theory of sets of points.

Certain classifications can be made at once. It is evident that, *if a set of intervals is dense everywhere, the complementary points are dense nowhere, and vice versa;* while, if the set of intervals be not dense everywhere, the complementary points certainly fill up some partial segment, and *vice versa.* If the given set of points contains any such whole segment, its potency is evidently c, and the process of derivation leaves that segment unaltered. Hence, from the point of view of the succeeding articles, the case when the intervals are dense everywhere and the complementary points dense nowhere is the only interesting case, and it will in future be assumed that this is the case unless the contrary is expressly stated.

3. *Potency of a Closed Set of Points.*

The first question to be answered is as to the potency of a set of points. The theory of sets of intervals enables us to answer this question fully for a closed set of points. *A closed set of points is either finite or countably infinite or has the potency of the linear continuum.*

This follows from the last section, and *loc. cit.*, Theorem 4 and § 28.

Two cases can be at once disposed of in detail:

If the set of complementary intervals consist of a finite number only, the closed set is finite, and vice versa. (*Loc. cit.*, Theorem 2.)

If the content of the set of intervals be less than l,* *the set of points has the potency* c. (*Loc. cit.*, Theorem 3′.)

There remains over the case when the set of complementary intervals is not finite, but has the content l. In this case the potency of the points may be a or c, and it depends entirely on the "ultimate set" (*loc. cit.*, Theorem 4) which of these will be the case. We shall return to the discussion of this ultimate set in § 6.

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* l being the length of (A, B).

4. *Content of a Closed Set of Points.*

I propose now to give the following definition of the content of a closed set of points :—*The content I_P of a closed set of points is defined as $l - I_\delta$, where I_δ is the content of the complementary intervals.*

We notice that this definition agrees with our fundamental notions when the set of points is dense everywhere in a partial segment. The investigations of the paper on "Sets of Intervals" show that *the content, so defined, is always $\geqslant 0$ and $\leqslant l$,* and that, while it cannot be 0 if the set be dense everywhere in any partial segment, it is possible that the content of a set which is nowhere dense may be greater than zero ; indeed, it is possible to construct such sets with content as near the maximum as possible. This maximum l, the length of the segment (A, B) in which we are operating, can only be attained when the set consists of *all* the points of the segment (A, B). *The content of a finite or countably infinite closed set of points is, by Theorem 4, always zero ;* but these are not the only sets of content zero: for instance, H. J. S. Smith's sets of the first kind have zero content.

The following is the fundamental property of the content and might have been taken as the definition and the above deduced from it: historically, this is what was done :—*If we determine a finite number of non-overlapping intervals, such that every point of a given closed set is internal to one of the intervals, the content of these intervals is always greater than I_P, but may be made as near as we please to I_P, by taking the intervals small enough.*

First, it is obvious from our theory how to constuct such intervals. We only have to take any small quantity ϵ, and determine the finite number k of black intervals $> \epsilon$. These leave over a finite number of complementary intervals, and, if we lengthen each of these at each end by as small a quantity as we please, we get such a finite set of intervals as we are in search of.

Secondly, suppose we have in any way determined such a finite set of intervals $d_1, d_2, \ldots$. Then we must be able to fix a limit ϵ, such that all the black intervals $\leqslant \epsilon$ of the given set lie inside the d-intervals, and all the black intervals $> \epsilon$ lie entirely or partially outside them.* Suppose any one of the intervals d_r to overlap at one or both ends into a black interval $> \epsilon$. Then, without freeing any but internal points of this black interval, we can ensure, by curtailing d_r if desirable, that the distance of the end-point of the overlapped piece from the overlapping end-point of d_r should be less than $\frac{\sigma}{4r+4}$. The content of the overlapped parts being now less than $\sum_1^\infty \frac{\sigma}{2r+2}$, *i.e.*, less than $\frac{1}{2}\sigma$, and, the content of the black intervals $\leqslant \epsilon$ being denoted by $R(\epsilon)$, the content of those parts of the black intervals that lie outside the intervals d_r will now lie between $I_\delta - R(\epsilon)$ and $I_\delta - R(\epsilon) - \frac{1}{2}\sigma$. Hence the content of the intervals d_r will lie between $I_p + R(\epsilon)$ and $I_P + R(\epsilon) + \frac{1}{2}\sigma$.

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* For otherwise we could determine a sequence of the black intervals having a single limiting point, external to all the intervals d, or an end-point of one of them, which is inconsistent with the hypothesis, since such a point must be a point of the given set.

Now, if we diminish the length of each interval d_r, ϵ must decrease without limit.* $R(\epsilon)$ can therefore be made as small as we please by sufficiently decreasing the lengths of the intervals d_r. Hence we can choose the lengths of the intervals d_r so that the sum of them, though always greater than I_P, may differ by less than any assigned quantity from I_P. Q. E. D.

It may be remarked in this connexion that it is not possible to determine an infinite set of non-overlapping intervals such that each point of a closed set is internal to one of the intervals, and no interval is free of points of the set. For an infinite set of intervals has always at least one external or semi-external limiting point, and such a point would by the second assumption be a limiting point of the set and therefore belong to it, contrary to the first assumption.

Another form of the preceding, used by Hankel and Cantor as the basis of their original definitions, and frequently very useful, is the following:— *If we enclose each point of a closed set of points in a small interval of which it is the middle point, the limit of the sum of the finite number of non-overlapping intervals filled up by these small overlapping intervals, when the lengths of these latter are indefinitely decreased, is the content* I_P *of the set of points.*

In a note on "Sets of Overlapping Intervals" † I showed, in fact, that such a set of overlapping intervals enclosing all the points of a closed set can be replaced by a finite number of them; so that this theorem is seen to be merely a different form of the preceding.

5. *Limiting Points.*

A limiting point of a set of points may be either a limit for the set when approached from one side only or from both sides. We may express this difference by speaking of "limiting points on one side" and "limit-

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* For, if not, let η be the limit of ϵ; then, however we construct our intervals d_r, there is always a black interval $\geqslant \eta$ inside them, which is obviously nonsense.

† *Proc. London Math. Soc.*, Vol. xxxv., p. 387.

ing points on both sides." If a limiting point be a limit on one side only, it is evidently an end point of a black interval, and, if it is a limit on both sides, it is external to the black intervals. Hence we see that *the limiting points of a closed set on one side only are the same as the semi-external points of the black intervals and are therefore, at most, countably infinite in number.* The limits on both sides are identical with the external points of the black intervals.* The potency of these latter, if not zero or a natural number, is a or c, and may be any one of these when the set is dense nowhere; when it is dense everywhere in any partial segment the potency is evidently always c.

6. *Derived Sets, Deduced Sets, and the Nucleus.*

We shall denote by E a set of points, and by E_1, E_2, ... the successive derived sets of E. If E is closed, E_1 is contained in E and those points of E which are not points of E_1 are end-points of abutting black intervals of E. The connexion between E and E_1 is, indeed, precisely that of the sets (1) and (2) of § 1. [Whether E is closed or not, E_1 and therefore all the successive derived sets are closed.] In passing from the black intervals of E to those of E_1 therefore, we amalgamate all abutting intervals of E. Starting with any particular black interval δ_r of E, the amalgamation process is only arrested by the first limiting points to which we reach on either side of δ_r, and we blacken up to these inclusive; let us denote this black interval of E_1 by δ'_r.

Now we saw (*loc. cit.*, p. 266) how it was possible, systematically and uniquely, to obtain from a given set of intervals an ultimate set of *non-abutting* intervals having the same content, and, with the exception of at most a countably infinite set of points, the same external points. The ultimate set is then either such a set as was contemplated in § 28, *loc. cit.*, or consists of the whole continuum,† and that, if these external points were not more than countable in number, the ultimate set of intervals would consist of the whole infinite straight line.

The closed set of points consisting of the end-points and external points of the ultimate set of intervals we shall call the "Nucleus of E," and denote by E_Ω. The Nucleus is evidently perfect‡ (except when E is countable, when it evanesces), since, having no abutting black intervals, it has no isolated points. Let us denote the black interval of E_Ω which contains any particular interval δ_r of E by δ_r^Ω. Then, since in amalgamating the intervals of E with δ_r so as to form δ_r' we blacken out at most two limiting points, it is evident that δ_r' *will also lie inside* δ_r^Ω, though it may, of course, coincide with it.

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* *Loc. cit.*, p. 253.

† In § 7, *loc. cit.*, it was pointed out that it was sometimes convenient to regard the part of the straight line exterior to (A, B) as black; from our present point of view this is so, and this interval must be considered as one of the black intervals.

‡ Dense in itself and closed.

Similarly the corresponding interval δ_r^n *of* E_n *lies inside* δ_r^Ω *for all values of* n.

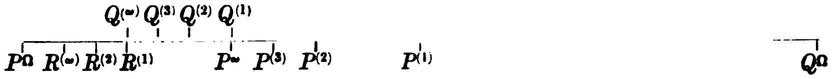

FIG. 1.

Let $P^{(1)}, P^{(2)}, \ldots, P^{(n)}$ be the left-hand end-points of $\delta_r^{(1)}, \delta_r^{(2)}, \ldots, \delta_r^{(n)}$. Then either from and after a definite integer m all the $P^{(n)}$s coincide, or else they define a limiting point to the left of all of them. In either case let us denote the point so obtained by $P^{(\omega)}$.

Now $P^{(n+1)}, P^{(n+2)}, \ldots$ are all points of E_n, and therefore $P^{(\omega)}$ is a limiting point of E_n, that is, since E_n is closed, a point of E_n. *Hence* $P^{(\omega)}$ *is a point of* E_n *for every value of* n.

The process of forming from any number of sets the largest set which is contained in all of them is usually called "the process of finding the H.C.F. of the sets." But, as the process is a very important one, we require a simpler term for expressing it; and, as the term has not been otherwise appropriated, I propose to call it "deduction." Thus: *Given any finite or infinite number of sets of points, that set which contains all the points which belong to every set, and no other points, is called the deduced set.* If we denote the given sets by $E_1, E_2 \ldots, E_n, \ldots$, the deduced set will usually be denoted by E_ω.

We notice that the process of deduction applied to any *finite* number of derived sets $E_1, E_2, \ldots, E_n$ gives us E_n and nothing new. The deduced set of an infinite number of derived sets* is, however, not necessarily identical with any one of them. It is convenient to give here the following example, due to Cantor, which we shall have occasion subsequently to refer to.

7. *Cantor's Classical Example.*

Let T denote the set of points obtained by bisecting any segment, and then bisecting the right-hand segment, and so on.

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* It can easily be shown that such a set always exists, since $E_1, E_2, \ldots, E_n, \ldots$ are closed sets.

We start with T in the segment (0, 1). The corresponding binary fractions are 0, 1, $\cdot 1$, $\cdot 11$, $\cdot 1^2 1$, ..., $1^n 1$, ...

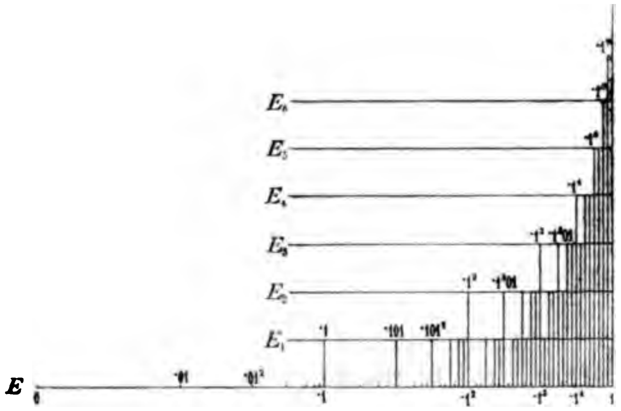

FIG. 2.

Now erect at each point $\cdot 1^r$ of T an ordinate of length r. In each of the black intervals of T place a set of points similar to T, and at all these points in any segment ($\cdot 1^r$, $\cdot 1^{r+1}$) erect ordinates of length r. In the largest segment these will only be dots. This largest segment we now leave undisturbed, and in each of the segments between each consecutive pair of points already marked we insert a set similar to T, and erect ordinates of length r in any segment ($\cdot 1^{r+1}$, $\cdot 1^{r+2}$). In the segment ($\cdot 1$, $\cdot 11$) these will only be dots; this segment is subsequently left undisturbed.

Thus we go on, leaving undisturbed segments in which we have already inserted dots, and placing in all the rest sets of the type T. At each stage we erect perpendiculars of length one less than those erected in the same segment at the preceding stage of proceedings.

We see that this process, carried on *ad infinitum*, gives us on the axis a closed countable set of points E. If we draw parallels to the axis at heights 1, 2, 3, ..., the first, at height 1, will be cut by the ordinates which we erected in the first derived set E_1, the second in E_2, and so on. Every successive parallel is cut by the ordinates more and more to the right, the points crowding up to the point $\cdot \dot{1}$ or 1, which alone is common to all the derived sets. Thus E_ω consists of this one point, $\cdot \dot{1}$ or 1, while E_n contains an infinite number of points.

The binary numbers of our set are evidently all those of the form $1^m 01^n 01^p \dots 01^z$, where $m, n, \dots, z$ are any integers (including the case when any of these integers are absent, and the corresponding power of 1 drops out), the number of zeroes being at most $(m+1)$.

8. Returning now to the discussion of § 6, we not unnaturally want to know whether the set E_ω is identical with E_Ω, and it is obvious from the above that this is not so. In the above example E_Ω (which is, of course, evanescent) is the first derived of E_ω, but it is easy to construct more complicated examples on the same principle as the above, taking as T a more complicated set, *e.g.*, the set constructed in the preceding article; in this case E_ω consist of the set T itself. If, therefore, we take the series of derived sets of E_ω, the deduced set will consist of the point 1 alone, and the first derived set of this will be E_Ω.

It is evident that here again we have not arrived at the most general case. For we might have taken as T the more complicated set of the last section, and we should then have arrived at a more complicated E_ω, which, however, by an extended process of derivation and deduction, could be reduced to E_Ω.

On this principle we can evidently set up whole classes of countable sets of points, in each of which E_Ω is obtained after a complicated series of repetitions of the operations of derivation and deduction from the derived sets.

Here, however, we are not necessarily at an end. For, denoting the point called P_ω, by Q_1 (Fig. 1), we get, after deriving E_ω and then deducing a new set, another point Q_2, which, by precisely the same reasoning as before, lies to the right of P^Ω, if it does not coincide with it. If we have not now come to an end, we obtain another point Q_3, and Q_4, and so on— a whole sequence of points which will have a limiting point if they do not all coincide after a time. Such a limiting point we may denote by Q_ω. It will evidently belong to all the sets which we have so far obtained by derivation and deduction; that is, to the set obtained by deduction from all these sets, which we may, if we prefer, regard as deduced from the deduced sets alone.

Starting afresh with this set, we may evidently derive and deduce exactly as before. Denoting by R_1 the limiting point just obtained and denoted by Q_ω, we may get a whole sequence of points R_1, R_2, ... leading to a limiting point R_ω, which will again lie to the right of P^Ω if it do not coincide with it, since by none of our processes of amalgamation have we blackened out a more than countable set of external points.

Furthermore, having realized to ourselves any particular series of the operations of derivation and deduction, we can easily, on the principles laid down in the examples, construct a set of points in which that series of derivations and deductions can actually be performed, and give a set which either is E_Ω, or which leads to E_Ω when we subject it itself to the operations of derivation and deduction.

9. Special as this mode of constructing examples may seem, it can now be proved quite easily, by the principles established in the paper on "Sets of Intervals," that this is quite a general theorem which may be enunciated as follows :—

The Nucleus E_Ω of a given closed set of points E can always be obtained by means of a finite or countably infinite series of the operations of derivation and deduction.

For let us consider a black interval (P^Ω, Q^Ω) of the ultimate set, and, starting with any interval δ_r of E inside it, let us form the successive intervals $\delta_r^{(1)}$, $\delta_r^{(2)}$, ... of all the derived and deduced sets possible. The left-hand end-points $P_1, P_2, \ldots$ of these form a countable set of points with all sorts and varieties of limiting points as in our examples, and may need all the letters of the alphabet, and of many alphabets, and all the indices over and over again to characterize them in their natural order. But they have the two properties :—(i.) they lead always to the left, (ii.) they never pass P^Ω. There are, therefore, only two possibilities :—(*a*) we can assign a point P to the right of P^Ω such that there is no one of the points P_1, P_2, ... to the left of P, while no such point can be assigned more to the right than P; or (*b*) P^Ω itself is the only point which can be so assigned. Whichever of these cases is true, we will first show that, as we amalgamate our intervals, we can assign a definite stage of proceedings at which the point P has actually been reached, and is no longer a mere limit to which we continually approach.

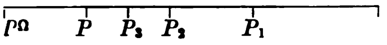

FIG 3.

To show this, let P_1 be the left-hand end-point of $\delta_r^{(1)}$. Then either P_1 coincides with P or lies to the right of it. In the former case we have proved our point. In the latter case we bisect (P, P_1) at M_1. Then we can assign one of the derived or deduced sets, which we may denote by $E^{(2)}$, whose corresponding black interval $\delta_r^{(2)}$ extends beyond M_1 to the left. Let P_2 be its left-hand end-point, which either coincides with P or lies between P and M_1. We now reason with respect to P_2 as we did with respect to P_1. In this way either we determine a finite number of the sets E_1, $E^{(2)}$, ..., $E^{(n)}$, ... chosen from among the derived and deduced sets on these principles, and such that the black interval $\delta_r^{(n)}$ has its left-hand end-point at P, which proves our theorem, or else we obtain a countable set E_1, $E^{(2)}$, ..., $E^{(n)}$, ... and a corresponding set of points P_1, P_2, ..., P_n, ... having P as sole and only limiting point, and such that P_n is the left-hand end-point of $\delta_r^{(n)}$, and therefore the first point

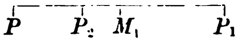

FIG. 4.

of $E^{(n)}$ at which we arrive in passing from P_1 towards P; so that P_n is certainly not a point of $E^{(n+1)}$. Let $E^{(\omega)}$ denote the deduced set of E_1, $E^{(2)}, \ldots, E^{(n)}, \ldots$. Then there is certainly no point of $E^{(\omega)}$ to the right of P; but we know that the black interval $\delta_r^{(\omega)}$ does not extend beyond P, hence P must be its left-hand end-point, so that we have determined a definite stage at which P is reached, as was asserted.

Now let us suppose that P lies to the right of P^{Ω}. Similarly we determine a point Q actually attained and never passed on the right of δ_r. Q may coincide with Q^{Ω}. Now, if from the black intervals of E lying in (P^{Ω}, P) we choose one and enlarge it by amalgamation as we did δ_r, we get a definite black interval (P_1, Q_1) actually attained, but never surpassed, during the processes of derivation and deduction.

FIG. 5.

Now P_1 may coincide with P^{Ω}, but Q_1 cannot coincide with P, because, if it did, one single derivation would amalgamate the abutting intervals (P_1, Q_1) and (P, Q), contrary to the hypothesis that P could not be passed starting from δ_r.

Continuing this process in the segments (P^{Ω}, P_1), (Q_1, P), (Q, Q^{Ω}), one at least of which must actually exist, viz., (Q_1, P), we ultimately get a set of non-abutting intervals, dense everywhere in (P^{Ω}, Q^{Ω}), since the intervals of E were dense everywhere in (P^{Ω}, Q^{Ω}). But by § 28, *loc. cit.*, we know that such a set of intervals has a more than countable set of external points in (P^{Ω}, Q^{Ω}). These external points are, however, certainly external points of E by our construction, and E has at most a countably infinite set of external points in (P^{Ω}, Q^{Ω}). (*Loc. cit.*, p. 267.)

Thus this is impossible; P must coincide with P^{Ω}, and Q with Q^{Ω}, and both P^{Ω} and Q^{Ω} are actually attained at a definite stage of proceedings.

We have still to show that we can assign a definite stage of proceedings at which *every* (P^{Ω}, Q^{Ω}) *has* been attained. If this were not possible, then, choosing any one of the sets obtained by derivation and deduction, and calling it E_1, we must be able to assign an interval of E_{Ω}, say $\delta_{1,\Omega}$, which has not not yet been actually attained. By what has been proved, however, we can assign a definite stage at which $\delta_{1,\Omega}$ has been attained. Let us denote by E_2 a definite set which has $\delta_{1,\Omega}$ as a black interval. E_2 is then certainly obtained from E_1 by derivation and deduction, since E_2 could not have occurred previously to E_1, by the hypothesis that $\delta_{1,\Omega}$ is not yet attained at the stage of E_1, and is attained at the stage of E_2.

So we go on, and form a simply infinite series of sets $E_1, E_2, \ldots, E_n$, from a corresponding series of the black intervals of E_Ω, viz., $\delta_{1,\Omega}, \delta_{2,\Omega}, \ldots, \delta_{n,\Omega}, \ldots$, where $\delta_{n,\Omega}$ does not belong to E_n, but does belong to E_{n+1}. Now it follows that, for all values of n, $\delta_{n,\Omega}$ will belong to the deduced set of $E_1, E_2 \ldots, E_n, \ldots$.

We might, then, have started with this set as $E_1^{(1)}$; since then every one of the countably infinite set just dealt with, ($\delta_{2,\Omega}, \delta_{3,\Omega}, \ldots$), would already have been attained, we should have to seek a $\delta_{2,\Omega}^{(1)}$ among the remaining black intervals of E_Ω. At the end of another infinite set of processes we should get to another set, which we might have taken originally as $E_1^{(2)}$. Since the whole number of intervals of E_Ω is countable, we cannot go on putting aside countably infinite sets of intervals without exhausting them after at most a countably infinite series of steps. Thus the deduced set of $E_1^{(1)}, E_1^{(2)}, \ldots, E_1^{(n)}, \ldots$ must coincide with E_Ω.

The theorem is now proved, and may be restated as follows:—

Any closed set of points E can be reduced by a finite, or at most countably infinite, series of the operations of derivation and deduction to one of two forms :—(i.) *no points at all,* (ii.) *a perfect set. This set, using the word in a somewhat extended sense if* (i.) *be the case, we call the Nucleus and denote it by* E_Ω. E_Ω *has the same content (in the first case zero) as E, and, with the possible exception of a finite, or at most countably infinite, set of points, the same external points as E.* (i.) *occurs if, and only if, the set E be countable.*

The following is also an immediate consequence of our reasoning:—

Given any finite or countably infinite series of the operations of derivation and deduction, a countable set of points can be constructed on which this series can actually be performed, giving at each stage of proceedings a new set of points, which evanesces at the last of the processes.

10. *Cantor's Transfinite Numbers of the First Potency.*

We notice that the ideas of derivation and deduction, and of the possible series of these operations which lead from closed sets of points to their Nuclei, gives us a conception of sequence which, though perfectly definite, goes beyond what we can qualify by means of the ordinal numbers first, second, third, and so on. Even if we use a whole alphabet, or many alphabets, to distinguish the new sets which arise, we find the symbols at our disposal inadequate, and the idea of sequence which we possess becomes hazy in a mist of symbols to which our minds are not accustomed as expressing sequence. Beyond the ordinary series of ordinal numbers, first, second, ..., or, which comes to the same thing, only is more restricted, the sequence of letters of the alphabet, our mind is accustomed to the sequence expressed by the points of a straight line in their natural order, or, which is the same thing, the sequence of all the numbers rational and irrational: *e.g.*, all the points of the segment (0, 1) of the x-axis, or the sequence of all the numbers from 0 to 1 expressed in any scale of notation, for instance, the ternary scale.

The potency of these numbers is c, and the question is whether we can employ some or all of these numbers, in an *ordinal* sense, to qualify the sets of points obtained from every conceivable E by derivation and deduction. It is, of course, quite easy to do so in any special case. Given, in fact, any finite or countably infinite series of the operations of derivation and deduction in their natural order, we know that we can, as in the examples, set up a countable set of points in the segment (0, 1) on which that series of operations can actually be effected. We might then choose the corresponding binary fractions as indices of the corresponding E's.

For instance, to be more precise, taking as basis the example of § 7 (see Fig. 2), we could denote the derived sets by $E_{\cdot 01}$, $E_{\cdot 01^2}$, $E_{\cdot 01^3}$, ... ; the set deduced from these by $E_{\cdot 0\dot{1}}$ or $E_{\cdot 1}$; the derived sets of E_1 by E_{1001}, E_{1001^2}, ..., $E_{\cdot 1001^n}$, ... ; and the set deduced from these by $E_{\cdot 101}$.

Proceeding on this principle, generally the derived sets of E_{101^m} will be denoted by $E_{\cdot 101^m 01}$, $E_{\cdot 101^m 01^2}$, ..., $E_{\cdot 101^m 01^n}$, ... ; and the set deduced from these by $E_{\cdot 101^{m+1}}$.

Further, the set deduced from all these, or, which comes to the same thing, from $E_{\cdot 101}$, $E_{\cdot 101^2}$, ..., $E_{\cdot 101^n}$, ... will be denoted by $E_{\cdot 1^2}$.

As we proceed further, the general form of the index is $\cdot 1^m\, 01^n 0 \ldots 01^z$, where m, n, ..., z are any integers (including zero, and, if zero, the corresponding figure is to be altogether omitted) and the number of zeros which occur between the 1's is at most $m+1$. These numbers present to our minds the same idea of sequence as the operations themselves, and, if we should speak of the $\cdot 1^3 0100111$-th set, we should know at once what was meant, and where the set occurred in the sequence under discussion.

But, although in any special case, however complicated, we can assign such a set of binary fractions, and the notation is a very convenient one for such purposes, the method does not permit us to discuss simultaneously two or more special cases, since sets which we recognize in our minds as the same, *i.e.*, having the same place in their respective sequences counted from the beginning, will in the two cases have different indices, while, on the other hand, the same index will sometimes refer to sets which we recognize as distinct.

The question as to whether it is possible to assign a system of fixed numerical indices to the E's so as to characterize the order of the whole set of them in every conceivable case has not been answered. A preliminary step would be to determine whether $\aleph_1$, the potency of all the E's, is greater, equal to, or less than that of the continuum. If it were greater than the continuum, the above question would certainly have to be answered in the negative. It has been proved* that the potency is not greater than c; it is still an open question whether or not it is equal to c.

Before proceeding to the proof of this theorem, however, I propose to describe the notation introduced by G. Cantor, and which for the earlier sets of the series is so convenient and obvious that it has met with universal acceptation.

The derived sets of E are denoted by the indices $1, 2, \ldots, n, \ldots$, the deduced set of these by ω; the derived sets of E_ω by $\omega+1$, $\omega+2$, $\ldots$, $\omega+n$, $\ldots$, and the deduced set of these by $\omega+\omega$ or $\omega \,.\, 2$; and so on. The set deduced from $E_\omega, E_{\omega 2}, \ldots, E_{\omega n}$ has the index ω^2, that deduced from $E_{\omega^2}, E_{\omega^3}, \ldots, E_{\omega^n}$ the index ω^ω, that deduced from the E's with indices $\omega^\omega, \omega^{2\omega}, \ldots, \omega^{n\omega}$ the index ω^{ω^2}, and that deduced from the E's with indices $\omega^\omega, \omega^{\omega^2}, \omega^{\omega^3}, \ldots$ the index ω^{ω^ω}. For the set deduced from the sets with indices $\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$ the notation breaks down, but the principle can be carried on *ad infinitum*. This principle, or more properly these *two* principles, are—(1) to every number a there is a *next* number, which shall be denoted by $a+1$; (2) to every infinite set of numbers there is a *next* number, which shall be called a "limiting number" (*Limeszahl*).

Following out these principles, if we have obtained any Cantor number β, then the set which is got by performing on E_β that operation which led from E to E_a, a being any Cantor number, will be denoted by $\beta+a$. We notice that, if a precedes β, $E_{\beta+a}$ is different from E_β, and from any set preceding E_β; but, if a follows β, then $E_{\beta+a}$ is the same as E_β. Hence the Cantor numbers do not follow the commutative law: that is, $a+\beta \neq \beta+a$.

Similarly, $2 \,.\, \omega \neq \omega \,.\, 2$. Since $E_{2\,.\,\omega}$ means that we derive E_2 twice, giving $E_{2\,.\,2}$, then twice more, giving $E_{2\,.\,3}$, and so on; finally $E_{2\,.\,\omega}$ we obtain by deduction from this series, but this is none other than E_ω, so that $2\omega = \omega$. Generally $a\beta \neq \beta a$.

The ideal symbols defined by these two principles are called Cantor's

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* F. Bernstein, Inaug. Diss.: "Untersuchungen aus des Mengenlehre," Göttingen, 1901.

"transfinite numbers of the first potency." Cantor uses the notation $\aleph_1$ or *Aleph-eins* for the potency of his numbers of the first potency, while $\aleph_0$ or *Aleph-null* denotes the potency of a countable set of points; this notation we shall keep.

11. *Order represented by Diagrams. Ordinal Types.*

F. Bernstein has pointed out that the idea of *sequence*, or, to use Cantor's expression, of "ordinal type," is nothing more than a discontinuous function of two variables, and may be graphically represented by means of the diagram of a rectangular trellis, so familiar in the theory of numbers.

If we take any countably infinite set, whether it be of points on the straight line, or of the operations of derivation and deduction, or anything else, and arrange them in countable order, say $E_1, E_2, \ldots, E_n, \ldots$ (this may generally be done in a variety of ways, but we choose out one particular arrangement), the idea of the *natural order* of these E's is completely embodied by giving a law by which we can say whether or no E_i came after E_j originally. E_i could, of course, only coincide with E_j in the one case if it did in the other, *i.e.*, if $i \equiv j$. Unless we can give such a law, we cannot speak of a "natural order" at all; *vice versa*, given such a law, we can determine the position of any element E_i with respect to any other one E_j, and so can always say whether or no it lies between any two assigned elements: this is what we mean by saying we know the natural order, or the ordinal type.

The diagrammatic representation of the natural order depends on the customary representation of the pair of integers (i, j) by means of the cross points of a rectangular trellis, so that the point (i, j) is the point whose coordinates are i and j. Each such point (i, j) we mark with a black spot if E_j comes before E_i (in symbols, if $E_j < E_i$), and with a small circle if $j \equiv i$. It is obvious that in this way we have a diagram such that the reflection of any black spot in the line of small circles is a trellis point which has not been spotted. We may therefore, if we prefer, use only the wedge-shaped diagram bounded by the small circles, as it gives us all the information we require.

We may transform this diagram into a numerical representation by interpreting the black spots by the figure 1, and the unspotted trellis-points by 0, and reading the columns in order. Prefixing a point and interpreting in the binary scale, we get a binary fraction corresponding to each diagram, and *vice versa*. A given Cantor number will in general be represented by a variety of diagrams ; hence $\aleph_1 \leqslant c$. Fig. 6 is one of the diagrams representing the natural order of the derived and deduced sets up to E_ω, inclusive, the countable order being obtained from the diagram

$$\begin{array}{cccccccc}
\omega^2 & \omega & 1 & 2 & 3 & 4 & 5 & \ldots \\
 & & 2\omega & \omega+1 & \omega+2 & \omega+3 & \omega+4 & \ldots \\
 & & & 3\omega & 2\omega+1 & 2\omega+2 & 2\omega+3 & \ldots \\
 & & & & 4\omega & 3\omega+1 & 3\omega+2 & \ldots \\
 & & & & & 5\omega & 4\omega+1 & \ldots \\
 & & & & & & 6\omega & \ldots
\end{array}$$

the columns being read in order, thus

$$\omega^2,\ \omega,\ 1,\ 2\omega,\ 2,\ \omega+1,\ 3\omega,\ 3,\ \omega+2,\ 2\omega+1,\ 4\omega,\ 4,\ \omega+3,\ \ldots.$$

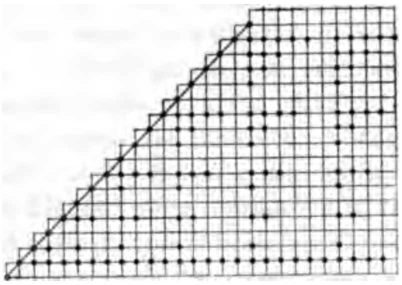

FIG. 6.

The number equivalent to the diagram is

$$\cdot 1\ 11\ 001\ 1011\ 01001\ 000001\ 1101011\ 01001001\ \ldots.$$

If we start with any generic binary fraction, or, which is the same thing, with any generic diagram, we shall not in general have a corresponding Cantor number. The only sets whose ordinal types are Cantor numbers are such as have a definite *first* element, and such that every element has a definite element immediately following it, as is seen by means of the sets of derived and deduced sets. Such sets are said to be "well ordered"; a set which corresponds to a generic binary fraction is said to be "simply ordered." For instance, the binary number $\cdot 1\ 11\ 111\ \ldots$, or $\cdot\dot{1}$, or 1, represents the natural order of the negative integers ..., $-n$, $-n+1$, ..., -2, -1, in the countable order -1, -2, ..., $-n+1$, $-n$, ..., the corresponding diagram having dots at all the trellis points. This ordinal type is denoted by Cantor by ω^*. Thus the potency of the ordinal types of all simply ordered sets is less than or equal to c.

A simple proof of G. Cantor's enables us to go further and assert that this potency is c. Denoting by Π a set whose ordinal type is that of all the negative and positive integers, we take corresponding to any given binary fraction $.e_1 e_2 e_3 \ldots$, where the e's are either ones or zeros, the simply ordered set got by inserting a set Π between e_1 isolated points and e_2 isolated points, then another set Π, then e_3 isolated points and so on. It is easily proved that no set so obtained could be obtained in that order from any different ternary fraction. Hence c is less than or equal to the potency of the ordinal types of all the simply ordered sets. It follows then by a theorem of Cantor's proved by Bernstein and by Schröder, and later by Zermelo, *that c is the potency of the simply ordered sets.* This is the last word which has so far been said on this subject; it was to be hoped that a further investigation of the connexion between the set of simply ordered sets and the partial set of well ordered sets would lead to a determination of $\aleph_1$, but so far this is not the case.

12. *Contents of Closed Sets of Points.*

In amalgamating the black intervals of E to form those of E_1, E_2, ..., E_Ω, we blacken out at most a countably infinite set of limiting points; hence, by Theorem 3′ of the paper on "Sets of Intervals," all these sets of black intervals have the same content. Thus we see *the operations of derivation and deduction leave the content unaltered.*

In dealing with the content of closed sets of points, we need therefore only consider perfect sets, since E_Ω is always perfect, unless it evanesces, *i.e.*, unless the set E is countable. H. J. S. Smith's sets of points of the first kind give us examples of sets of points of potency c and content 0. They are not themselves perfect, but their first derived sets are perfect, *i.e.*, E_Ω is the first derived set. This set was set up in § 14, *loc. cit.*, and in the ternary case consists of all the binary fractions interpreted in the ternary scale; in the general case it consists of all the $(m-1)$-ary fractions interpreted in the m-ary scale. Cantor's perfect set of points with zero content (*loc. cit.*, § 23) consists of all the ternary fractions which do not involve the figure 1.

H. J. S. Smith's sets of the second kind have positive content, which always lies between $\frac{m-1}{m}$ and $\frac{m-2}{m-1}$, where m is the base of the scale adopted. The law of construction of the set E_Ω or E_1 for such a set was proved (*loc. cit.*, p. 260), and runs as follows in the case $m = 3$.

Let $. e_1 \mid e_2 e_3 \mid \ldots \mid e_{\frac{1}{2}[n(n-1)]+1} e_{\frac{1}{2}[n(n-1)]+2} \ldots e_{\frac{1}{2}[n(n+1)]} \mid$ denote any ternary fraction, and let A_n denote the conditions that for no positive integral value of $r \leqslant n$ all the figures from $e_{\frac{1}{2}[r(r-1)]+1}$ to $e_{\frac{1}{2}[r(r+1)]}$, both inclusive, should be 2's.

Then the set of numbers in question consists of all the ternary fractions which violate the conditions A_n for some value of n.

We have in this way got types of closed sets with contents ranging from 0 to l, the latter not inclusive. A closed set of content l is necessarily the whole continuum (A, B).

This follows at once from the definition, since a set of intervals of content zero is an impossibility, unless the whole set evanesces.

[*Note.*—A proof by Schoenflies of what I have called the "Theorem of the Nucleus," based upon the amalgamation of abutting intervals, which involves, however, the properties of well-ordered sets, but avoids the use of Cantor's numbers of the *third* class, appeared this year in the *Göttinger Nachrichten*. The idea of amalgamation as such, and of its application to the simplification of the proof in question, was, I believe, new when I communicated my paper on "Sets of Intervals" to this Society on November 13th, 1902. (See p. 246, lines 4–9, and p. 267 ; also §§ 14–16.) The present paper had, as there indicated, been written at the same time.]

ELEMENTARY NOTE ON THE WEDDLE QUARTIC SURFACE

By H. F. BAKER.

[Received October 8th, 1903.—Read November 12th, 1903.]

[Introduction.—§ 1. Weddle's Definition of the Surface.—§ 2. Geiser's Definition of the Surface.—§ 3. Forms of its Equation and Identities arising therefrom.—§ 4. Statement of the Configuration of Thirty-two Points under consideration.—§ 5. Summary of their Relative Positions.—§ 6. Coordinates of the Points.—§ 7. Group of Thirty-two Transformations arising therefrom.—§ 8. Correspondences of different Weddle Surfaces.]

THE quartic surface which is the locus of the vertices of quadric cones passing through six given points is mentioned by Weddle,* as a correction of a theorem of Chasles, *Camb. and Dubl. Math. Journal*, Vol. v., 1850, p. 69, note; *cf.* Cayley, *Collected Works*, Vol. v., p. 4. The same surface occurs in a paper by Geiser, *Crelle*, Vol. LXVII., 1867, p. 78, who gives a reference to *Züricherischen Vierteljahrschrift*, Vol. x.; considering quadric surfaces passing through six given points, those of these which pass through a seventh point P pass through an eighth point P'; the locus of the points P which coincide with their corresponding points P' is the surface in question as obtained by Geiser. The surface is further considered in a remarkable paper by Schottky, *Crelle*, Vol. cv., 1889, in connexion with the relations among the double theta functions; and the birational relation of the surface with Kummer's sixteen-nodal quartic surface is there developed, a generalisation to particular theta functions of three variables being given in a second paper in the same volume of the *Journal*. In a most interesting paper by Mr. Richmond, *Quarterly Journal*, Vol. XXXIV., 1902, p. 117 (see also *Quarterly Journal*, Vol. XXXI., 1899, p. 125), it is shown that Kummer's surface can be obtained by simple constructions in space of four dimensions; reading this paper, I have noticed that the Weddle surface can be obtained with equal facility, and that the birational relation between the two surfaces is interpretable as a linear projection in four dimensions; and then that from any point of the Weddle surface can be derived by linear projections in three dimensions a remarkable figure of thirty-two points lying on the surface, all variable with the first point, these points lying upon ninety-six straight lines passing in sixes through the thirty-two points, while the thirty-two rational formulæ defining the other points from the first point constitute an Abelian group of thirty-two transformations, definable in abstract sense by the products of six commutable symbols, each of period two, whose product is identity. It is possible that this figure is well known, or that it is derivable at once by simple constructions in four dimensions, while in any case the existence of one-to-one correspondences of points on the surface is an obvious corollary* from the expression of the surface by double theta functions; but it appears to me that the elementary geometry of the surface should precede and lead up to any such mode of expression, and I have thought it may be worth while to collect the results. Some remarks are added at the end as to the birational transformation of different Weddle surfaces into one another, which are interesting, I think, in connexion with the study of the manifoldness constituted by the aggregate of all hyper-elliptic Riemann surfaces.

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* For other references see p. 261 below.

1. Let x, y, z be three-dimensioned rectangular coordinates, the origin be O, the infinite ends of the axes of coordinates be P, Q, R. If (a, b, c), (a_1, b_1, c_1) be any two further points, and l, m, n, l_1, m_1, n_1 be the line coordinates of the line joining them, and for momentary abbreviation, we put

$$\xi = l_1+mz-ny, \qquad \eta = m_1+nx-lz, \qquad \zeta = x_1+ly-mx,$$

$$\tau = -(l_1x+m_1y+n_1z),$$

then the locus of the vertices of the cones contained in the system of quadrics

$$Ax\xi+By\eta+Cz\zeta+l_1aa_1yz+m_1bb_1zx+n_1cc_1xy = 0$$

through the six points is found to be

$$\frac{aa_1\xi}{x}+\frac{bb_1\eta}{y}+\frac{cc_1\zeta}{z}+\tau = 0.$$

2. In what follows we shall put $a_1 = 1$, $b_1 = 1$, $c_1 = 1$, and

$$a' = 1-a, \quad b' = 1-b, \quad c' = 1-c, \quad x' = 1-x, \quad y' = 1-y, \quad z' = 1-z,$$

and $\xi = yz'$, $\xi' = y'z$, $\eta = zx'$, $\eta' = z'x$, $\zeta = xy'$, $\zeta' = x'y$,

$$\alpha = bc', \quad \alpha' = b'c, \quad \beta = ca', \quad \beta' = c'a, \quad \gamma = ab', \quad \gamma' = a'b;$$

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* See the reference to M. Humbert's paper, p. 261 below.

then the general quadric surface through the six points O, P, Q, R, (a, b, c), $(1, 1, 1)$ may be written

$$A(\alpha\xi'-\alpha'\xi)+B(\beta\eta'-\beta'\eta)+C(\gamma\zeta'-\gamma'\zeta) = \frac{\eta-\eta'}{\beta-\beta'}-\frac{\zeta-\zeta'}{\gamma-\gamma'}.$$

We have $\xi+\eta+\zeta = \xi'+\eta'+\zeta'$, $\xi\eta\zeta = \xi'\eta'\zeta'$, and, if we interpret ξ, η, ζ, ξ', η', ζ' as homogeneous planar coordinates in four dimensions, the points of our original space of three dimensions are, with certain exceptions, related to the points of a cubic surface in four dimensions, which is, in fact, that described by Mr. Richmond as Segre's cubic variety, the converse formulæ, when the ratios of ξ, η, ζ, ξ', η', ζ' are given, being

$$x = \frac{\eta'(\zeta'-\zeta)}{\xi(\eta-\eta')}, \quad y = \frac{\zeta'(\xi'-\eta)}{\eta(\zeta-\zeta')}, \quad z = \frac{\xi'(\eta'-\xi)}{\zeta(\xi-\xi')};$$

by this transformation the quadric surfaces through the six given points in three dimensions become the intersections of Segre's cubic variety with arbitrary planar threefolds passing through the point $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ of the variety. Those of these quadric surfaces which pass through a particular seventh point (x_1, y_1, z_1), or P_1, have an equation

$$[A(\alpha\xi'-\alpha'\xi)+B(\beta\eta'-\beta'\eta)+C(\gamma\zeta'-\gamma'\zeta)]\left[\frac{\eta_1-\eta_1'}{\beta-\beta'}-\frac{\zeta_1-\zeta_1'}{\gamma-\gamma'}\right]$$
$$=\left[\frac{\eta-\eta'}{\beta-\beta'}-\frac{\zeta-\zeta'}{\gamma-\gamma'}\right][A(\alpha\xi_1'-\alpha'\xi_1)+B(\beta\eta_1'-\beta'\eta_1)+C(\gamma\zeta_1'-\gamma'\zeta_1)],$$

where

$$\xi_1 = y_1z_1', \quad \xi_1' = y_1'z_1, \quad \eta_1 = z_1x_1', \quad \eta_1' = z_1'x_1, \quad \zeta_1 = x_1y_1', \quad \zeta_1' = x_1'y_1,$$

and these quadric surfaces all pass through an eighth point (x, y, z), or P, such that

$$\frac{\alpha\xi'-\alpha'\xi}{\alpha\xi_1'-\alpha'\xi_1} = \frac{\beta\eta'-\beta'\eta}{\beta\eta_1'-\beta'\eta_1} = \frac{\gamma\zeta'-\gamma'\zeta}{\gamma\zeta_1'-\gamma'\zeta_1} = \frac{\dfrac{\eta-\eta'}{\beta-\beta'}-\dfrac{\zeta-\zeta'}{\gamma-\gamma'}}{\dfrac{\eta_1-\eta_1'}{\beta-\beta'}-\dfrac{\zeta_1-\zeta_1'}{\gamma-\gamma'}};$$

these three equations clearly determine the straight line in four dimensions joining the points $(\alpha, \beta, \ldots)$, $(\xi_1, \eta_1, \ldots)$ of Segre's variety, and the remaining intersection of this with the variety given by $(\alpha+\lambda\xi, \beta+\lambda\eta, \ldots)$, where

$$\lambda(\alpha\eta_1\zeta_1+\beta\zeta_1\xi_1+\gamma\xi_1\eta_1-\alpha'\eta_1'\zeta_1'-\beta'\zeta_1'\xi_1'-\gamma'\xi_1'\eta_1')$$
$$+\xi_1\beta\gamma+\eta_1\gamma\alpha+\zeta_1\alpha\beta-\xi_1'\beta'\gamma'-\eta_1'\gamma'\alpha'-\zeta_1'\alpha'\beta' = 0,$$

gives the eighth intersection P of the quadric surfaces in three dimensions. The points P_1, P will thus coincide when the straight line in four dimensions touches the variety, and will then satisfy the equation of the polar quadric (which makes $\lambda = \infty$),

$$\frac{\alpha}{\xi}+\frac{\beta}{\eta}+\frac{\gamma}{\zeta}=\frac{\alpha'}{\xi'}+\frac{\beta'}{\eta'}+\frac{\gamma'}{\zeta'};$$

putting $\xi = yz'$, ..., $\alpha = bc'$, ..., this equation gives another form for the Weddle surface which on reduction is the same as that given in § 1.

The Weddle surface thus arises as the interpretation in three dimensions of the twofold of contact of the enveloping cone to Segre's variety, whose vertex is an arbitrary point of this. Mr. Richmond shows that the intersection of this cone with an arbitrary planar threefold in the space of four dimensions is Kummer's sixteen-nodal quartic surface; the birational transformation between the surfaces has thus a simple and interesting geometrical interpretation as a linear projection.

3. The equation of the Weddle surface is capable of a great variety of forms of which the following are some. Throughout we put

$$x' = 1-x, \quad a' = 1-a, \quad \dots :$$

$$(\alpha) \quad xx'(bc'y'z-b'cyz')+yy'(ca'z'x-c'azx')+zz'(ab'x'y-a'bxy') = 0,$$

$$(\beta) \quad xyz\Sigma x(b-c)+\Sigma yza(yc'-zb')+\Sigma bcx(y-z) = 0,$$

$$(\gamma) \quad xyz\Sigma x(b-c)-\Sigma ayz(y'c'-z'b') = 0,$$

$$(\delta) \quad \Sigma x'(y-z)(bcx+yza) = 0,$$

$$(\epsilon) \quad \begin{vmatrix} a/x & x & a & 1 \\ b/y & y & b & 1 \\ c/z & z & c & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

$$(\zeta) \quad \frac{yz(y'c'-z'b')}{bc'y'z-b'cyz'}=\frac{zx(z'a'-x'c')}{ca'z'x-c'azx'}=\frac{xy(x'b'-y'a')}{ab'x'y-a'bxy'}, \ = H(x, y, z), \text{ say},$$

so that also $$H(x, y, z) = -\frac{\Sigma x(b-c)}{\Sigma \dfrac{a(b-c)}{x}},$$

$$(\iota) \quad \frac{(b-y)(c-z)(y-z)}{y'z'(yc-zb)}=\frac{(c-z)(a-x)(z-x)}{z'x'(za-xc)}=\frac{(a-x)(b-y)(x-y)}{x'y'(xb-ya)},$$

$$= K(x, y, z), \text{ say};$$

(κ) $$\left(1-\frac{aH}{x}\right)\frac{x'}{a'}=\left(1-\frac{bH}{y}\right)\frac{y'}{b'}=\left(1-\frac{cH}{z}\right)\frac{z'}{c'}=-\frac{\Sigma x'(b-c)}{\Sigma\dfrac{a'(b-c)}{x'}},$$

$$\left(1-\frac{H}{x}\right)\frac{a-x}{a'}=\left(1-\frac{H}{y}\right)\frac{b-y}{b'}=\left(1-\frac{H}{z}\right)\frac{c-z}{c'}=-\frac{\Sigma x(b-c)}{\Sigma\dfrac{aa'(b-c)}{x-a}},$$

where H denotes $H(x, y, z)$, a notation constantly employed.

(λ) If we denote by $\vartheta(x)$, $\phi(x)$, $\psi(x)$ three points derived from any point (x, y, z) on the Weddle surface by the formulæ

$$\vartheta(x)=\left(\frac{a}{x},\ \frac{b}{y},\ \frac{c}{z}\right),\qquad \phi(x)=\left(\frac{aH}{x},\ \frac{bH}{y},\ \frac{cH}{z}\right),$$

$$\psi(x)=\left(\frac{a-x}{x'},\ \frac{b-y}{y'},\ \frac{c-z}{z'}\right),$$

it will be found that these are also points of the surface, whose geometrical derivation from (x, y, z) will appear below; and, further, that on repetition of these formulæ we shall have

$$\vartheta\{\vartheta(x)\} \text{ or } \vartheta^2(x)=(x, y, z),\qquad \phi^2(x)=(x, y, z),\qquad \psi^2(x)=(x, y, z),$$

and $\vartheta\phi(x)=\phi\vartheta(x)$, $\quad\vartheta\psi(x)=\psi\vartheta(x)$, $\quad\phi\psi(x)=\psi\phi(x)$,

these latter being

$$\vartheta\phi(x)=\left(\frac{x}{H},\ \frac{y}{H},\ \frac{z}{H}\right),\qquad \vartheta\psi(x)=\left(\frac{ax'}{a-x},\ \frac{by'}{b-y},\ \frac{cz'}{c-z}\right),$$

$$\phi\psi(x)=(x_1, y_1, z_1),$$

where, as throughout in what follows, we use x_1, y_1, z_1 to denote

$$x_1=\frac{a\left(1-\dfrac{H}{x}\right)}{1-\dfrac{aH}{x}},\qquad y_1=\frac{b\left(1-\dfrac{H}{y}\right)}{1-\dfrac{bH}{y}},\qquad z_1=\frac{c\left(1-\dfrac{H}{z}\right)}{1-\dfrac{cH}{z}},$$

so that $\dfrac{aH}{x}=\dfrac{a-x_1}{x_1}$, $\quad\phi(x)=\left(\dfrac{a-x_1}{x_1'},\ \dfrac{b-y_1}{y_1'},\ \dfrac{c-z_1}{z_1'}\right)$.

With this notation we shall find, if ϕH denote $H\{\phi(x)\}$, ..., that

$$\phi H=H,\qquad \vartheta H=\frac{1}{H},\qquad \psi H=K,$$

$$\phi K=K,\qquad \vartheta K=\frac{1}{K},\qquad \psi K=H,$$

K being the function occurring with H under (ζ), (ι) above; these give rise to a variety of identities and reciprocities, such as

$$H\left(\frac{a}{x},\ \frac{b}{y},\ \frac{c}{z}\right)=\frac{1}{H(x,y,z)},\qquad H\left(\frac{aH}{x},\ \frac{bH}{y},\ \frac{cH}{z}\right)=H(x,\ y,\ z),$$

$$K\left(\frac{a}{x},\ \frac{b}{y},\ \frac{c}{z}\right)=\frac{1}{K(x,y,z)},\qquad K\left(\frac{aH}{x},\ \frac{bH}{y},\ \frac{cH}{z}\right)=K(x,\ y,\ z),$$

$$\begin{aligned} H(x_1, y_1, z_1) &= H\left(\frac{a-x}{x'},\ \frac{b-y}{y'},\ \frac{c-z}{z'}\right) \\ &= K(x,y,z)=x_1\frac{a-x}{ax'}=y_1\frac{b-y}{by'}=z_1\frac{c-z}{cz'}, \\ K(x_1, y_1, z_1) &= K\left(\frac{a-x}{x'},\ \frac{b-y}{y'},\ \frac{c-z}{z'}\right) \\ &= H(x,y,z)=x\frac{a-x_1}{ax_1'}=y\frac{b-y_1}{by_1'}=z\frac{c-z_1}{cz_1'}. \end{aligned}$$

(μ) The function H is not symmetrical in regard to the four points O, P, Q, R. If we put x/t, y/t, z/t, a/d, b/d, c/d respectively for x, y, z, a, b, c and denote H by H_t, so that H_x will denote the similar function of y, z, t, ..., we find

$$\frac{x^2}{a}H_x=\frac{y^2}{b}H_y=\frac{z^2}{c}H_z=\frac{t^2}{d}H_t=\Omega,$$

where Ω is capable of the four forms, of 2 dimensions in x, y, z, t and -1 dimensions in a, b, c, d,

$$\begin{aligned} \Omega &= -\frac{(b-c)x+(c-a)y+(a-b)z}{\dfrac{a(b-c)}{x}+\dfrac{b(c-a)}{y}+\dfrac{c(a-b)}{z}} \\ &= \ldots = \ldots = -\frac{(b-c)t+(c-d)y+(d-b)z}{\dfrac{d(b-c)}{t}+\dfrac{b(c-d)}{y}+\dfrac{c(d-b)}{z}}. \end{aligned}$$

(ϖ) Finally an irrational form of which the equation of the Weddle surface is capable is

$$\Sigma a'\sqrt{x(y'c'-z'b')(bc'y'z-b'cyz')}=0,$$

which in the notation above is the same as

$$\Sigma\sqrt{\frac{a'^2}{a}(\beta'\zeta-\gamma\eta')(\alpha\xi'-\alpha'\xi)}=0,$$

or, putting

$$X=\alpha\xi'-\alpha'\xi,\quad Y=\beta\eta'-\beta'\eta,\quad Z=\gamma\zeta'-\gamma'\zeta,$$
$$T=a(\xi-\xi')+b(\eta-\eta')+c(\zeta-\zeta'),$$

the same as

$$\Sigma\sqrt{a'(b-c)X\left[aT+\frac{c-a}{b'}Z-\frac{a-b}{c'}Y\right]}=0,$$

whereby a transformation is effected to a Kummer 16-nodal surface. It is found on rationalisation that the irrational form gives, without extraneous factors, the result

$$[xx'(bc'y'z-b'cyz')+yy'(ca'z'x-c'azx')+zz'(ab'x'y-a'bxy')]^2=0.$$

4. The Weddle surface has six nodes, at O, P, Q, R and (a, b, c), $(1, 1, 1)$; and these enter symmetrically, as is obvious from the definition of the surface. It contains the fifteen straight lines joining these nodes in pairs and the ten straight lines in which the plane of any three of these nodes meets the plane of the remaining three, of which, for instance, one is the line at infinity lying on the plane $\Sigma x(b-c)=0$. It also contains the cubic curve in space through the six nodes whose points are given in terms of a parameter θ by the equations

$$x=\frac{a}{1-a'\theta},\quad y=\frac{b}{1-b'\theta},\quad z=\frac{c}{1-c'\theta}.$$

The tangent cone of the surface at any one of the nodes is the quadric cone having this point for vertex containing the other five nodes; thus this cone contains the twisted cubic just spoken of. Hence it follows that the quadric cone projecting the twisted cubic from any point C of itself has six generators which touch the Weddle surface at the six nodes. Consider now the section of the surface by the plane containing C and any two of the nodes; besides the joining line of these nodes it will consist of a plane cubic curve passing through the two nodes, and the tangent lines of this plane cubic at the two nodes will intersect on the curve at C. Hence it follows from the theory of a plane cubic curve, though sufficient independent verification of the result arises below, that, if any point ω of the plane cubic be joined to the two nodes, and the remaining intersections of the joining lines with the curve be crosswise joined to the same two nodes, the last two joins will intersect in a point of the curve. Since the point C may be taken anywhere on the twisted cubic, we have therefore the result:

Let N_1, N_2 be any two of the six nodes of the quartic surface, and ω any point of the surface. Let the remaining intersection with the surface of the line $N_1\omega$ be called (N_1), and, similarly, let (N_2) be the remaining intersection of $N_2\omega$. Then the straight line joining N_1 to (N_2) and the straight line joining N_2 to (N_1) intersect one another on the surface. This point of intersection we may denote by (N_1N_2) or (N_2N_1).

Now, let N_3 be any other node of the surface. By constructing similarly with the pairs of nodes N_1, N_3 and N_2, N_3, we obtain three other points (N_3), (N_1N_3), (N_2N_3) lying upon the surface, and it is a further theorem, to be verified by the formulæ below, that the remaining intersections with the surface of the three straight lines joining respectively N_1 to (N_2N_3), N_2 to (N_3N_1), and N_3 to (N_1N_2) are one and the same point. This point we may denote by $(N_1N_2N_3)$, where the order of the letters is indifferent.

If, finally, M_1, M_2, M_3 be the other three nodes of the surface, we can derive similarly by projection from the same point ω of the surface seven other points (M_1), ..., (M_1M_3), ..., (M_2M_3), terminating with $(M_1M_2M_3)$; and it is a further theorem that this point $(M_1M_2M_3)$ coincides with $(N_1N_2N_3)$. Thus the attempt to obtain points, which would be denoted by symbols containing four letters, by projecting $(N_1N_2N_3)$ from any one of the nodes M_1, M_2, M_3 will only lead to the points denoted by symbols containing less than three of the letters M_1, M_2, M_3.

The total of the points obtained in this way is therefore thirty-two, namely,

ω,
(N_1) .... (6 symbols),
(N_1N_2) ...... (15 symbols),
$(N_1N_2N_3)$ ... (10 distinct symbols),

each of the last ten symbols being replaceable by the symbol containing the three complementary letters. These thirty-two points are variable with ω, and derived from it by linear projection, as has been explained. We shall denote them by (x), (0), (P), (Q), (R), (1), (a), (QR), $(1P)$, $(1a)$, $(1aP)$, ...

In obtaining these thirty-two points we have used ninety-six straight lines: namely, each of the six nodes is joined to (x, y, z), giving the six points (0), (P), (Q), (R), (1), (a); then each pair of these latter is joined crosswise to the corresponding pair of nodes, requiring thirty straight lines in addition to the first six; then each of the doublet points so obtained, such as (QR), is joined to the four nodes whose symbols are complementary to that of the doublet, requiring in all sixty further straight lines. Of these straight lines six pass through (x, y, z); through such a point as (P) there pass lines from each of the nodes other than P, together with the line to (x, y, z) which goes through P, so that six of the straight lines pass through (P); through a doublet point, such as (QR), there pass the lines to (Q), (R) which pass respectively to R and Q and the lines to the four nodes other than Q and R, again six in all; while through a triplet point such as (PQR) there pass the three lines joining P to (QR), Q to (RP), and R to (PQ), and, as $(PQR) = (0a1)$, also the three lines joining 0 to $(a, 1)$, (a, b, c) to $(0, 1)$, and $(1, 1, 1)$ to $(0a)$, also six altogether. The ninety-six straight lines pass then in sixes through the thirty-two points, and, accordingly, there are two of the points upon any one of the straight lines. Besides this, each straight line passes through one of the nodes, and through each node there are sixteen of the straight lines. Further, as the quartic surface is the locus of the vertices of quadric cones passing through the six nodes, the six lines through any one of the points are generators of a quadric cone, and developments are possible by utilising Pascal's theorem.

It is at once seen that the thirty-two points are symmetrical with one another in the sense that, if we start with any one of them instead of (x, y, z) and make the same linear projections, we shall arrive only at the same points in a different order. The substitutions are given by supposing in the symbols of the derived points, such as (PQ), that the individual symbols P, Q, ..., of the six nodes, are added together, a repetition being self-destructive; it is necessary to bear in mind that a triplet such as (PQR) is the same as $(0a1)$, and to assign zero as the symbol of (x, y, z). Thus, for example, if (0) be substituted for (x, y, z), the points (P), (QR), $(0QR)$, (PQR) change respectively to $(0P)$, $(0QR)$, (QR), $(1a)$.

5. To grasp the geometrical relations of the thirty-two points we may summarize them in various ways.

Thus from the three infinite nodes are derived eight points, namely, (x, y, z), (P), (Q), (R), (QR), (RP), (PQ), (PQR). These form, in fact, the corners of a rectangular parallelepiped, having (x, y, z) and (PQR) as opposite ends of a diagonal; and the twelve diagonals of the faces meet in twos in six points lying in twos on the sides of the triangle PQR, which is at infinity. The set of eight points similarly associated with any other triplet of nodes form a figure of the same descriptive character as this one. The twenty such hexahedral figures have all one common corner, at (x, y, z), and of the diagonals through this point there are ten, each common to two of the figures.

Another description which gives the thirty-two points is as follows:— Any two points in space say A and B, can be used to define a rectangular parallelepiped of which AB is a diagonal, namely by drawing planes from each of A, B to each of the lines at infinity QR, RP, PQ, where, as before, P, Q, R are the infinite ends of the rectangular axes of coordinates. In a similar way a hexahedral figure is defined by drawing planes from each of A, B to each of the sides of any triangle $M_1M_2M_3$. Now, let

M_1, M_2, M_3 and N_1, N_2, N_3 be the six nodes of the surface in any order; obtain by projection of (x, y, z) from N_1, N_2, N_3, as before, the eight derived points including (x, y, z); divide these into the four pairs

$$(x, y, z),\ (N_1N_2N_3) \mid (N_1),\ (N_2N_3) \mid (N_2),\ (N_3N_1) \mid (N_3),\ (N_1N_2);$$

now complete each of these pairs into eight points, the corners of such a hexahedral figure as before described, by drawing planes from each constituent of a pair to each of the straight lines M_2M_3, M_3M_1, M_1M_2. The thirty-two corners of the four hexahedral figures so obtained are the thirty-two points in question.

Thus, if we take for N_1, N_2, N_3 the infinite nodes P, Q, R, the four pairs of points above put down are the extremities of the four diagonals of a rectangular parallelepiped whose coordinates are given below; taking the eight triangular pyramids having as common base the triangle whose angular points are the three nodes (0, 0, 0), (a, b, c), (1, 1, 1), and as vertices the eight corners of the rectangular parallelepiped, the six faces of the two pyramids whose vertices are the extremities of any diagonal of the parallelepiped give a hexahedral figure with eight corners; these 4×8 corners, corresponding to the four diagonals, are the 32 points in question.

If, on the other hand, we take for N_1, N_2, N_3 the three finite nodes (0, 0, 0), (a, b, c), (1, 1, 1), the four hexahedral figures formed by joining the eight points above to the lines QR, RP, PQ are actually rectangular parallelepipeds. This is the arrangement adopted in the table of the coordinates of the thirty-two points now to be given.

6. With the notation of § 3,

$$\vartheta(x) = \left(\frac{a}{x}, \frac{b}{y}, \frac{c}{z}\right), \qquad \phi(x) = \left(\frac{aH}{x}, \frac{bH}{y}, \frac{cH}{z}\right),$$

$$\psi(x) = \left(\frac{a-x}{x'}, \frac{b-y}{y'}, \frac{c-z}{z'}\right),$$

it is found that the eight points, including (x, y, z), derived as explained by linear projection from the three nodes (0, 0, 0), (a, b, c), (1, 1, 1), namely,

$$(x),\ (0a1) \mid (1a),\ (0) \mid (1),\ (a0) \mid (a),\ (01),$$

are respectively

$$(x),\ \phi(x) \mid \vartheta(x),\ \vartheta\phi(x) \mid \phi\psi(x),\ \psi(x) \mid \vartheta\phi\psi(x),\ \vartheta\psi(x),$$

of which the coordinates are given in § 3, save that

$$\vartheta\phi\psi(x) = \left\{\frac{1-\dfrac{aH}{x}}{1-\dfrac{H}{x}}, \frac{1-\dfrac{bH}{y}}{1-\dfrac{H}{y}}, \frac{1-\dfrac{cH}{z}}{1-\dfrac{H}{z}}\right\};$$

making now each of these couples the extremities of a diagonal of a rectangular parallelepiped whose faces pass through the lines at infinity QR, RP, PQ, that is, are parallel to the coordinate planes, the thirty-two points constituted by the corners of these are as follows :—

$$(x, y, z) = (x, y, z), \qquad (1a) = \left(\frac{a}{x}, \frac{b}{y}, \frac{c}{z}\right),$$

$$(P) = \left(\frac{aH}{x}, y, z\right), \qquad (P1a) = (0QR) = \left(\frac{x}{H}, \frac{b}{y}, \frac{c}{z}\right),$$

$$(Q) = \left(x, \frac{bH}{y}, z\right), \qquad (Q1a) = (0RP) = \left(\frac{a}{x}, \frac{y}{H}, \frac{c}{z}\right),$$

$$(R) = \left(x, y, \frac{cH}{z}\right), \qquad (R1a) = (0PQ) = \left(\frac{a}{x}, \frac{b}{y}, \frac{z}{H}\right),$$

$$(QR) = \left(x, \frac{bH}{y}, \frac{cH}{z}\right), \qquad (0P) = \left(\frac{a}{x}, \frac{y}{H}, \frac{z}{H}\right),$$

$$(RP) = \left(\frac{aH}{x}, y, \frac{cH}{z}\right), \qquad (0Q) = \left(\frac{x}{H}, \frac{b}{y}, \frac{z}{H}\right),$$

$$(PQ) = \left(\frac{aH}{x}, \frac{bH}{y}, z\right), \qquad (0R) = \left(\frac{x}{H}, \frac{y}{H}, \frac{c}{z}\right),$$

$$(0a1) = (PQR) = \left(\frac{aH}{x}, \frac{bH}{y}, \frac{cH}{z}\right), \qquad (0) = \left(\frac{x}{H}, \frac{y}{H}, \frac{z}{H}\right),$$

$$(1) = (x_1, y_1, z_1), \qquad (a) = \left(\frac{a}{x_1}, \frac{b}{y_1}, \frac{c}{z_1}\right),$$

$$(1P) = \left(\frac{a-x}{x'}, y_1, z_1\right), \qquad (aP) = \left(\frac{ax'}{a-x}, \frac{b}{y_1}, \frac{c}{z_1}\right),$$

$$(1Q) = \left(x_1, \frac{b-y}{y'}, z_1\right), \qquad (aQ) = \left(\frac{a}{x_1}, \frac{by'}{b-y}, \frac{c}{z_1}\right),$$

$$(1R) = \left(x_1, y_1, \frac{c-z}{z'}\right), \qquad (aR) = \left(\frac{a}{x_1}, \frac{b}{y_1}, \frac{cz'}{c-z}\right),$$

$$(1QR) = (0Pa) = \left(x_1, \frac{b-y}{y'}, \frac{c-z}{z'}\right), \qquad (0P1) = (QRa) = \left(\frac{a}{x_1}, \frac{by'}{b-y}, \frac{cz'}{c-z}\right),$$

$$(1RP) = (0Qa) = \left(\frac{a-x}{x'}, y_1, \frac{c-z}{z'}\right), \qquad (0Q1) = (RPa) = \left(\frac{ax'}{a-x}, \frac{b}{y_1}, \frac{cz'}{c-z}\right),$$

$$(1PQ) = (0Ra) = \left(\frac{a-x}{x'}, \frac{b-y}{y'}, z_1\right), \qquad (0R1) = (PQa) = \left(\frac{ax'}{a-x}, \frac{by'}{b-y}, \frac{c}{z_1}\right),$$

$$(a0) = \left(\frac{a-x}{x'}, \frac{b-y}{y'}, \frac{c-z}{z'}\right), \qquad (01) = \left(\frac{ax'}{a-x}, \frac{by'}{b-y}, \frac{cz'}{c-z}\right),$$

where, as before,

$$x_1 = \frac{a\left(1-\dfrac{H}{x}\right)}{1-\dfrac{aH}{x}}, \qquad y_1 = \frac{b\left(1-\dfrac{H}{y}\right)}{1-\dfrac{bH}{y}}, \qquad z_1 = \frac{c\left(1-\dfrac{H}{z}\right)}{1-\dfrac{cH}{z}},$$

and $$\frac{aH(x, y, z)}{x} = \frac{a-x_1}{x_1'}, \qquad \frac{aH(x_1, y_1, z_1)}{x_1} = \frac{a-x}{x'}, \qquad \dots .$$

Further, it will easily be found that the values of each of the functions $H(x, y, z)$, $K(x, y, z)$ are the same at all the eight points of any one column, these values being

First column.	*Second column.*	*Third column.*	*Fourth column.*
$H = H(x, y, z)$,	$H = \dfrac{1}{H(x, y, z)}$,	$H = K(x, y, z)$,	$H = \dfrac{1}{K(x, y, z)}$,
$K = K(x, y, z)$,	$K = \dfrac{1}{K(x, y, z)}$,	$K = H(x, y, z)$,	$K = \dfrac{1}{H(x, y, z)}$;

so that also any symmetric function of the four quantities $H(x, y, z)$, $\dfrac{1}{H(x, y, z)}$, $K(x, y, z)$, $\dfrac{1}{K(x, y, z)}$ has the same value at all the thirty-two points.

7. In accordance with what was said in § 4 as to the possibility of obtaining this system of thirty-two points starting from any one of them, the table of their coordinates which has just been given defines a group of thirty-two transformations of the quantities x, y, z connected by the equation of the quartic surface. The group may be defined in an abstract sense as constituted by the products of six commutative symbols of which each is equal to its own inverse, while the product of the six symbols is identity, thus

1;

A_1 .... (6 symbols);

A_1A_2 ...... (15 symbols);

$A_1A_2A_3$ ... (10 distinguishable symbols).

The group has sub-groups of two operations, such as 1, (P); it has sub-groups of four operations, such as 1, (P), (Q), (PQ); also sub-groups of eight operations, such as 1, (P), (Q), (R), (QR), (RP), (PQ), (PQR); also sub-groups of sixteen operations, such as

$$1, (P), (Q), (R), (1), (QR), (RP), (PQ), (1P), (1Q), (1R),$$
$$(PQR), (1QR), (1RP), (1PQ), (PQR1) = (0a);$$

and, finally, can itself be generated as from the five operations (P), (Q), (R), (1), (a), thus

$$1, (P), (Q), (R), (1), (a), (QR), (RP), (PQ), (1P), (1Q), (1R), (aP), (aQ), (aR), (a1),$$
$$(PQR), (1QR), (1RP), (1PQ), (aQR), (aRP), (aPQ), (a1P), (a1Q), (a1R),$$
$$(1PQR) = (0a), \quad (aPQR) = (01), \quad (QR1a) = (0P), \quad (RP1a) = (0Q),$$
$$(PQ1a) = (0R), \quad (PQR1a) = (0).$$

It may be remarked that one of these thirty-two transformations is suggested at once by one of the forms into which the equation of the quartic surface can be put; it is obvious in fact that, if

$$\frac{bc'}{yz'}+\frac{ca'}{zx'}+\frac{ab'}{xy'}=\frac{b'c}{y'z}+\frac{c'a}{z'x}+\frac{a'b}{x'y},$$

we can put $\quad \dfrac{bc'}{yz'} = \sigma YZ', \quad \dfrac{ca'}{zx'} = \sigma ZX', \quad \dfrac{ab'}{xy'} = \sigma XY',$

$$\frac{b'c}{y'z} = \sigma Y'Z, \quad \frac{c'a}{z'x} = \sigma Z'X, \quad \frac{a'b}{x'y} = \sigma X'Y,$$

where σ is a factor of proportionality, and obtain

$$\frac{bc'}{YZ'}+\frac{ca'}{ZX'}+\frac{ab'}{XY'}=\frac{b'c}{Y'Z}+\frac{c'a}{Z'X}+\frac{a'b}{X'Y}.$$

On solving the equations for X, Y, Z, we find

$$X = \frac{aH}{x}, \quad Y = \frac{bH}{y}, \quad Z = \frac{cH}{z},$$

which are the formulæ whereby we pass from (x, y, z) to $(0a1)$ or (PQR).

8. At the beginning of this paper we obtained the Weddle quartic surface by projection in four dimensions as one of ∞^3 such surfaces, according to the point of projection (a, b, c), these ∞^3 surfaces having five nodes in common. For the sake of comparison with what has preceded, brief reference should be made to a group of transformations of these ∞^3 surfaces each into another of the set; or, say, to a group of transformations for the six quantities x, y, z, a, b, c which leaves unaltered the equation

$$xx'(bc'y'z-b'cyz')+yy'(ca'z'x-c'azx')+zz'(ab'x'y-a'bxy') = 0,$$

We have put (§ 2)

$$\xi = yz', \quad \eta = zx', \quad \zeta = xy', \quad \xi' = y'z, \quad \eta' = z'x, \quad \zeta' = x'y;$$

if we now put

$$x_1 = \tfrac{1}{2}(\eta+\zeta-\xi), \quad x_2 = \tfrac{1}{2}(\zeta+\xi-\eta), \quad x_3 = \tfrac{1}{2}(\xi+\eta-\zeta),$$
$$x_4 = \tfrac{1}{2}(\xi'-\eta'-\zeta'), \quad x_5 = \tfrac{1}{2}(\eta'-\zeta'-\xi'), \quad x_6 = \tfrac{1}{2}(\zeta'-\xi'-\eta'),$$

giving

$$\xi = x_2+x_3, \qquad \eta = x_3+x_1, \qquad \zeta = x_1+x_2,$$
$$\xi' = -(x_5+x_6), \qquad \eta' = -(x_6+x_4), \qquad \zeta' = -(x_4+x_5),$$

we find from

$$\xi+\eta+\zeta = \xi'+\eta'+\zeta', \qquad \xi\eta\zeta = \xi'\eta'\zeta',$$

the equations for Segre's cubic variety, which are fundamental in Mr. Richmond's paper (*loc. cit.*),

$$x_1+x_2+x_3+x_4+x_5+x_6 = 0, \qquad x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3 = 0,$$

while, if $(g_1, \ldots, g_6)$ be the corresponding coordinates for the particular values $x = a$, $y = b$, $z = c$, the equation of the first polar, to give the Weddle quartic surface, becomes

$$g_1x_1^2+g_2x_2^2+g_3x_3^2+g_4x_4^2+g_5x_5^2+g_6x_6^2 = 0.$$

These equations are unaltered by any interchange of two of the six coordinates $(x_1, \ldots, x_6)$, provided a like interchange be made of two of the coordinates $(g_1, \ldots, g_6)$. Every such interchange gives a formula of transformation for (x, y, z) and a like formula for (a, b, c), which together leave the Weddle surface unaltered, and so furnishes a point-to-point correspondence between the Weddle surfaces arising on Segre's cubic variety by taking two different suitably related points of projection (a, b, c).

Now every such permutation of $(x_1, \ldots, x_6)$ may be obtained by combinations of operations of the following characters:—

(i.) An interchange of two of the three coordinates x_1, x_2, x_3.

(ii.) The simultaneous interchange of these three coordinates respectively with x_4, x_5, x_6.

(iii.) The interchange of the coordinates x_1 and x_4.

Thus the character of the transformations for (x, y, z) and (a, b, c) will be sufficiently represented by the three following facts, of which it is unnecessary to give the verification.

(i.) The interchange of x_2 and x_3 is equivalent to replacing x, y, z respectively by $\dfrac{x-z}{y-z}$, $\dfrac{y(x-z)}{x(y-z)}$, $\dfrac{y'(x-z)}{x'(y-z)}$; and after this replacement, with the corresponding change for a, b, c, the Weddle surface is unchanged.

(ii.) The interchange of x_1, x_2, x_3 respectively and simultaneously with

x_4, x_5, x_6 is equivalent to replacing x, y, z respectively by $x'\,(=1-x)$, $y'\,(=1-y)$, $z'\,(=1-z)$; and the Weddle surface is unaltered thereby, if at the same time a, b, c are replaced by a', b', c'.

(iii.) The interchange of x_1 and x_4 is equivalent to replacing x, y, z respectively by x, $\frac{xz'}{x-z}$, $\frac{xy'}{x-y}$; and this with the corresponding change for a, b, c leaves the Weddle surface unaltered.

[*December* 11*th*, 1903.—For the Weddle surface see also: Hierholzer, *Math. Ann.*, Vol. IV., 1871; Reye, *Crelle*, Vol. LXXXVI., pp. 87, 98; Hunyady, *Crelle*, Vol. XCII., 1882; Darboux, *Bulletin des Sci. math.*, 1e Sér., t. I., p. 354; Caspary, *Compt. Rend.*, Vol. CXII., 1891; Caspary, *Bulletin des Sci. math.*, 2e Sér., t. XV., 1891, where equation IV. *a*, p. 314, compared with equation (ϵ), § 3, of the preceding note, gives at once a representation of the surface by hyperelliptic functions; and, finally, Humbert, *Liouville*, Vol. IX., 1893, where a comparison of pp. 470 and 112 gives the relation between the hyperelliptic parameters of the points denoted in the preceding note by (O), (P), (Q), (R), (1), (a), from which the existence of the group of thirty-two points follows at once. By this, however, the interest of approaching the parametric representation from an elementary point of view appears rather to be increased than diminished.]

ON SEQUENCES OF SETS OF INTERVALS CONTAINING A GIVEN SET OF POINTS

By W. H. YOUNG.

[Received September 17th, 1903.—Read November 12th, 1903.]

1. In the prosecution of investigations into sets of intervals on the straight line, some of which are to be found in the *Proceedings of the London Mathematical Society*, I was naturally led to consider sequences of sets of intervals [*] formed in the following manner:—Taking any set of points E, describe round each point an interval having that point as internal point; diminish the length of each interval, according to any law, in such a way that all tend simultaneously towards zero, the original set of points always remaining internal to their corresponding intervals. What do we know about those points which are always internal to the intervals of every set (inner limiting set)? The chief properties of such a set may, I find, be summed up in the following six theorems:—

(1) The inner limiting set consists of E, together with certain points of the first derived set E'; the latter points may sometimes be absent.

(2) The inner limiting set may contain every point of E'.

(3) If the content of the intervals [†] is ever less than that of E', there is a more than countable set of points E' not contained in the inner limiting set.

(4) The potency of the inner limiting set is the same as the potency of E', unless E contains no component dense in itself, while E' is more than countable.

(5) If E contains no component dense in itself, while E' is more than countable, the inner limiting set may be either countably infinite

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[*] A special case of this was considered by Borel in his *Leçons sur la Théorie des Fonctions*; cf. also Baire's sets of the second category, and the general theory of content and kindred subjects as developed by Riemann, Cantor, H. J. S. Smith, and others.

[†] That is the content of a set of non-overlapping intervals having the same internal points as the given set (*Proc. London Math. Soc.*, Vol. xxxv., p. 386). It is well known that when E is countable the intervals can be so constructed that their content becomes less than *any* assignable quantity.

or have the potency c; and we can so arrange the intervals that the inner limiting set consists of E alone.

(6) In general, we can so arrange the intervals that those points of the inner limiting set which are not points of E are limiting points only of U, the greatest component of E which is dense in itself.*

Theorems 1, 2, and 3 are almost obvious; for convenience I give the formal proof. Theorem 4 and the first part of Theorem 5 are to be found in a paper presented to the Sächsischer Gesellschaft der Wissenschaft†; Theorem 6, which includes the latter part of Theorem 5 as a special case, is here stated and proved for the first time. The theorem employed requires some consideration of the analysis of sets of points into what Cantor calls *adherents and coherents*, and I have found it necessary to go shortly into this matter for the purposes of the proof.

2. *Proof of Theorem* 1.

Suppose P to be a point of the inner limiting set not contained in E; then we can assign an interval from each successive set containing P as internal point, and each of these intervals contains a point of E; since the length of these intervals decreases without limit, it follows that P is a limiting point of points of E. Q. E. D.

3. *Proof of Theorem* 2.

Take all the intervals of any set equal in length, and let that length diminish without limit for the successive sets; then every limiting point of E must be internal to intervals of every set, and therefore the inner limiting set consists of E together with all those points of E' which were not points of E. Q. E. D.

4. *Proof of Theorem* 3.

Let I be the content of E', and let $J < I$ be the content of one of the sets of intervals, and e any small positive quantity less than $J-I$.

If possible, let those points of E' which do not belong to the inner limiting set be arranged in countable order $P_1, P_2, \ldots$. Round each point P_i as centre describe an interval of length $e/2^{i+1}$, and add these intervals to those of the set in question; then the content J of the latter is increased at most by $\frac{1}{2}e$. Thus we have shut up all the points of the closed set E' in a set of intervals of content less than I, which is known to be impossible. Hence the points in question cannot be countable. Q. E. D.

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* As in Theorem 4, if the content of the intervals be ever less than that of U' (the first derived of U), there will be a more than countable set of limiting points of U not contained in the inner limiting set. A similar theorem evidently holds for any closed component of E'.

† "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipz. Ber.*, pp. 287-293.—*August 1st*, 1903.

5. *Adherences and Coherences.*

Given any set of points E, Cantor denotes as an adherent of E any isolated point of E, and as a coherent any limiting point contained in E. The set of all the adherents he calls the *adherence*, and of all the coherents the *coherence*, and denotes these symbolically by the addition of an a or a c respectively to the symbol for the whole set: thus, $E = E_a + E_c$. If E_c is neither an isolated set nor dense in itself, it will have both adherents and coherents, and we may proceed a stage further, $E_c = E_{ca} + E_{cc}$. We notice that the points of E_{ca} *must* be limiting points of E_a.

Similarly, if E_{cc} be neither an isolated set nor dense in itself, we can proceed a stage further and, generally, as long as we do not arrive at a coherence which is either an isolated set or dense in itself, we can proceed on another stage in our analysis. During the process *each new adherence consists of points which are limiting points of every preceding adherence.* If the process never comes to an end, we can examine whether, or no, the infinite series of coherences

$$E_c,\ E_{cc},\ E_{ccc},\ \ldots \tag{1}$$

has any common point. Since these sets are not necessarily closed, they may have no common point, and in this case the original set consists only of the countably infinite series of adherences

$$E_a,\ E_{ca},\ E_{cca},\ \ldots, \tag{2}$$

and is, therefore, countable, and has no component dense in itself.

If, however, the coherences (1) have at least one point common, we can "deduce" from them a set, viz., that consisting of all their common points. E will then consist of the series of adherences (2), together with this "deduced coherence." We can then continue the process with this deduced coherence, and so on. It can then be shown that the adherence of this deduced coherence consists, as before, of points which are limiting points of every preceding adherence.

If, now, E have any component dense in itself, it is easily seen that it has a definite largest component dense in itself, U, such that U contains every component of E which is dense in itself, and that those points of E which are not points of U are countable; if there be no such component U, however, E is itself countable.

From this it is easy to show that, after a countable set of stages, the process of analysis into adherences and a coherence comes to an end, leaving us either with U as residuary coherence or, if there be no U, without any residuary coherence at all.

Arranging the successive adherences then in *countable* order $A_1, A_2, \ldots$, *we have the following general analysis* :—$E = U + A_1 + A_2 + \ldots$ (*where U may, of course, in a special case be absent, and the series of A's be finite or countably infinite*), *U denoting the greatest component of E which is dense in itself, and A_1, A_2, ... isolated, and therefore countable, sets (adherences) such that each A_i consists of points which are limiting points of every adherence A_k which, in the natural order, precedes it.*

6. *Proof of Theorem* 6.

Having arranged the coherences in countable order, let us arrange the points in each adherence also in countable order, and let P_{ij} denote the j-th point of A_i.

We will now show how to construct the intervals in such a way that the only extraneous points which can remain in the inner limiting set are limiting points of U.

Let us assume any finite positive quantity l. Then, since A_i contains none of its limiting points, we can assign a definite largest interval, of length less than l, say $2d_{ij}$, with P_{ij} as centre, such that inside this interval there is no point of A_i, except P_{ij}; one or both of the end-points of this interval may be points of A_i or limiting points of A_i.

The law of intervals is now that, round each point P_{ij} as centre, we describe an interval of length d_{ij} for the first set of intervals, and $e^n d_{ij}$ for the n-th set of intervals, where e is any assigned small positive quantity less than 1.

The law of intervals for points of U (if it exists) may be any we please.

Now, if there be any point of the inner limiting set not belonging to E, it is, by Theorem 1, a point of the first derived set E'. Let then Q be any point of E' which is not a point of E.

If Q be not a limiting point of E_a, it cannot be a limiting point of any adherence, whence, considering all the stages successively, it follows that Q must be a limiting point of U. Assuming then that Q is not a limiting point of U, it must be a limiting point of E_a, and may be a limiting point of some adherent subsequent to E_a in the natural order. Let A_i be any adherence of which Q is a limiting point; then Q is a limiting point of every adherence A_k preceding A_i in the natural order. Hence, by the construction, Q is external to all the intervals d_{ij} whose centres belong to A_i, or to any adherence preceding A_i in the natural order.

Thus, if Q be a limit for every A_i, Q cannot be a point of the inner limiting set without remaining always interior to intervals described round points of U, which it cannot do, since, by hypothesis, Q was not a limiting point of U. Such a Q could not then belong to the inner limiting set.

We must then be able to assign an integer h, such that Q is a limiting point of A_h, but not of any adherence subsequent to A_h in the natural order. Let A_i denote the adherence next after A_h in the natural order. Then we can assign a definite largest interval, of length say d, with Q as centre, such that inside this interval there is no point of A_i and therefore (since any point of an adherence subsequent to A_i is a limiting point of A_i) no point of any adherence subsequent to A_i in the natural order.

Let us now determine an integer m, so that $e^m l < d$; then, for all values of i and j, $e^m d_{ij} < d$; therefore Q is external to all the intervals $e^m d_{ij}$ of the m-th set of intervals whose centres P_{ij} belong to adherences subsequent to A_h in the natural order. Also, since Q was shown to be external to all the remaining intervals of the m-th set, it follows that, if Q be a point of the inner limiting set, it must, from and after the m-th set, be interior to intervals described round points of U, which again is contrary to the hypothesis that Q was not a limiting point of U. Thus it is shown that no point Q belonging to E', but not to the first derived U' of U, can be a point of the inner limiting set of the intervals as we have constructed them. Q. E. D.

This is the proof of Theorem 6, and, by supposing U to be non-existent, of Theorem 5 also.

ON SPHERICAL CURVES

By HAROLD HILTON.

[Received September 25th, 1903.—Read November 12th, 1903.]

1. Since any curve on a sphere can be stereographically projected (from a point on the sphere on to the plane through the centre parallel to the tangent plane at that point) into a plane curve, we see that we can develop the theory of spherical curves just as we develop the theory of plane curves. From the properties of plane curves we deduce properties of spherical curves, and from the properties of spherical curves we deduce properties of plane curves. Darboux has discussed the properties of spherical curves of the fourth degree (see § 8), and has shown how to deduce properties of plane bicircular quartics (*Mém. Soc. Sci. phys. et nat. de Bordeaux*, Vol. VIII., 1870, pp. 316 *et seq.*); while Clifford (*Collected Papers*, p. 343) suggests the application of the same method to tricircular sextics. We propose to extend their results in a systematic manner.

2. We shall take the equation of the sphere (of radius c) on which the curve lies as $x^2+y^2+z^2 = c^2$. If O' is the vertex of projection, O the diametrically opposite point, and o the centre of the sphere, and A and B any points on the sphere, then $oa = c \tan \frac{1}{2}OA$,

$$ab = c \sin \tfrac{1}{2}AB \sec \tfrac{1}{2}OA \sec \tfrac{1}{2}OB = c \sin \tfrac{1}{2}AB \operatorname{cosec} \tfrac{1}{2}O'A \operatorname{cosec} \tfrac{1}{2}O'B,$$

$$\sin^2 \tfrac{1}{2}AB = \frac{ab \,.\, a'b'}{aa' \,.\, bb'}, \qquad \cos^2 \tfrac{1}{2}AB = \frac{ab' \,.\, a'b}{aa' \,.\, bb'},$$

where a, b, a', b' are the projections of A, B and the points diametrically opposite to A and B. By "projection" we shall mean stereographic projection unless otherwise stated. If (ρ, θ) are the polar coordinates of a point on the sphere, O being the origin of coordinates, and (r, θ) the polar coordinates of its projection, then $r = m \tan \frac{1}{2}\rho$.

3. Let the polar coordinates of a point P on the sphere be (ρ, θ) when O is taken as origin, and (ρ_1, θ_1) when O_1 is taken, the great circle OO_1 being the prime meridian in each case. Let the arc $OO_1 = \alpha$. Then in the spherical triangle OPO_1

$$OP = \rho, \quad O_1P = \rho_1, \quad PO_1O = \theta_1, \quad POO_1 = \pi-\theta, \quad OO_1 = \alpha.$$

Hence we can express ρ and θ in terms of ρ_1 and θ_1. If the polar coordinates of the projections of P from the points diametrically opposite to O and O_1 are (r, θ), (r_1, θ_1) respectively, then

$$r = c\left\{\frac{c^2\sin^2\tfrac{1}{2}a + r_1^2\cos^2\tfrac{1}{2}a - 2cr_1\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a\cos\theta_1}{c^2\cos^2\tfrac{1}{2}a + r_1^2\sin^2\tfrac{1}{2}a + 2cr_1\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a\cos\theta_1}\right\}^{\frac{1}{2}},$$

$$\theta = \cot^{-1}\left\{\frac{2cr_1\cos\theta_1\cos a + (r_1^2 - c^2)\sin a}{2mr_1\sin\theta_1}\right\}.$$

4. For example: since the equation of the projection of the rhumb-lines on the Earth from the North or South Pole is a family of equi-angular spirals $r = be^{\theta\cot\beta}$, therefore the equation of their projection from a point on the Equator is

$$a + \log(c^2 + r^2 - 2cr\cos\theta)$$
$$= \log(c^2 + r^2 + 2cr\cos\theta) + 2\cot\beta\cot^{-1}\{(r^2 - c^2) \div 2cr\sin\theta\}.$$

This is the family of curves cutting the coaxial system of circles through $(\pm c, 0)$ at an angle β, and is therefore the integral of the equation

$$\frac{r\,d\theta}{dr} = \frac{c^2\sin(\beta - \theta) - r^2\sin(\beta + \theta)}{c^2\cos(\beta - \theta) - r^2\cos(\beta + \theta)}.$$

The projection of the rhumb-lines from any other point on the Earth is of the same form, if $2c$ denotes the distance between the projections of the Poles.

5. If we project any curve on the sphere on to two different planes h and k, intersecting in the line l, and rotate the plane k about l until it coincides with h, then one of the projections is derived from the other by inverting with respect to a circle (whose centre is the projection on h of the pole of k) and then reflecting in the line l. This follows from the equations of § 3, or otherwise.

6. It is now evident that, if the projection of a spherical curve from one point of the sphere is an algebraic curve, so is its projection from any other point, and conversely. Spherical curves whose projections are algebraic will be called "algebraic"; we shall devote this paper (except § 22) to the discussion of such curves. The intersection of the sphere with any algebraic surface is such a curve; for the stereographic projection from the point $(0, 0, -c)$ of the intersection of $x^2+y^2+z^2 = c^2$ and $\phi(x, y, z) = 0$ is $0 = z = \phi\left(\frac{2c^2x}{x^2+y^2+c^2}, \frac{2c^2y}{x^2+y^2+c^2}, \frac{c(c^2-x^2-y^2)}{x^2+y^2+c^2}\right)$, which is an alge-

braic plane curve of the $2k$-th degree, if $\phi(x, y, z) = 0$ is a rational algebraic surface of the k-th degree.

7. Let the projection of an algebraic spherical curve from some point be

$$f(x, y) \equiv v_0(x^2+y^2)^p+v_1(x^2+y^2)^{p-1}+v_2(x^2+y^2)^{p-2}+\ldots$$
$$+v_{p-1}(x^2+y^2)+u_p+u_{p-1}+\ldots+u_2+u_1+u_0 = 0, \quad \text{(i.)}$$

where v_k and u_k are homogeneous of the k-th degree. (Any of the quantities $v_0, v_1, v_2, \ldots, u_0, u_1, u_2, \ldots$ may be identically zero—so that (i.) can represent *any* plane algebraic curve—though this is not the case in general.)

Since the inverse of (i.) with respect to any circle in its plane and its reflexion in any straight line in its plane are curves of the same type, we see that the projection of the spherical curve from any other point is of the same type in general, *i.e.*, is a $2p$-ic (a curve of the $2p$-th degree) with a p-ple point (a multiple point with p branches) at each circule. If we project from a k-ple point on the spherical curve, the projection is a $(2p-k)$-ic with a $(p-k)$-ple point at each circule, and k asymptotes not passing through the circules (proved by § 5).

8. The spherical curve which projects in general into a $2p$-ic of the type (i.) is said to be of the $2p$-th degree; we shall write n for $2p$. The curve is cut by any circle on the sphere in n points.

9. A multiple point on the spherical curve evidently projects in general into a similar multiple point on the plane curve; therefore the number of nodes and cusps (δ, κ) of the spherical curve is equal to the number of finite nodes and cusps possessed by any projection. The maximum number of double points which the spherical curve can have is

$$\tfrac{1}{2}[(n-1)(n-2)]-p(p-1) = \tfrac{1}{4}(n-2)^2;$$

its " deficiency " is $\frac{1}{4}(n-2)^2-(\delta+\kappa)$.

10. The number of points on the sphere through which an n-ic with δ nodes and κ cusps can be drawn is

$$\tfrac{1}{2}[n(n+3)]-p(p+1)-\delta-2\kappa = \tfrac{1}{4}[n(n+4)]-\delta-2\kappa.$$

If $\frac{1}{2}[u(n-u+3)]$ of the points lie on a u-ic, the n-ic breaks up into a u-ic and an $(n-u)$-ic.

11. The number (m) of circles which pass through two fixed points on the sphere and touch the curve is equal to the number of tangents which

pass through a point and touch a curve of the type (i.). (Project from either of the fixed points.) The number of tangent circles through the two fixed points is therefore $\frac{1}{2}n^2-2\delta-3\kappa$.

12. The number of small circles which pass through two fixed points on the sphere and cut the curve orthogonally is also m. (Proof as in § 11.)

13. The number (ι) of circles of curvature of a spherical curve which cut a given circle orthogonally is $\frac{1}{2}[3n(n-2)]-6\delta-8\kappa$. For the points of contact of circles of curvature of (i.) which cut the circle

$$(x-X)^2+(y-Y)^2 = d^2 \qquad \text{(ii.)}$$

orthogonally lie on the curve

$$(AM^2-2HLM+BL^2)[(x-X)^2+(y-Y)^2-d^2]$$
$$= 2(L^2+M^2)[(x-X)L+(y-Y)M],$$

where $L = \frac{\partial f}{\partial x}, \quad M = \frac{\partial f}{\partial y}, \quad A = \frac{\partial^2 f}{\partial x^2}, \quad H = \frac{\partial^2 f}{\partial x \partial y}, \quad B = \frac{\partial^2 f}{\partial y^2}.$ (iii.)

(iii.) is a $(3n-2)$-ic cutting (i.) six times at each finite double point, eight times at each cusp, $p(3p+1)$ times at each circule, and $\frac{1}{2}[3n(n-2)]-6\delta-8\kappa$ times at other finite points.

14. There are some special cases of interest. If $d = 0$ and (X, Y) lies on (iii.), (i.) and (iii.) have three-point contact at (X, Y), and the number of circles of curvature passing through that point is

$$\tfrac{1}{2}[3(n^2-2n-2)]-6\delta-8\kappa.$$

If (ii.) cuts (i.) orthogonally at P, (i.) and (iii.) touch at P. If, in addition, the circle of curvature of (i.) at P has four-point contact, (i.) and (iii.) have three-point contact at P.

15. The number (τ) of circles touching a spherical curve at two distinct points and cutting a given circle orthogonally is given by

$$n+2\tau+3\iota = m(m-1).$$

For consider the locus of the centres of circles cutting a given circle (ii.) orthogonally and touching (i.). To each of its nodes corresponds a circle touching (i.) twice and cutting (ii.) orthogonally; to each of its cusps corresponds a circle of curvature of (i.) cutting (ii.) orthogonally. The locus has therefore τ nodes and ι cusps. To each point in which the locus cuts the line at infinity corresponds a tangent to (i.) from the centre of (ii.); therefore the degree of the locus is m. The only normals to the locus from the centre of (ii.) are evidently the normals from this centre to (i.) and the radii joining this centre to the n finite intersections of (i.) and (ii.). Hence $m^2-2\tau-3\iota = n+m$ (cf. § 12).

16. There are some special cases of interest. If (ii.) cuts (i.) orthogonally in k points, the locus breaks up into an $(m-k)$-ic and the k tangents to (ii.) at the points. Some of the bitangent circles cutting (ii.) orthogonally are derived from the intersections of the k tangents with the $(m-k)$-ic, some from the double points of the $(m-k)$-ic, and some from the mutual intersections of the k tangents.

17. If we denote by n, m, δ, κ, τ, ι the degree of the spherical curve, the number of great circles which can be drawn through any point to touch it, the number of nodes, the number of cusps, the number of great circles touching it in two points, the number of great circles of curvature, then we have

$$m = \tfrac{1}{2}n^2-2\delta-3\kappa, \qquad n = m(m-1)-2\tau-3\iota,$$

$$\iota = \tfrac{1}{2}[3n(n-2)]-6\delta-8\kappa, \qquad \kappa = 3m(m-2)-6\tau-8\iota,$$

$$\text{degree deficiency} = \tfrac{1}{4}(n-2)^2-(\delta+\kappa) = \tfrac{1}{2}[(m-1)(m-2)]-(\tau+\iota)$$

$$= \text{class deficiency}.$$

We obtain these equations (corresponding to Plücker's equations for plane curves) from §§ 9–16, noticing that all great circles are orthogonal to a fixed circle, namely, the circle at infinity.

18. The foci of a spherical curve are the intersections of those imaginary generating lines of the sphere which touch the curve. In many ways it may be shown that their projections are the foci of the projection of the spherical curve. Hence they are $(m-n)^2$ in number, $m-n$ being *real*.

19. If a spherical curve is projected from a real focus, the projection has only $m-n-1$ real finite foci, but two of the tangents to the projection at each circule coincide.

20. If a spherical curve has a real double focus (the intersection of two generating lines each touching the curve twice), so has every projection, except the projection from the double focus, which has only $m-n-2$ real finite foci, but has two pairs of tangents coincident at each circule.

21. The evolute of a curve is the locus of the poles of the circles of curvature. The length of any arc of the evolute is in general equal to the difference between the great circle normals drawn from its extremities. The degree of the evolute is 2ι, its class is m (cf. §§ 13 and 12). Every focus of a spherical curve is a focus of its evolute. For (since a generating line of a sphere is perpendicular to itself) a generating line touching a spherical curve makes up with the parallel generating line a great circle normal to the curve; and therefore a generating line which touches a curve touches its evolute.

22. If (ρ, θ) are the polar coordinates of any point P on a spherical curve, the origin being O, then the angle between the curve and the great circle PO is $\tan^{-1}\left(\sin\rho\,\frac{d\theta}{d\rho}\right)$. The locus of the intersection of a circle touching the curve at P and passing through the point diametrically opposite to O, with the orthogonal great circle through O, may be called the "first pedal of the curve with respect to O." A curve and its first pedal are equally inclined to the great circles joining O to corresponding points. Other properties of first pedals, &c., can be readily deduced.

23. Two points A, B on a sphere are said to be inverse with respect to the circle μ in which the polar plane of any point on the line AB cuts the sphere. Their projections are readily seen to be inverse with respect to the projection of μ. (In general, if the sphere is inverted from *any* point, the points inverse to A and B are inverse with respect to the inverse of μ.)

24. The simplest algebraic curve on a sphere is a 2-ic, *i.e.*, a circle. Very many well-known properties of circles on a sphere can be proved by projecting on to a plane. The circles in which the polar planes of all points on a line l cut the sphere form a coaxial system. The points A, B in which l cuts the sphere project into the limiting points of the projection of the system. All small circles through A and B cut every circle of the coaxial system orthogonally; if H is any point on a circle touching two circles of the system at K and M, the circles HKA, HMA are orthogonal, &c.

25. Algebraic curves of the fourth degree on a sphere have very many interesting properties. First consider curves with deficiency zero. If the curve has a cusp, it has only one focus, and that *real*. The projection from the cusp is a parabola. Hence the points of contact of two ortho-

gonal circles which touch the curve and pass through the cusp lie on a circle through the cusp and focus; if three circles are drawn touching the curve and passing through the cusp, a circle can be drawn through the focus and their other three intersections, &c. The curve is "anallagmatic" (*i.e.*, its own inverse) with respect to only one circle, namely, the circle touching the curve at the cusp and passing through the focus. (For the parabola is its own inverse with respect to the axis considered as the limit of a circle of infinite radius.)

26. If the curve has a double point, it has two real and two imaginary foci. The projection from the double point is a central conic. Hence, if S, H are the foci, N the double point, and P any point on the curve, the circles SPN, HPN are equally inclined to the tangent at P. Only two 4-ics can be described through P with double points at N, and S, H as foci; and those two cut orthogonally. Again,

$$\sin \tfrac{1}{2}PS \operatorname{cosec} \tfrac{1}{2}NS \pm \sin \tfrac{1}{2}PH \operatorname{cosec} \tfrac{1}{2}NH = d \sin \tfrac{1}{2}NP,$$

where d is a constant (see § 2), &c. The curve is anallagmatic with respect to two circles, one through the real and the other through the imaginary foci, passing through the double point and bisecting the angles between the tangents at that point. (For the conic is its own inverse with respect to either principal axis.)

27. Now consider a 4-ic with no double point. Its projection is a bicircular 4-ic of the eighth class which is anallagmatic with respect to four mutually orthogonal circles.* Hence every spherical 4-ic with no double point is anallagmatic with respect to four mutually orthogonal circles on the sphere; and therefore lies on four cones of the second degree, whose vertices are the corners of a self-conjugate tetrahedron. Making use of this fact, many properties of such curves can be deduced. (Cf. Darboux, *loc. cit.*, p. 324; see also § 32.)

28. Many theorems can be obtained for spherical curves from known properties of plane curves. For example:

> If a spherical 6-ic with a triple point N has three real circles of curvature passing through N, then the three points of contact lie on a circle through N, &c.

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* The equation of a circular 3-ic referred to one of the four points of contact of tangents parallel to the asymptote as origin shows that the 3-ic is anallagmatic with respect to four circles whose centres are these four points. It follows at once that these circles are mutually orthogonal, and that each meets the four tangents from a circule to the 3-ic in four foci of the 3-ic. Inverting with respect to any point, we see that a bicircular 4-ic has similar properties.

29. Some striking theorems can be proved for n-ics which are anallagmatic with respect to a circle j. Such curves lie on a cone whose vertex is the pole of the plane of j, and cut j orthogonally at n points in general. For such curves we must modify the results of §§ 11–16. Let the cone be of degree p and class m', and have δ' nodal lines, κ' cuspidal lines, ι' inflexional tangent planes; then in general $n = 2p$, $\delta = 2\delta'$, $\kappa = 2\kappa'$.

Let H, H' be the poles of j. Then through H, H' can be drawn n circles touching the curve at its intersections with j, and m' circles touching the curve at *two* points each. We shall count each of these latter circles as *two* tangent circles through H and H'; so that the total number of tangent circles through H and H' is

$$n+2m' = n+2\{p(p-1)-2\delta'-3\kappa'\} = \tfrac{1}{2}n^2-2\delta-3\kappa,$$

which is consistent with § 17.

Again, we may suppose *three* circles of curvature cutting j orthogonally at each intersection of the curve and j (§ 14); also there are ι' circles having three-point contact with the curve at two distinct points which cut j orthogonally. We count each of the latter as *two* circles of curvature cutting j orthogonally; so that the total number of circles of curvature which cut j orthogonally is

$$3n+2\iota' = 3n+2\{3p(p-2)-6\delta'-8\kappa'\} = \tfrac{1}{2}[3n(n-2)]-6\delta-8\kappa;$$

which is consistent with § 17.

The bitangent circles cutting j orthogonally may be discussed similarly.

If the cone touches the sphere in δ_1 points, the circle j passes in general through δ_1 nodes of the intersection of the cone and the sphere, and cuts the intersection orthogonally in $n-2\delta_1$ other points. Then $\delta = \delta_1+2\delta'$, $\kappa = 2\kappa'$. The number of tangent circles through H and H' is

$$(n-2\delta_1)+2\{p(p-1)-2\delta'-3\kappa'\} = \tfrac{1}{2}n^2-2\delta-3\kappa,$$

and the number of circles of curvature cutting j orthogonally is

$$3(n-2\delta_1)+2\{3p(p-2)-6\delta'-8\kappa'\} = \tfrac{1}{2}[3n(n-2)]-6\delta-8\kappa,$$

as before. δ_1 is $\not> p$, for otherwise j would meet the intersection in more than n points; if the cone has its maximum number $\tfrac{1}{2}[(p-1)(p-2)]$ of double lines and $\delta_1 = p$, the intersection breaks up; for its deficiency is $\tfrac{1}{4}(n-2)^2-p-(p-1)(p-2) = -1$: for example, the intersection of a sphere with a doubly tangent cone of the second degree is a pair of circles.

We shall discuss only two cases: (1) when the curve lies on a real cone whose vertex lies outside the sphere, (2) when it lies on a real cone whose vertex lies inside the sphere. Certain properties are common to both cases, for instance:—

> All spherical n-ics which lie on a cone whose vertex is V and pass through $\frac{1}{8}(n^2+6n-8)$ fixed points no two of which are collinear with V pass through $\frac{1}{4}[(n-2)(n-4)]$ other fixed points.

Hence also, by projection:—

> All plane n-ics which are anallagmatic with respect to the same circle and pass through $\frac{1}{8}(n^2+6n-8)$ fixed points no two of which are inverse with respect to the circle pass through $\frac{1}{4}[(n-2)(n-4)]$ other fixed points.

30. In case (1) the curve is anallagmatic with respect to a real circle on which lie $\frac{1}{2}[n(n-2)]-2\delta-3\kappa$ foci (all of which *may* be real), which are the points of contact with the sphere of tangent planes common to the cone and the sphere (excluding the points, if any, at which the cone and sphere touch). Projecting on to a plane, we have a plane n-ic anallagmatic with respect to a real circle; and, projecting back on to a suitable sphere, we have an n-ic spherical curve anallagmatic with respect to a great circle, *i.e.*, symmetrical about a plane and the intersection of a sphere with a cylinder. Therefore, to find properties of spherical curves which are the intersections of a sphere and a cone whose vertex lies outside the sphere, it is sufficient to investigate the properties of the intersection of a sphere and a cylinder. For instance, we know that four points can be found such that the products of the perpendiculars from three of them on any tangent to a plane curve of the third class bears a constant ratio to the perpendicular from the fourth point on the tangent. Hence:—

> If we have a spherical 6-ic with two cusps and anallagmatic with respect to a great circle j, then four points A, B, C, P can be found on the sphere such that $UA.UB.UC \div UP$, and therefore $\sin\frac{1}{2}UA\,\sin\frac{1}{2}UB\,\sin\frac{1}{2}UC\,\operatorname{cosec}\frac{1}{2}UP$, is the same for each of the six foci U which lie on j.

Projecting on to a plane from the pole of j, we have:—

> Any plane 6-ic with two cusps and anallagmatic with respect to a real circle j is such that points A, B, C, P can be found for which $\dfrac{UA.UB.UC.UA_1.UB_1.UC_1}{UP.UP_1}$ is the same for each of the six foci lying on j, where A and A_1, B and B_1, C and C_1, P and P_1 are inverse points with respect to j (cf. § 2).

Again, projecting back on to any other sphere from a point T on the sphere, we get a spherical 6-ic with two cusps and anallagmatic with respect to a real circle j, and see that:—

> Points A, B, C, P can be found such that
>
> $$\sin \tfrac{1}{2}UA \sin \tfrac{1}{2}UB \sin \tfrac{1}{2}UC \sin \tfrac{1}{2}UA_1 \sin \tfrac{1}{2}UB_1 \sin \tfrac{1}{2}UC_1 \times \operatorname{cosec} \tfrac{1}{2}UP \operatorname{cosec} \tfrac{1}{2}UP_1 \operatorname{cosec}^4 \tfrac{1}{2}TU$$
>
> is the same for each focus U on the circle j, where A and A_1, B and B_1, C and C_1, P and P_1 are inverse points with respect to the circle j and T is any point on the sphere.

31. If the square roots of the perpendiculars from any point on a sphere on any number of tangent planes are connected by a linear relation, so are the distances of the point from the points of contact. Again, if the distances of a point P from any points $A, B, C, \ldots$ are connected by a homogeneous linear relation, so are the distances of the inverse of P from the inverses of $A, B, C, \ldots$. By the use of the first of these principles many theorems can be proved for the intersection of a sphere and a cylinder, and then by the use of the second principle theorems can be deduced for any plane or spherical curve anallagmatic with respect to a real circle. Thus, since the square roots of the perpendiculars from any point of a cylinder of the second degree on any three tangent planes are connected by a homogeneous linear relation, therefore the distances of any point of a bicircular quartic from three real foci lying on a circle with respect to which the curve is anallagmatic are connected by a homogeneous linear relation (Darboux, *loc. cit.*, p. 332). Again, since the square roots of the perpendiculars from any point of a plane 4-ic with three nodes on four real double tangents are connected by a homogeneous linear relation, the distances of any point on a plane or spherical 8-ic with six double points from four real double foci lying on a circle j, with respect to which the curve is anallagmatic, are connected by a homogeneous linear relation.*

Similarly, if a plane (spherical) 8-ic has the six intersections of the lines joining the circules to (generating lines through) three real points A, B, C as imaginary cusps, and is anallagmatic with respect to the circle ABC, then the reciprocals of the distances of any point on the 8-ic from A, B, C are connected by a homogeneous linear relation.

32. In case (2) we see, by reasoning similar to that at the beginning of

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* [There are six circles orthogonal to j, each osculating the 8-ic at two distinct points, and the twelve points of osculation lie on a 4-ic anallagmatic with respect to j.—*November* 16*th*.]

§ 30, that we may obtain properties of all intersections of a sphere with a real cone of the p-th degree whose vertex lies inside the sphere by considering properties of the intersection of the sphere and a cone of the p-th degree whose vertex is the centre of the sphere. Such an intersection will be called a "sphero-p-ic." The equation of the cone may be put in the shape $(x^2+y^2+z^2)u_{p-2}+(lx+my+nz)u_{p-1}=0$, where u_{p-1}, u_{p-2} are homogeneous quantics in x, y, z of the $(p-1)$-th and $(p-2)$-th degree respectively; $lx+my+nz=0$ will be called an "asymptotic plane." Any plane parallel to an asymptotic plane cuts the cone in a curve passing through the circules, as is easily seen by transforming the axes of reference, so that $z=0$ is an asymptotic plane. A line perpendicular to an asymptotic plane of the reciprocal cone (which touches the planes through the vertex perpendicular to the generating lines of the given cone) will be called a "focal line" of the given cone. The sections of a cone and the reciprocal cone by any plane are polar reciprocals of each other with respect to a circle whose centre is the foot of the perpendicular from the common vertex on the plane; and therefore a focal line cuts a plane perpendicular to it in a focus of the section of the plane and cone. Now any tangent plane q to a sphere at a point Q cuts the sphere in two generating lines which meet the sphero-p-ic and the section of the cone by q in the same points, and which pass through the circules in q. Hence, if Q is a focus of the sphero-curve, it is a focus of the section of the cone by q; and therefore the diameter through Q is a focal line of the cone, and *vice versa*.

Every sphero-p-ic has $2k$ real foci, $2k$ imaginary foci on the circle at infinity, and $4k(k-1)$ other imaginary foci, where

$$2k \equiv \tfrac{1}{2}[n(n-2)] - 2\delta - 3\kappa.$$

Therefore every cone of the p-th degree with δ' double lines and κ' cuspidal lines has k real focal lines and $2k(k-1)$ imaginary focal lines, where $k \equiv p(p-1)-2\delta'-3\kappa'$. The degree of the reciprocal cone is, however, k; hence we see that every real cone of degree p has in general p real asymptotic planes and $2p(p-1)$ imaginary asymptotic planes.

It readily follows that the equation of any cone of degree p can be put in the form

$$\begin{aligned} 0 = \alpha_1\alpha_2\alpha_3 \dots \alpha_p + a_1(x^2+y^2+z^2)\,\beta_1\beta_2 \dots \beta_{p-2} \\ + a_2(x^2+y^2+z^2)^2\,\gamma_1\gamma_2 \dots \gamma_{p-4} + \dots, \quad \text{(iv.)} \end{aligned}$$

where the a's are $\tfrac{1}{4}[2p-1+(-1)^p]$ real constants, and the α's, β's, γ's, ... are real linear expressions of the form $(ax+by+cz)/\sqrt{(a^2+b^2+c^2)}$.

The planes $\alpha_1 = \alpha_2 = \dots, = \alpha_p = 0$ are the real asymptotic planes, the planes $\beta_1 = \beta_2 = \ldots = \beta_{p-2} = 0$, $\gamma_1 = \gamma_2 = \ldots \gamma_{p-4} = 0$, ... will be called the "guiding-planes of the first, second, ..., orders." They meet the sphere in the "asymptotic arcs" and the "guiding arcs of the first, second, ..., orders." The perpendiculars to the asymptotic planes of the reciprocal cone meet the sphere in the foci, and the perpendiculars to the guiding planes of the reciprocal cone meet the sphere in the "ruling points of the first, second, ..., orders."

The geometrical interpretation of (iv.) is :—

> The products of the sines of the perpendiculars from any point of a sphero-p-ic on the asymptotic arcs and guiding arcs of the first, second, ..., orders are connected by a linear relation ;

and reciprocally :—

> The products of the sines of the perpendiculars on any tangent to a sphero-p-ic from the real foci and the ruling points of the first, second, ..., orders are connected by a linear relation.

33. From equation (iv.) or from properties of a plane p-ic many properties of a sphero-p-ic can be deduced; by projection properties of a plane n-ic which is its own inverse with respect to a circle whose centre is real and whose radius is purely imaginary can be deduced. Projecting back on to another sphere, we have properties of the curve of intersection of a sphere with any real cone of the p-th degree whose vertex lies inside the sphere. For example, we obtain readily from (iv.) the theorem :—

> If any great circle cuts a sphero-p-ic in the p points $Q, R, S, \ldots$ and the diametrically opposite points, and cuts the asymptotic arcs and the guiding arcs of the first, second, ... orders in $A_1, A_2, \ldots, A_p$; $B_1, B_2, \ldots, B_{p-2}$; $C_1, C_2, \ldots, C_{p-4}$; ... and the diametrically opposite points, then every determinant of the array

$$\left\Vert \begin{array}{cccc} \sin QA_1 \sin QA_2 \ldots \sin QA_p, & \sin RA_1 \sin RA_2 \ldots \sin RA_p, & \sin SA_1 \sin SA_2 \ldots \sin SA_p, & \ldots \\ \sin QB_1 \sin QB_2 \ldots \sin QB_{p-2}, & \sin RB_1 \sin RB_2 \ldots \sin RB_{p-2}, & \sin SB_1 \sin SB_2 \ldots \sin SB_{p-2}, & \ldots \\ \sin QC_1 \sin QC_2 \ldots \sin QC_{p-4}, & \sin RC_1 \sin RC_2 \ldots \sin RC_{p-4}, & \sin SC_1 \sin SC_2 \ldots \sin SC_{p-4}, & \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \end{array} \right\Vert$$

> must vanish.

Reciprocally :—

> If $a_1, a_2, \ldots, a_p$; $b_1, b_2, \ldots, b_{p-2}$; $c_1, c_2, \ldots, c_{p-4}$; ... are the great circles joining any point to the foci and the ruling points of the first, second, ..., orders of a sphero-curve with $2p$ real foci, and if

$q, r, s, \ldots$ are the p great circles passing through the point and touching the curve, then every determinant of the array

$$\left\| \begin{array}{llll} \sin(qa_1)\sin(qa_2)\ldots\sin(qa_p), & \sin(ra_1)\sin(ra_2)\ldots\sin(ra_p), & \sin(sa_1)\sin(sa_2)\ldots\sin(sa_p), & \ldots \\ \sin(qb_1)\sin(qb_2)\ldots\sin(qb_{p-2}), & \sin(rb_1)\sin(rb_2)\ldots\sin(rb_{p-2}), & \sin(sb_1)\sin(sb_2)\ldots\sin(sb_{p-2}), & \ldots \\ \sin(qc_1)\sin(qc_2)\ldots\sin(qc_{p-4}), & \sin(rc_1)\sin(rc_2)\ldots\sin(rc_{p-4}), & \sin(sc_1)\sin(sc_2)\ldots\sin(sc_{p-4}), & \ldots \\ \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots\quad\ldots\quad\ldots\quad\ldots & \ldots \end{array} \right\|$$

must vanish, where (qa_1) denote the angle between the arcs q and a_1, &c.

Hence:—

If a plane curve with $2p$ real foci is anallagmatic with respect to an imaginary circle j, and if $q, r, s, \ldots$ denote the circles through any point P touching the curve and orthogonal to j, and $a_1, a_2, \ldots, a_p$; $b_1, b_2, \ldots, b_{p-2}$; $c_1, c_2, \ldots, c_{p-4}$; $\ldots$ the circles joining P to the foci and certain other fixed points and orthogonal to j, then every determinant of the array last written must vanish.

Finally, projecting back on to another sphere, we have:—

If the intersection of a sphere with a cone whose vertex V lies inside the sphere has $2p$ real foci, and if $q, r, s, \ldots$ denote the circles touching the curve whose planes pass through P and V, and if $a_1, a_2, \ldots, a_p$ denote the circles whose planes pass through P, V, and the foci, and if $b_1, b_2, \ldots, b_{p-2}$; $c_1, c_2, \ldots, c_{p-4}$; $\ldots$ denote the circles whose planes pass through P, V, and certain other fixed points, then every determinant of the array last written vanishes.

34. The projection of a sphero-conic is a bicircular 4-ic (of which a circular 3-ic or Cartesian oval is a particular case) which has its four real foci concyclic; and, conversely, any bicircular 4-ic with four real concyclic foci is the projection of a sphero-conic. A very large number of theorems concerning such 4-ics can be proved by known properties of sphero-conics (as suggested by Darboux), and by plane projection properties of 3-ics and binodal 4-ics can be obtained. Projecting back the bicircular 4-ic on to another sphere, we have properties of any spherical 4-ic whose four real foci are concyclic.

Thus, since a tangent to a sphero-conic makes equal angles with the radii vectores drawn from the foci to the point of contact, therefore:—

If P is any point on a bicircular 4-ic whose four real foci S, S', H, H' lie on a circle, and are such that the straight lines SS', HH'

meet *inside* the circle, then the circles SPS', HPH' make equal angles with the tangent at P.

Again :—

If P is any point on a spherical 4-ic whose four real foci S, S', H, H' lie on a circle, and are such that the lines SS', HH' meet *inside* the circle, then the circles SPS', HPH' make equal angles with the tangent at P.

Since two orthogonal confocal sphero-conics can be drawn through any point, therefore :—

Two bicircular 4-ics can be drawn through any point with the same four real concyclic foci, and they cut each other orthogonally.

Again :—

Two spherical 4-ics can be drawn through any point with the same four real concyclic foci, and they cut each other orthogonally.

Again :—

Two plane 4-ics can be drawn through any point with the same two imaginary nodes, which are the double points of a real involution I lying on a real line l, and touching the imaginary lines joining the nodes to four real points on a conic which has I as an involution of conjugate points ; the tangents at the points to the two 4-ics cut l in a pair of points of the involution I.

35. The equation of a cone of the third degree may be put in the form $0 = \alpha\beta\gamma + (x^2+y^2+z^2)\mu$, where $\alpha = \beta = \gamma = 0$ are the asymptotic planes and $\mu = 0$ is the guiding plane (cf. § 32). Hence we see that the asymptotic arcs of a sphero-3-ic each meet the curve in two real diametrically opposite points, and that these six points lie on the guiding arc. Reciprocally, three real great circle tangents can be drawn from the real foci to a sphero-curve with six real foci, and these three great circles are coaxial. Again, if any great circle meets the asymptotic arcs of a sphero-3-ic in A, B, C, the guiding arc in M, and the curve in P, Q, R (no two of these points being diametrically opposite), then

$$\frac{\sin AP \sin BP \sin CP}{\sin MP} = \frac{\sin AQ \sin BQ \sin CQ}{\sin MQ} = \frac{\sin AR \sin BR \sin CR}{\sin MR}$$

(a particular case of the first theorem of § 33).

If R coincides with C and M, we have $\sin AP \sin BP = \sin AQ \sin BQ$, and therefore $AP = BQ$. Hence :—

The tangent from the real intersection of a sphero-3-ic and an asymptotic arc is divided into two equal parts by its point of contact and its intersection with the other asymptotic arcs.

Reciprocally:—

If the tangent from a focus to a sphero-curve with six real foci cuts the curve at a point P, the tangent at P is equally inclined to the two other arcs joining P to the real foci.

Hence:—

If a plane curve is its own inverse with respect to an imaginary circle whose centre is O, and has six real foci S, S', H, H', F, F', such that O is collinear with S and S', H and H', F and F', then the real circles which touch the curve and pass through S and S', H and H', F and F' are coaxial. If the circle through S and S' touching the curve also cuts it in P, then the tangent at P is equally inclined to the circles HPH', FPF'.

Again:—

If the intersection of a sphere and a real cone whose vertex V lies inside the sphere has six real foci collinear in pairs with V, then the circles touching the curve whose planes pass through V and the foci are coaxial. If one of them *cuts* the curve at P, the tangent at P is equally inclined to the two circles whose planes pass through P, V, and the other foci.

Many similar theorems can be obtained, but the above may serve as examples.

36. To many of the theorems proved for the sphere correspond theorems for a conicoid. If we project stereographically (*i.e.*, from a point on the conicoid on to the plane through the centre parallel to the tangent plane at the point*), all conics on the conicoid are projected into conics through the same two points i and j at infinity. If two curves on the conicoid meet at P, the tangents at P form with the generating lines through P a pencil having the same cross-ratio as the pencil formed by the tangents to their projections at p, the projection of P, and the lines pi, pj. The intersections of generating lines which touch a curve on a conicoid are the "foci" of the curve. Two points A, B on the conicoid are "inverse" with respect to the conic in which the polar plane of any point on the line AB cuts the conicoid (cf. § 23).

---

* In the case of the paraboloid we take *any* plane parallel to the tangent plane.

37. To obtain theorems for curves on the conicoid we take a known theorem for the sphere, project it stereographically on to a plane, project it orthogonally on to another plane, and then project it stereographically on to a suitable conicoid. For example, since two circles on a sphere one of which passes through two points inverse with respect to the other cut each other orthogonally, therefore :—

> The tangents at a point of intersection P of two conics on a conicoid one of which passes through two points inverse with respect to the other are parallel to conjugate diameters of the indicatrix at P.

Again, since two orthogonal spherical 4-ics can be drawn through any point with the same four real concyclic foci, therefore :—

> Two curves of the fourth degree can be drawn on an ellipsoid passing through any point P and having the same four real coplanar foci; the tangents at P to the curves are parallel to conjugate diameters of the indicatrix at P.

ADDITION TO THE PAPER* ON THE FOUR KNOWN SIMPLE LINEAR GROUPS OF ORDER 25920

By L. E. DICKSON.

[Read June 11th, 1903.—Received in abbreviated form December, 1903.]

THE simple group $A(4, 3)$ defined by the quaternary Abelian group modulo 3 finds application in various geometrical problems. The question of resolvent equations for the latter is answered by the determination of the sub-groups of the former. To show that $A(4, 3)$ *contains a single set of conjugate sub-groups simply isomorphic with the symmetric group on six letters*, we make use of the isomorphic abstract group G_{720} generated by B_1, B_2, B_3, B_4, B_5, each of period 2, and such that B_1B_2, B_2B_3, B_3B_4, and B_4B_5 are of period 3, while B_1B_3, B_1B_4, B_1B_5, B_2B_4, B_2B_5, and B_3B_5 are of period 2. According to the paper cited, the operators of period 2 of $A(4, 3)$ fall into two distinct sets of conjugates, one set of 45 operators conjugate with $T_{1,-1}$ (and hence with P_{12}), the second set of 270 operators conjugate with M_1M_2. Now $B_1, \ldots, B_5$ are conjugate within G_{720}, and hence are all to be selected from the same set. The case in which they are selected from the first set was treated in the paper cited, pp. 59–65, the resulting sub-groups of type G_{720} being shown to be conjugate within $A(4, 3)$. I have recently verified the calculations,† and undertaken the remaining case in which $B_1, \ldots, B_5$ are selected from the second set. I find that, *within $A(4, 3)$, every sub-group of the symmetric type G_{720} whose generators B_i are conjugate with M_1M_2 is conjugate with the group generated by*

$$B_1 = M_1M_2, \qquad B_2 = \left\{\begin{matrix} 1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & -1 \end{matrix}\right\},$$

---

* *Proc. London Math. Soc.*, Vol. XXXI., pp. 30–68. At the top of p. 65, γ_1 should be replaced by $-\gamma_1$.

† While the theorem stated on p. 65 is correct in view of this note, the proof there given leads only to the theorem obtained by replacing "contain a substitution" by "is generated by substitutions $B_1, \ldots, B_5$."

$$B_3 = \left\{\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{matrix}\right\}, \qquad B_4 = \left\{\begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{matrix}\right\},$$

$$B_5 = \left\{\begin{matrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}\right\}.$$

Now $B_1 B_3 B_5 = T_{2,-1}$ belongs to the first set. Moreover, the isomorphism of G_{720} with the symmetric group on 1, 2, ..., 6 may be established by either of the following sets of correspondences of generators:— $B_1 \sim (12)$, $B_2 \sim (23)$, $B_3 \sim (34)$, $B_4 \sim (45)$, $B_5 \sim (56)$; $B_1 \sim (12)(34)(56)$, $B_2 \sim (13)(45)(26)$, $B_3 \sim (12)(35)(46)$, $B_4 \sim (13)(24)(56)$, $B_5 \sim (12)(36)(45)$. Hence the sub-groups simply isomorphic with G_{720} are conjugate.

A GENERAL THEOREM CONCERNING ABSOLUTELY CONVERGENT SERIES

By G. H. Hardy.

[Received October 9th, 1903.—Read November 12th, 1903.]

1. My object in this paper is to prove a theorem which contains as special cases a number of well known theorems concerning absolutely convergent series. Among these are, for instance, the following theorems:—

> (i.) the terms of an absolutely convergent simple series may be permuted in any manner without altering the sum of the series;
>
> (ii.) an absolutely convergent double series may be rearranged in any manner as a simple series;
>
> (iii.) an absolutely convergent double series may be summed indifferently by rows or by columns (Cauchy's double-series theorem).

It occurred to me recently that all these theorems are very particular cases of a general theorem concerning the rearrangement of series. This theorem not only includes all the known theorems, but is, as I shall show, the most general possible theorem of its kind; and, in my opinion, its introduction lends to the theory a simplicity and generality which is otherwise lacking.

Series of Type β.[*]

2. The ordinary simply infinite series is *of type* ω, that is to say, its "ordinal type" is that of the class of numbers $< \omega$. The sum of two infinite series $a_0+a_1+\ldots+b_0+b_1+\ldots$ is of type $\omega.2$. A doubly infinite series, summed by rows or columns, is of type ω^2; summed by diagonals, it becomes of type ω. As the class of numbers $<\beta$ is enumerable, it is clear that we can arrange any enumerably infinite set of terms in type β.

---

[*] In what follows I shall denote by Greek letters $\alpha, \beta, \gamma, \ldots$ numbers of Cantor's *first and second* classes $0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega.2, \ldots, \omega^2, \ldots, \omega^\omega, \ldots$. When it is necessary to distinguish specially the finite numbers (numbers of the *first* class) I shall use $m, n, p, \ldots$.

Suppose then that we have a series of type β, and, for the present, that its terms $u_0, u_1, \ldots, u_\omega, \ldots, u_{\omega.2}, \ldots, u_{\omega^2}, \ldots, u_{\omega^\omega}, \ldots$ are all $\geqslant 0$. There may or may not be a last term.

Sum of a Series of Type β.

3. Since all the u's are positive,

$$s_n = u_0+u_1+\ldots+u_{n-1} = \sum_{\gamma<n} u_\gamma$$

tends to a limit for $n = \infty$. We call this s_ω or $\sum\limits_{\gamma<\omega} u_\gamma$. Of course it may be that $s_\omega = \infty$. If s_ω is finite, $s_{\omega+n} = s_\omega+u_\omega+\ldots+u_{\omega+n-1}$ has a limit for $n = \infty$, which we call $s_{\omega.2}$. This again may $= \infty$, and, if $s_\omega = \infty$, we regard $s_{\omega.2}$ as also $= \infty$. Thus we may define $s_{\omega.n}$ for any value of n, and it is clear that $s_\omega \leqslant s_{\omega.2} \leqslant s_{\omega.3} \ldots$; so that $\lim\limits_{n=\infty} s_{\omega.n}$ is determinate. We call this s_{ω^2}. It is clear that we may proceed thus and define s_α for all values of $\alpha \leqslant \beta$. We call s_β the sum of the series. If any $s_\alpha = \infty$, we agree that $s_{\alpha'} = \infty$ $(\alpha \leqslant \alpha' \leqslant \beta)$. In this case we say that the series is *divergent*. If s_β is finite, we say that the series is *convergent in type β*, and write $s_\beta = \sum\limits_{\gamma<\beta} u_\gamma$.

The object of this paper is to prove the following

THEOREM.—*If a series is absolutely convergent in type β, it remains absolutely convergent when its terms are rearranged in another type β', and its sums in the two types are the same.*

I may observe that the theorem covers the case of any rearrangement of the series in the *same* type.

Elementary Properties of Series of Type β.

4. The series thus defined possess some of the characteristic properties of ordinary series. Thus it is easy to prove that

$$(1) \quad \sum_{\gamma<\beta} (u_\gamma+v_\gamma) = \sum_{\gamma<\beta} u_\gamma + \sum_{\gamma<\beta} v_\gamma$$

if Σu_γ, Σv_γ are convergent, and

$$(2) \quad \sum_{\gamma<\beta} \kappa u_\gamma = \kappa \sum_{\gamma<\beta} u_\gamma .$$

These propositions follow readily from the equations

$$(a)\quad \sum_{\gamma<\alpha+1} u_\gamma = \sum_{\gamma<\alpha} u_\gamma + u_\alpha$$

and

$$(b)\quad \sum_{\gamma<\alpha} u_\gamma = \lim_{n=\infty} \sum_{\gamma<\alpha_n} u_\gamma \quad (\alpha = \lim \alpha_n)$$

by induction.*

Again,

(3) *If* $\sum_{\gamma<\beta} u_\gamma$ *is convergent, and* $v_\gamma \leqslant u_\gamma$ *for all values of* γ, *then* Σv_γ *is convergent, and its sum* $\leqslant s_\beta$.

The proofs of these theorems are so simple that I need hardly write them out at length. In each case we show by means of the equations (*a*) and (*b*) that (i.), if they hold for series of type α, they hold for series of type $\alpha+1$; and (ii.) that, if they hold for series of types α_1, α_2, ... ($\alpha_1 < \alpha_2 < \dots$, $\lim \alpha_n = \alpha$), they hold for series of type α. As they evidently hold for series containing only one term, they hold generally.

Proof of the Theorem.

5. In order to prove the theorem of § 3 we must establish a series of lemmas.

LEMMA 1.—*If a convergent series of type* β *is arranged in type* ω, *the new series is convergent and its sum* $\leqslant s_\beta$.

For the sum of any finite number of terms of the series $\leqslant s_\beta$. This follows from (3), § 4, if we replace all the other terms of the series by zeros. The lemma follows immediately.

LEMMA 2.—*If a series is convergent in type* β, *we can find a finite number of terms whose sum* $> s_\beta - \delta$, *where* δ *is an arbitrarily small positive quantity ; and, if it is divergent, we can find a finite number of terms whose sum* $> G$, *an arbitrarily large quantity.*

For suppose the first part true for series of type α. Then it is true for series of type $\alpha+1$. For, if $u_{\gamma_1}+u_{\gamma_2}+\dots+u_{\gamma_N} > s_\alpha-\delta$,

$$u_{\gamma_1}+u_{\gamma_2}+\dots+u_{\gamma_N}+u_{\alpha+1} > s_{\alpha+1}-\delta.$$

Again, suppose the theorem true for series of the types α_1, α_2, ..., and $\alpha = \lim_{n=\infty} \alpha_n$. Then, since $s_\alpha = \lim_{n=\infty} s_{\alpha_n}$, we can choose n so that

$$s_{\alpha_n} > s_\alpha - \tfrac{1}{2}\delta.$$

---

* One fundamental property of an ordinary series which is obviously not possessed by the generalized series is that in an ordinary series we can assign a rank after which the terms are as small as we please.

But we can choose N numbers $\gamma_1, \ldots, \gamma_N < \alpha_n$ so that

$$u_{\gamma_1}+\ldots+u_{\gamma_N} > s_{\alpha_n}-\tfrac{1}{2}\delta > s_{\alpha_n}-\delta.$$

As the first part of the lemma is true for series of one term, it is true generally. A similar proof applies to the second part.

LEMMA 3.—*If a series is convergent in type ω and its sum is s_ω, it will be convergent when arranged in type β and its sum will be $\leqslant s_\omega$.*

This follows at once by a *reductio ad absurdum* from Lemma 2.

LEMMA 4.—*If a series is convergent in type β, it will be convergent when arranged in type ω, and the sums will be the same, and conversely.*

Suppose it convergent in type β. By Lemma 1 it is convergent in type ω, and its sum $\leqslant s_\beta$. But, if its sum $< s_\beta$, it follows from Lemma 3 that its sum in type $\beta < s_\beta$, which is absurd. The converse is proved similarly. From Lemma 4 the theorem follows at once.

A well known particular case of the theorem is the theorem concerning the summation of a doubly infinite series by rows, columns, or diagonals. In this case we have to deal with two arrangements in type ω^2 and one in type ω. Similar theorems for multiple series of any order follow at once.

[It is perhaps worth while at this point to emphasize the distinction between a *multiple* and a *repeated* series, as it is only recently that the notion of a multiple series has been made at all precise. According to Jordan the double series $\sum_{m,n=1}^{\infty} u_{m,n}$ is convergent only if *all* possible simply infinite series formed out of it are convergent. According to Pringsheim and Stolz it is convergent if *a certain large class* of them are. The point of importance for my present purpose is that in any case to assert the convergence of a double or multiple series is not really to assert the convergence of *one* series of a type different from that of the ordinary simple series, but to assert the convergence of a *whole class* of series of the ordinary type. But to assert the convergence of a repeated series *is* to assert the convergence of one series of a type in any case $> \omega$. The theorem applies to either kind of series.]

6. The theorem is easily extended to any absolutely convergent series, real or complex.

An absolutely convergent series of type β is a series of type β which is convergent when every term is replaced by its modulus. In the first place it is to be observed that the definitions and explanations of § 3 were given only for series of positive and zero terms. They apply, however, to series of arbitrary terms, except that when the terms are not all positive we cannot be sure that a definite limit (finite or infinite) will be found at each stage of the process. However, we can prove that *an absolutely convergent series of type* β *is convergent;* that is to say, a definite limit s_β results if we carry out the process of § 3.

For suppose first that the terms are all real, and define two new series Σv_γ, Σw_γ by the equations

$$v_\gamma = u_\gamma \;\; (u_\gamma > 0) = 0 \;\; (u_\gamma < 0), \qquad w_\gamma = 0 \;\; (u_\gamma > 0) = -u_\gamma \;\; (u_\gamma < 0);$$

then $u_\gamma = v_\gamma - w_\gamma$. Also, since $|v_\gamma| \leqslant |u_\gamma|$, $w_\gamma \leqslant |u_\gamma|$, Σv_γ, Σw_γ are convergent. Moreover, from (1) of § 4, $\sum\limits_{\gamma<\alpha} u_\gamma$ is convergent and

$$\sum_{\gamma<\alpha} u_\gamma = \sum_{\gamma<\alpha} v_\gamma - \sum_{\gamma<\alpha} w_\gamma.$$

Now apply the theorem of the last paragraph to each of these series separately. We find that $\Sigma u_\gamma = \Sigma' v_\gamma - \Sigma' w_\gamma$, Σ' denoting a series of the new type. And the right-hand is $\Sigma' u_\gamma$.

Next suppose the terms u_γ complex, and let $u_\gamma = v_\gamma + i w_\gamma$. Then, since $|v_\gamma| \leqslant |u_\gamma|$, $|w_\gamma| \leqslant |u_\gamma|$, the series Σv_γ, Σw_γ are absolutely convergent; and the theorem follows as before. The theorem is therefore proved for all absolutely convergent series.

7. It only remains for me to justify the assertion that this theorem is the most general possible theorem of its kind. This is evident if we consider that *any* rearrangement of an ordinary series of type ω must lead to an arrangement in *some* type β. The possibility of a series with a non-enumerable infinity of positive terms is excluded by Cantor's theorem that any set of intervals on a straight line must be enumerable.*

8. In conclusion, I give an example, suggested by a passage in Schoenflies's *Mengenlehre*, of an arrangement of the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

---

* See *e.g.*, W. H. Young, "On Sets of Intervals," *Proc. London Math. Soc.*, Vol. xxxv., p. 248. Another question is whether it is possible to define the sum of a series with a non-enumerable number of terms which are *not* all positive. It seems to me that the answer is probably *no;* but, as the question involves considerations of some logical difficulty, I shall not at present attempt to justify this answer.

in type β, where β is at any rate $> \omega^\omega$. It is

$$\begin{aligned}
&\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\dots\\
&+\frac{1}{2^2.3^2}+\frac{1}{2^2.5^2}+\frac{1}{2^2.7^2}+\dots\\
&+\frac{1}{3^2.5^2}+\frac{1}{3^2.7^2}+\dots\\
&+\frac{1}{5^2.7^2}+\dots\\
&+\dots\\
&+\frac{1}{2^2.3^2.5^2}+\frac{1}{2^2.3^2.7^2}+\frac{1}{2^2.3^2.11^2}+\dots\\
&+\frac{1}{2^2.5^2.7^2}+\frac{1}{2^2.5^2.11^2}+\dots\\
&+\frac{1}{2^2.7^2.11^2}+\dots\\
&+\dots\\
&+\frac{1}{3^2.5^2.7^2}+\dots\\
&+\dots.
\end{aligned}$$

It is easy to see that, if we carry out this scheme systematically, $\sum_{\gamma<\omega^2}$ contains every $1/n^2$ for which n is a product of two primes (including 1), $\sum_{\gamma<\omega^3}$ includes every $1/n^2$ for which n is a product of three primes, and so on. Hence $\sum_{\gamma<\omega^\omega}$ includes only those terms for which n is a product of different primes. The complete series (however we go on with it) is therefore of type $\beta > \omega^\omega$.

THE PROPAGATION OF WAVE-MOTION IN AN ISOTROPIC ELASTIC SOLID MEDIUM

By A. E. H. LOVE.

[Received and Read November 12th, 1903.]

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1. The investigations contained in this paper originated in the discovery of an oversight in the analysis in Stokes's justly famous memoir on the "Dynamical Theory of Diffraction."* To make this oversight apparent it is necessary to describe in some detail the contents of the second section of that memoir. This section deals with the propagation of an arbitrary disturbance in an elastic medium. The medium is taken to be an isotropic elastic solid, and the disturbance is supposed to be confined initially to a portion of it which occupies a finite volume. The dilatation Δ and the components of the rotation ϖ satisfy respectively the partial differential equations $\partial^2\Delta/\partial t^2 = a^2\nabla^2\Delta$ and $\partial^2\varpi/\partial t^2 = b^2\nabla^2\varpi$, and they are calculated from them by means of Poisson's integral formula.† According to this formula any function f which satisfies an

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* *Cambridge Phil. Soc. Trans.*, Vol. IX., 1849. The memoir is reprinted in Stokes's *Mathematical and Physical Papers*, Vol II., p. 243 (Cambridge, 1883). The detailed references in the sequel are to the pages of the reprint.

† Given by Poisson, *Mém. de l'Institut*, Paris, t. III., 1820.

equation of the form $\partial^2 f/\partial t^2 = a^2\nabla^2 f$ can be expressed in terms of initial values by the equation

$$f = \frac{t}{4\pi}\iint \dot{f}_0(at)\,d\sigma + \frac{\partial}{\partial t}\left\{\frac{t}{4\pi}\iint f_0(at)\,d\sigma\right\}, \tag{A}$$

in which the integration refers to angular space about the point at which f is estimated, and $f_0(at)$ and $\dot{f}_0(at)$ denote the initial values of f and $\partial f/\partial t$ on a sphere of radius at with its centre at the point. The components of displacement are calculated from the values obtained for the dilatation and rotation by the intervention of a scalar potential correlated with the dilatation and a vector potential correlated with the rotation. The process of calculation involves considerations of some delicacy. From the expressions obtained for the displacement due to initial disturbance Stokes deduced the effect of force operative at a point.

From this description it will be seen that all Stokes's results depend upon the employment of Poisson's integral formula to express the dilatation and the components of the rotation. In a recent paper* I have pointed out that this formula does not in general yield correct expressions for these quantities. In the same paper I identified the formula (A) with one that has been used by Poincaré† and others, viz.,

$$f = \frac{1}{4\pi}\iint \left(t\dot{f}_0 + f_0 + r\frac{\partial f_0}{\partial r}\right)_{r=at} d\sigma, \tag{B}$$

where r denotes distance from the point at which f is estimated.

The reason for the failure of such formulæ as (B) to represent the dilatation and the components of rotation is clear from an inspection of (B). When the point at which the disturbance is estimated is near the front of an advancing wave the sphere described about the point penetrates but a little way into the region within which the initial disturbance is confined, and the part of the sphere which is included in the integration is very small. Thus the formula cannot express any quantity which has a value different from zero at the front of an advancing wave. Now there is no kinematical or dynamical reason why the dilatation and rotation in an elastic solid should be supposed to vanish at the front of an advancing wave, and it appears therefore that Stokes's analysis is adequate to express the effects of particular types of initial disturbance, but not those of an arbitrary initial disturbance confined to a finite portion of the medium.

With the object of clearing up the difficulty thus encountered I begin by investigating the effects produced by force operative at a point. It is then possible to retrace the steps of Stokes's investigation and to arrive at a solution of the problem of arbitrary initial disturbances. It is also possible to obtain solutions of other problems relating to the propagation of wave-motions through an isotropic elastic solid medium. A summary of the results will be found at the end of the paper.

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* *Supra*, p. 37.

† *Théorie math. de la lumière*, t. I., Paris, 1889, p. 96. The conditions of identity of (A) and (B) are considered in Article 23 *infra*.

2. *Notation.*—The following notation will be used :—

ρ denotes the density of the medium.

λ and μ are the elastic constants of Lamé ; so that μ is the modulus of rigidity and $\lambda+\frac{2}{3}\mu$ is the modulus of compression.

a and b are the velocities of irrotational and equivoluminal waves; so that $a^2 = (\lambda+2\mu)/\rho$, $b^2 = \mu/\rho$.

(u, v, w) is the displacement at any point (x, y, z) at time t.

Δ is the dilatation; so that $\Delta = \text{div}\,(u, v, w) = \partial u/\partial x+\partial v/\partial y+\partial w/\partial z$.

$(\varpi', \varpi'', \varpi''')$ is the rotation; so that $2\,(\varpi', \varpi'', \varpi''') = \text{curl}\,(u, v, w)$, or $2\varpi' = \partial w/\partial y-\partial v/\partial z$, and so on.

ϕ is the scalar potential and (F, G, H) the vector potential; so that $(u, v, w) = -(\text{gradient of } \phi)+\text{curl}\,(F, G, H)$, where $\text{div}\,(F, G, H) = 0$, or $u = -\partial\phi/\partial x+\partial H/\partial y-\partial G/\partial z$, and so on; also $\nabla^2\phi = -\Delta$, $\nabla^2 F = -2\varpi'$, and so on.

∇^2 stands for the operator $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$.

r denotes the distance between a point (x, y, z) and a point (x', y', z').

A dot over a letter indicates differentiation with respect to the time.

3. *Conditions to be satisfied at a Wave-Boundary.**—When the displacement vanishes outside a wave the boundary of the wave is a surface Σ; on one side of Σ the displacement (u, v, w) vanishes everywhere; on the other side of Σ the components $u, \ldots$ are variable functions of x, y, z and of t. Since there can be no discontinuity of displacement without rupture of the material, the values assumed by u, v, w on Σ vanish. Let N denote the normal to Σ at any point, and s any direction in the tangent plane at the point. Since u vanishes all over Σ, we must have

$$\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}+\frac{\partial u}{\partial z}\frac{dz}{ds} = 0,$$

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* Cf. my paper already quoted.

and it follows that

$$\frac{\partial u/\partial x}{\cos(x, N)} = \frac{\partial u/\partial y}{\cos(y, N)} = \frac{\partial u/\partial z}{\cos(z, N)} = \frac{\partial u}{\partial N}. \tag{1}$$

Like equations hold with v and w in place of u.

Again Σ is not stationary, but moves normally to itself with a velocity which is either a or b. When it is a the equation $u = 0$ represents a surface which advances with velocity a, and we may take N to be in the direction in which the surface advances. Then we have

$$\frac{\partial u}{\partial t} + a\left\{\frac{\partial u}{\partial x}\cos(x, N) + \frac{\partial u}{\partial y}\cos(y, N) + \frac{\partial u}{\partial z}\cos(z, N)\right\} = 0,$$

and it follows from (1) that we must have

$$\frac{\partial u}{\partial N} = -a^{-1}\frac{\partial u}{\partial t}. \tag{2}$$

Like equations hold with v and w in place of u. If the surface Σ advances with velocity b, we must write b^{-1} in place of a^{-1}.

Equations (1) are simply kinematical conditions of continuity, but the equations of type (2) are dynamical conditions which connect the velocity near a wave-boundary with the strain in the same neighbourhood. When these equations are satisfied, we may say that the velocity at the boundary is "compensated" by the strain.

In the more general case, where the moving surface Σ separates two portions of the medium in which the components of displacement are expressed by different analytical functions, there are kinematical conditions of continuity of displacement and dynamical conditions connecting the differences of velocity with the differences of strain. The surface Σ moves normally to itself with a velocity which is either a or b. If u', u'' denote the values of u on the two sides of Σ, the conditions of continuity are of the type $u' = u''$ at Σ, and these lead us to equations of the form

$$\frac{\partial u'/\partial x - \partial u''/\partial x}{\cos(x, N)} = \frac{\partial u'/\partial y - \partial u''/\partial y}{\cos(y, N)} = \frac{\partial u'/\partial z - \partial u''/\partial z}{\cos(z, N)} = \frac{\partial u'}{\partial N} - \frac{\partial u''}{\partial N}, \tag{3}$$

and the dynamical conditions lead to additional relations of the type

$$\frac{\partial u'}{\partial N} - \frac{\partial u''}{\partial N} = -a^{-1}\left(\frac{\partial u'}{\partial t} - \frac{\partial u''}{\partial t}\right), \tag{4}$$

if the surface Σ moves with the velocity a. If it moves with the velocity b, we must write b^{-1} in place of a^{-1}.

When a wave with velocity a is propagated, the dilatation and the normal component velocity are connected by the equation

$$-a(\Delta'-\Delta'') = \{\dot{u}'\cos(x, N)+\dot{v}'\cos(y, N)+\dot{w}'\cos(z, N)\}$$
$$-\{\dot{u}''\cos(x, N)+\dot{v}''\cos(y, N)+\dot{w}''\cos(z, N)\}.$$

When a wave with velocity b is propagated the rotation and the tangential components of velocity are connected by equations of the type

$$-b\left\{\left(\frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z}\right)-\left(\frac{\partial w''}{\partial y}-\frac{\partial v''}{\partial z}\right)\right\}$$
$$= \{\dot{w}'\cos(y, N)-\dot{v}'\cos(z, N)\}-\{\dot{w}''\cos(y, N)-\dot{v}''\cos(z, N)\}.$$

4. *Displacement due to Body Forces.*—We consider the case where the motion of the medium is produced and maintained by the action of forces which are operative at the points within a certain region T. Let (X, Y, Z) be the body force per unit of mass. The equations of motion can be written in such forms as

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u+\rho X = \rho\frac{\partial^2 u}{\partial t^2}. \tag{5}$$

If (X, Y, Z) were zero, the solutions of these equations would represent free motions of the medium, such, for example, as can be due to forces which have ceased to act. Solutions representing free motions can be added to any particular solutions of the equations (5), containing (X, Y, Z), and the difference between such particular solutions will always represent a free motion. The effects produced by the forces can therefore be found by obtaining particular solutions of the equations (5).

For this purpose we express the field of force in terms of a scalar potential Φ and a vector potential (L, M, N), so that

$$(X, Y, Z) = -(\text{gradient of }\Phi)+\text{curl }(L, M, N).$$

With notation already introduced, equation (5) then becomes

$$-(\lambda+2\mu)\frac{\partial}{\partial x}\nabla^2\phi+\mu\left(\frac{\partial}{\partial y}\nabla^2 H-\frac{\partial}{\partial z}\nabla^2 G\right)-\rho\left(\frac{\partial\Phi}{\partial x}-\frac{\partial N}{\partial y}+\frac{\partial M}{\partial z}\right)$$
$$= -\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial\phi}{\partial x}-\frac{\partial H}{\partial y}+\frac{\partial G}{\partial z}\right),$$

and we have two other equations like this. It is clear that a particular solution will be obtained if we can solve the equations

$$\frac{\partial^2\phi}{\partial t^2}-a^2\nabla^2\phi = \Phi, \qquad \frac{\partial^2 F}{\partial t^2}-b^2\nabla^2 F = L, \qquad \dots. \tag{6}$$

It is well known that particular solutions of these equations can be expressed in the forms*

$$\left.\begin{aligned} \phi &= \frac{1}{4\pi a^2}\iiint \frac{1}{r}\,\Phi'\left(t-\frac{r}{a}\right) dx'\,dy'\,dz' \\ F &= \frac{1}{4\pi b^2}\iiint \frac{1}{r}\,L'\left(t-\frac{r}{b}\right) dx'\,dy'\,dz' \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\}, \tag{7}$$

in which $\Phi'(t-r/a)$ denotes that the value of Φ at the point (x', y', z') is to be calculated at the instant $t-r/a$, with a like signification for $L'(t-r/b)$, ..., and the integration extends through all space in which Φ', ... are different from zero at the times indicated.

Now Φ, L, M, N can be expressed in the forms

$$\left.\begin{aligned} \Phi &= \frac{1}{4\pi}\iiint \left(X'\frac{\partial r^{-1}}{\partial x}+Y'\frac{\partial r^{-1}}{\partial y}+Z'\frac{\partial r^{-1}}{\partial z}\right) dx'\,dy'\,dz' \\ L &= \frac{1}{4\pi}\iiint \left(Z'\frac{\partial r^{-1}}{\partial y}-Y'\frac{\partial r^{-1}}{\partial z}\right) dx'\,dy'\,dz' \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\}, \tag{8}$$

where the integration extends through the region T in which $(X, Y, Z) \neq 0$. In fact, it appears at once that, when these forms are assumed and the gradient of Φ and the curl of (L, M, N) are calculated from them,

$$-(\text{gradient of }\Phi)+\text{curl }(L, M, N) = (X, Y, Z) \text{ or zero,}$$

according as the point (x, y, z) is within T or not.

5. *Potentials of Disturbance due to Force at a Point.*—We may use our results to obtain the effect of force operative at one point. It will be sufficient to consider the case of a force of variable magnitude $\chi(t)$ acting at the point (x', y', z') in the direction of the axis of x. In equations (8) we pass to a limit by supposing all the linear dimensions of the region T to be diminished indefinitely, but the integral of X' through this region to remain finite, so that

$$\iiint \rho X'\,dx'\,dy'\,dz' = \chi(t), \qquad \iiint \rho Y'\,dx'\,dy'\,dz' = 0, \qquad \ldots.$$

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* L. Lorenz, *J. f. Math.* (Crelle), Bd. LVIII., 1861. The solution of equations of the type (6) in the case of forces which are simple harmonic functions of the time had been discussed by Helmholtz in the memoir "Theorie d. Luftschwingungen in Röhren mit offenen Enden" in the previous volume of the same journal.

Then we have

$$\Phi = \frac{1}{4\pi\rho}\chi(t)\frac{\partial r^{-1}}{\partial x}, \quad L = 0, \quad M = \frac{1}{4\pi\rho}\chi(t)\frac{\partial r^{-1}}{\partial z}, \quad N = -\frac{1}{4\pi\rho}\chi(t)\frac{\partial r^{-1}}{\partial y}. \tag{9}$$

We have to deduce the values of ϕ, F, G, H by means of equations (7). For this purpose we observe that in (7) x', y', z' are the coordinates of a point at which Φ, L, M, N are not all zero, and r is the distance of this point from (x, y, z); but in (9) x', y', z' are the coordinates of the point of application of the force $\chi(t)$, and r denotes the distance of this latter point from (x, y, z). We therefore rewrite (7) in the form

$$\phi = \frac{1}{4\pi a^2}\iiint \frac{1}{D}[\Phi]\,d\alpha\,d\beta\,d\gamma, \quad F = \frac{1}{4\pi b^2}\iiint \frac{1}{D}\{L\}\,d\alpha\,d\beta\,d\gamma, \quad \ldots, \tag{10}$$

in which D denotes the distance of the point (α, β, γ) from (x, y, z), and $[\Phi]$, $\{L\}$, ... denote the values of Φ, L, ... at the point (α, β, γ) at the instants $t-D/a$, $t-D/b$; so that we have

$$\left.\begin{aligned} [\Phi] &= \frac{1}{4\pi\rho}\chi\left(t-\frac{D}{a}\right)\frac{\partial R^{-1}}{\partial \alpha}, & \{L\} &= 0, \\ \{M\} &= \frac{1}{4\pi\rho}\chi\left(t-\frac{D}{b}\right)\frac{\partial R^{-1}}{\partial \gamma}, & \{N\} &= -\frac{1}{4\pi\rho}\chi\left(t-\frac{D}{b}\right)\frac{\partial R^{-1}}{\partial \beta} \end{aligned}\right\}, \tag{11}$$

where R denotes the distance of the point (α, β, γ) from the point (x', y', z') at which the force $\chi(t)$ is applied.

To perform the integrations in (10) we imagine space to be partitioned into thin sheets by means of spherical surfaces having the point (x, y, z) as centre. Let D be the radius of one such sphere, dS an element of its surface. We observe that $\chi(t-D/a)$ and $\chi(t-D/b)$ are constant over S, and thus we have

$$4\pi\rho\iiint D^{-1}[\Phi]\,d\alpha\,d\beta\,d\gamma = \int_0^\infty D^{-1}\chi\left(t-\frac{D}{a}\right)dD\iint\left(\frac{\partial R^{-1}}{\partial\alpha}\right)dS, \tag{12}$$

and like equations involving L, M, N. Now $\iint(\partial R^{-1}/\partial\alpha)\,dS$ is equal to zero if the point (x', y', z') is within S (so that $D > r$), but it is equal to $4\pi D^2(\partial r^{-1}/\partial x)$ if this point is outside S (so that $D < r$); for it may be conceived to represent a component of attraction at (x', y', z') of an uniform distribution of surface density on the sphere S, and this attraction vanishes at internal points, and at external points it is the same as if the whole distributed mass were concentrated at the centre of the sphere. Thus the upper limit of integration with respect to D may be replaced by

r, and the surface integrations can be performed. Similar considerations apply to L, M, N, and we have therefore

$$\phi = \frac{1}{4\pi a^2 \rho} \int_0^r \chi \left(t - \frac{D}{a}\right) \frac{\partial r^{-1}}{\partial x} D dD, \ldots .$$

These results can be expressed by the equations

$$\left.\begin{aligned} \phi &= \frac{1}{4\pi\rho} \frac{\partial r^{-1}}{\partial x} \int_0^{r/a} t' \chi(t-t') \, dt', & F &= 0, \\ G &= \frac{1}{4\pi\rho} \frac{\partial r^{-1}}{\partial z} \int_0^{r/b} t' \chi(t-t') \, dt', & H &= -\frac{1}{4\pi\rho} \frac{\partial r^{-1}}{\partial y} \int_0^{r/b} t' \chi(t-t') \, dt' \end{aligned}\right\}, \tag{13}$$

6. *Effect of Force at a Point.*—From the values of ϕ, F, ... we may deduce those of the components of displacement. The expression for u is simplified by observing that $\nabla^2 r^{-1} = 0$. The integrals are differentiated in accordance with the rules

$$\frac{\partial}{\partial x} \int_0^{r/a} t' \chi(t-t') \, dt' = \frac{r}{a} \chi \left(t - \frac{r}{a}\right) \frac{1}{a} \frac{\partial r}{\partial x}, \quad \ldots .$$

The results are expressed by the equations

$$\left.\begin{aligned} u &= \frac{1}{4\pi\rho} \frac{\partial^2 r^{-1}}{\partial x^2} \int_{r/a}^{r/b} t' \chi(t-t') \, dt' \\ &\quad + \frac{1}{4\pi\rho b^2 r} \chi \left(t - \frac{r}{b}\right) + \frac{1}{4\pi\rho r} \left(\frac{\partial r}{\partial x}\right)^2 \left\{ \frac{1}{a^2} \chi \left(t - \frac{r}{a}\right) - \frac{1}{b^2} \chi \left(t - \frac{r}{b}\right) \right\} \\ v &= \frac{1}{4\pi\rho} \frac{\partial^2 r^{-1}}{\partial x \partial y} \int_{r/a}^{r/b} t' \chi(t-t') \, dt' \\ &\qquad\qquad + \frac{1}{4\pi\rho r} \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} \left\{ \frac{1}{a^2} \chi \left(t - \frac{r}{a}\right) - \frac{1}{b^2} \chi \left(t - \frac{r}{b}\right) \right\} \\ w &= \frac{1}{4\pi\rho} \frac{\partial^2 r^{-1}}{\partial x \partial z} \int_{r/a}^{r/b} t' \chi(t-t') \, dt' \\ &\qquad\qquad + \frac{1}{4\pi\rho r} \frac{\partial r}{\partial x} \frac{\partial r}{\partial z} \left\{ \frac{1}{a^2} \chi \left(t - \frac{r}{a}\right) - \frac{1}{b^2} \chi \left(t - \frac{r}{b}\right) \right\} \end{aligned}\right\}. \tag{14}$$

These expressions for the displacement due to force at a point may be identified readily with those obtained by Stokes (p. 278). When $\chi(t)$ is a constant they are equivalent to the expressions given by Lord

Kelvin* to represent the statical effect of force at a point of an elastic solid body. I have verified also that the corresponding tractions over the surface of a spherical cavity with its centre at the origin of r are statically equivalent to a single force in the direction of the axis of x, and that, in the limit when the radius of the cavity is diminished indefinitely, this force is of magnitude $\chi(t)$.

It is convenient to observe that the dilatation and rotation calculated from (14) are given by the equations

$$\left.\begin{array}{ll}\Delta = \dfrac{1}{4\pi a^2\rho}\dfrac{\partial}{\partial x}\left\{\dfrac{1}{r}\chi\left(t-\dfrac{r}{a}\right)\right\}, & \varpi' = 0, \\ 2\varpi'' = \dfrac{1}{4\pi b^2\rho}\dfrac{\partial}{\partial z}\left\{\dfrac{1}{r}\chi\left(t-\dfrac{r}{b}\right)\right\}, & 2\varpi''' = -\dfrac{1}{4\pi b^2\rho}\dfrac{\partial}{\partial y}\left\{\dfrac{1}{r}\chi\left(t-\dfrac{r}{b}\right)\right\}\end{array}\right\}. \tag{15}$$

The results for forces in the directions of the axes of y and z can be written down by symmetry, and the results for an arbitrary direction of the force can be found by addition. When we wish to emphasize the relation of the expressions (14) to force parallel to the axis of x, and to an expression of the form $\chi(t)$ for the magnitude of the force, we shall write them $u_1(\chi)$, $v_1(\chi)$, $w_1(\chi)$, and a similar notation $\Delta_1(\chi)$, ... can be used in regard to the dilatation and rotation. Suffixes 2 and 3 will indicate in like manner that the forces are parallel to the axes of y and z. When χ is the same for all, we have the relations

$$w_2 = v_3, \quad u_3 = w_1, \quad v_1 = u_2; \tag{16}$$

also we have the relations

$$\varpi_2''' = -\varpi_3'', \quad \varpi_3' = -\varpi_1''', \quad \varpi_1'' = -\varpi_2', \quad \varpi_1' = \varpi_2'' = \varpi_3''' = 0. \tag{17}$$

7. *Establishment of the State of Strain due to Constant Force at a Point.*—A special example of some interest is afforded by the case in which the force begins to act at a particular instant, increases at an uniform rate from zero to a finite value X, attained at the end of an interval ϵ, and thereafter remains constant. The force will be taken to act in the

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* Sir W. Thomson, *Cambridge and Dublin Math. Journal*, 1848, or *Mathematical and Physical Papers*, Vol. I., p. 97 (Cambridge, 1882). Cf. Thomson and Tait, *Natural Philosophy*, Part II., p. 279, and J. Boussinesq, *Applications des Potentiels*, &c., p. 82 (Paris, 1885).

direction of the axis of x. The function $\chi(t)$ is determined by the equations

$$\chi(t) = 0 \quad \text{when} \quad t < 0,$$

$$\chi(t) = Xt/\epsilon \quad \text{when} \quad \epsilon > t > 0,$$

$$\chi(t) = X \quad \text{when} \quad t > \epsilon.$$

The formulæ (14) show that at a distance r from the point of application of the force the displacement vanishes at any instant t which is such that $t < r/a$. In the subsequent motion we have to distinguish the intervals* (i.) when $r/a+\epsilon > t > r/a$, (ii.) when $r/b > t > r/a+\epsilon$, (iii.) when $r/b+\epsilon > t > r/b$, (iv.) when $t > r/b+\epsilon$.

In the first of these intervals, *i.e.*, when $r/a+\epsilon > t > r/a$, we have

$$\int_{r/a}^{r/b} t'\chi(t-t')\,dt' = X\epsilon^{-1}\int_{r/a}^{t} t'(t-t')\,dt', \tag{18}$$

and we find, after a little algebraic rearrangement,

$$\left.\begin{aligned} u &= \frac{X}{4\pi\rho}\left\{\frac{(t-r/a)^2(t+2r/a)}{6\epsilon}\,\frac{\partial^2 r^{-1}}{\partial x^2} + \frac{t-r/a}{a^2\epsilon r}\left(\frac{\partial r}{\partial x}\right)^2\right\} \\ v &= \frac{X}{4\pi\rho}\left\{\frac{(t-r/a)^2(t+2r/a)}{6\epsilon}\,\frac{\partial^2 r^{-1}}{\partial x\,\partial y} + \frac{t-r/a}{a^2\epsilon r}\,\frac{\partial r}{\partial x}\,\frac{\partial r}{\partial y}\right\} \end{aligned}\right\}, \tag{19}$$

with a similar expression for w. Thus the displacement at the front ($r = at$) of the advancing wave vanishes, and there is no tendency for it to become large in the neighbourhood of this surface when ϵ is small. The velocity at the front of the wave does not vanish, but the equation $\partial u/\partial t + a\,\partial u/\partial r = 0$ is satisfied at that surface in accordance with the condition (2) of Article 3. Thus a wave having a definite front is generated. During an interval ϵ, beginning at the instant when the wave-front reaches a point, it may be shown by means of (19) that the displacement is derived from a scalar potential ϕ, so that

$$u = -\frac{\partial\phi}{\partial x},\quad \ldots,\quad \phi = -\frac{X}{4\pi\rho}\left\{\frac{t^3}{6\epsilon}\,\frac{\partial r^{-1}}{\partial x} + \frac{t}{2a^2\epsilon}\,\frac{\partial r}{\partial x} - \frac{x}{3a^3\epsilon}\right\}. \tag{20}$$

Hence the motion is irrotational and the dilatation is given by the equation

$$\Delta = \frac{Xt}{4\pi\rho a^2\epsilon}\,\frac{\partial r^{-1}}{\partial x}. \tag{21}$$

The dilatation does not vanish at the front of the advancing wave, and near this front it tends to become large if ϵ becomes very small, unless X also is very small.

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* Some modifications would be made in the results if ϵ is so long that these intervals overlap; but they are of no great interest.

In the second interval, *i.e.*, when $r/b > t > r/a+\epsilon$, we have

$$\int_{r/a}^{r/b} t'\chi(t-t')\,dt' = X\int_{r/a}^{t-\epsilon} t'\,dt' + X\epsilon^{-1}\int_{t-\epsilon}^{t} t'(t-t')\,dt', \tag{22}$$

and we find after a little reduction that the displacement is derived from a scalar potential ϕ, where

$$\phi = -\frac{X}{8\pi\rho}\left\{(t^2-t\epsilon+\tfrac{1}{3}\epsilon^2)\frac{\partial r^{-1}}{\partial x} + \frac{1}{a^2}\frac{\partial r}{\partial x}\right\}; \tag{23}$$

so that the motion is irrotational and the dilatation is given by the equation

$$4\pi\rho a^2\Delta = X\frac{\partial r^{-1}}{\partial x}. \tag{24}$$

Hence the state of dilatation associated with the operation of a constant force is established at any point at the end of the interval ϵ, beginning at the instant when the wave-front reaches the point.

In the third interval, *i.e.*, when $r/b+\epsilon > t > r/b$, we have

$$\int_{r/a}^{r/b} t'\chi(t-t')\,dt' = X\int_{r/a}^{t-\epsilon} t'\,dt' + X\epsilon^{-1}\int_{t-\epsilon}^{r/b} t'(t-t')\,dt',$$

and we find after a little reduction that

$$\left.\begin{aligned} u &= \frac{X}{4\pi\rho}\left[\left\{-\frac{(t-r/b)^2(t+2r/b)}{6\epsilon}+\tfrac{1}{2}t(t-\epsilon)+\tfrac{1}{6}\epsilon^2-\tfrac{1}{2}\frac{r^2}{a^2}\right\}\frac{\partial^2 r^{-1}}{\partial x^2}\right. \\ &\qquad\left.+\frac{1}{a^2 r}\left(\frac{\partial r}{\partial x}\right)^2+\frac{1}{b^2 r}\,\frac{t-r/b}{\epsilon}\left\{1-\left(\frac{\partial r}{\partial x}\right)^2\right\}\right] \\ v &= \frac{X}{4\pi\rho}\left[\left\{-\frac{(t-r/b)^2(t+2r/b)}{6\epsilon}+\tfrac{1}{2}t(t-\epsilon)+\tfrac{1}{6}\epsilon^2-\tfrac{1}{2}\frac{r^2}{a^2}\right\}\frac{\partial^2 r^{-1}}{\partial x\,\partial y}\right. \\ &\qquad\left.+\left(\frac{1}{a^2 r}-\frac{t-r/b}{b^2\epsilon r}\right)\frac{\partial r}{\partial x}\,\frac{\partial r}{\partial y}\right] \end{aligned}\right\}, \tag{25}$$

with a similar expression for w. The displacement is expressible in the form

$$u = -\frac{\partial\psi}{\partial x}+\frac{Xt}{4\pi b^2\rho\epsilon r}, \qquad v = -\frac{\partial\psi}{\partial y}, \qquad w = -\frac{\partial\psi}{\partial z}, \tag{26}$$

where

$$\psi = \frac{X}{4\pi\rho}\left\{\frac{(t-\epsilon)^3}{6\epsilon}\frac{\partial r^{-1}}{\partial x}-\frac{1}{2a^2}\frac{\partial r}{\partial x}+\frac{t}{2\epsilon b^2}\frac{\partial r}{\partial x}+\frac{2x}{3\epsilon b^3}\right\}; \tag{27}$$

so that the dilatation and the rotation are given by the equations

$$\Delta = \frac{X}{4\pi\rho a^2}\frac{\partial r^{-1}}{\partial x}, \quad \varpi' = 0, \quad 2\varpi'' = \frac{Xt}{4\pi\rho b^2\epsilon}\frac{\partial r^{-1}}{\partial z}, \quad 2\varpi''' = -\frac{Xt}{4\pi\rho b^2\epsilon}\frac{\partial r^{-1}}{\partial y}. \tag{28}$$

We see that the displacement does not tend to become large when ϵ is small, but that the rotation does; also the dilatation established in the second interval is unaltered.

Subsequent to the instant $t = r/b+\epsilon$, the state associated with the action of a constant force is maintained, that is to say, we have

$$\left.\begin{aligned} u = -\frac{X}{8\pi\rho}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{\partial^2 r}{\partial x^2}+\frac{X}{4\pi b^2\rho r}, \quad v = -\frac{X}{8\pi\rho}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{\partial^2 r}{\partial x\,\partial y} \\ w = -\frac{X}{8\pi\rho}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{\partial^2 r}{\partial x\,\partial z} \end{aligned}\right\}, \tag{29}$$

and the dilatation and the rotation are expressed by the equations

$$\Delta = \frac{X}{4\pi\rho a^2}\frac{\partial r^{-1}}{\partial x}, \quad \varpi' = 0, \quad 2\varpi'' = \frac{X}{4\pi\rho b^2}\frac{\partial r^{-1}}{\partial z}, \quad 2\varpi''' = -\frac{X}{4\pi\rho b^2}\frac{\partial r^{-1}}{\partial y}. \tag{30}$$

8. The wave generated by the force is characterized by the presence of four surfaces of discontinuity, viz.: (i.) the front of the irrotational wave, (ii.) the rear of the wave by which the dilatation is established, (iii.) the front of the rotational wave, (iv.) the rear of the wave by which the rotation is established. The first two move with the velocity a, and the remaining two with the velocity b. The displacement is continuous at all these surfaces. I have verified that the conditions of the type (cf. Article 3)

$$a\left(\frac{\partial u'}{\partial r}-\frac{\partial u''}{\partial r}\right)+\left(\frac{\partial u'}{\partial t}-\frac{\partial u''}{\partial t}\right) = 0$$

are satisfied at the rear of the wave by which the dilatation is established, *i.e.*, at the surface $r = a(t-\epsilon)$, and that the conditions of the type

$$b\left(\frac{\partial u'}{\partial r}-\frac{\partial u''}{\partial r}\right)+\left(\frac{\partial u'}{\partial t}-\frac{\partial u''}{\partial t}\right) = 0$$

are satisfied at the front of the rotational wave, and at the rear of the wave by which the rotation is established, *i.e.*, at the surfaces $r = bt$ and $r = b(t-\epsilon)$. Thus the necessary conditions of continuity and dynamical conditions connecting velocity and strain at wave-fronts are all satisfied.

9. *Graphic Representation of the Establishment of the Strain due to Constant Force.*—The results obtained in the last article may be illustrated by figures. For this purpose I consider points on the line of action of the force (say the "axis"), and on the plane through its point of application at right angles to the axis (say, the "equatorial" plane), and distinguish

the part of the axis drawn from the point of application in the sense towards which the force acts as in "front" of the point. At points on the axis the displacement is longitudinal, and at points on the equatorial plane it is transverse. I have taken $a = 2b$; so that Poisson's ratio for the material is $\frac{1}{3}$. The scale of the figures is fixed by the conventional equations

$$a = 20, \quad b = 10, \quad \epsilon = 1, \quad \frac{X}{4\pi\rho} = 30{,}000. \tag{31}$$

Fig. 1 shows the gradual establishment of the longitudinal displacement at points on the axis. The abscissæ represent distance from the point of application of the force. The ordinates of the curves represent the displacement at definite times. In the uppermost curve the displacement has reached its final value. In the curve which breaks off from this on the ordinate marked 30 the displacement has its value at the instant $t = 4$. It is zero beyond $r = 80$; the part of the curve between $r = 80$ and $r = 60$ represents the displacement in the wave by which the dilatation is established; at $r = 60$ the direction of the curve changes suddenly; the part of the curve between $r = 40$ and $r = 30$ represents the displacement in the wave by which the rotation is established; this curve has an inflexion; behind $r = 30$ the ultimate state is established when $t = 4$. In the curve which breaks off at the ordinate marked 20 the displacement has its value at the time $t = 3$. It is zero beyond $r = 60$; the wave by which the dilatation is established lies between $r = 60$ and $r = 40$, and the wave by which the rotation is established lies between $r = 30$ and $r = 20$; behind $r = 20$ the ultimate state is established when $t = 3$. In the curve which breaks off at the ordinate marked 10 the displacement has its value at the instant $t = 2$. It is zero beyond $r = 40$; the wave by which the dilatation is established lies between $r = 40$ and $r = 20$,

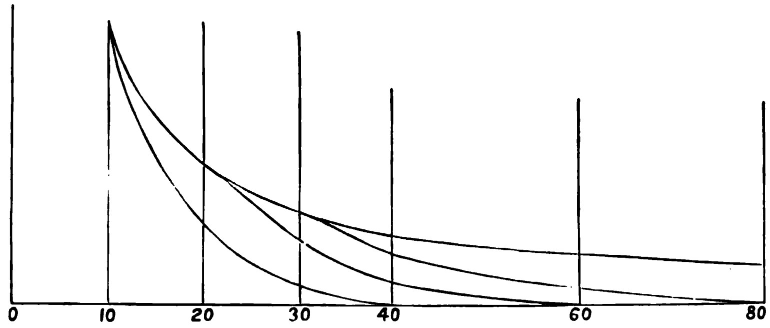

FIG. 1.

and the wave by which the rotation is established lies between $r = 20$ and $r = 10$; so that there is now no interval between the waves; behind $r = 10$ the ultimate state is established when $t = 2$.

Fig. 2 shows the transverse displacement at points on the equatorial plane at the time $t = 4$. The relative smallness of the displacement in the dilatational part of the wave and the result that it is opposite in direction to the force are noteworthy.

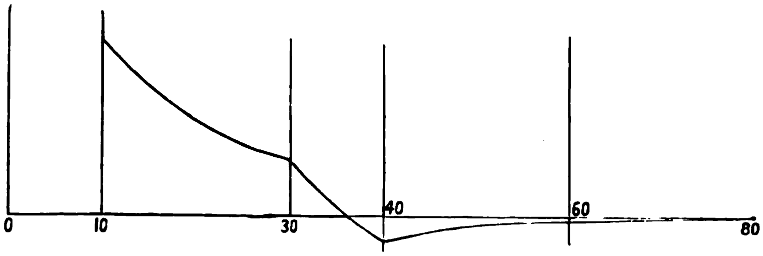

FIG. 2.

Fig. 3 shows the compression at points on the axis and in front of the point of application of the force and the rotation at points in the equatorial plane, each magnified 50 times from the values given by the conventional scheme of equations (31) adopted for Figs. 1 and 2. The compression is represented by the ordinates of the line which is broken at $r = 60$, and

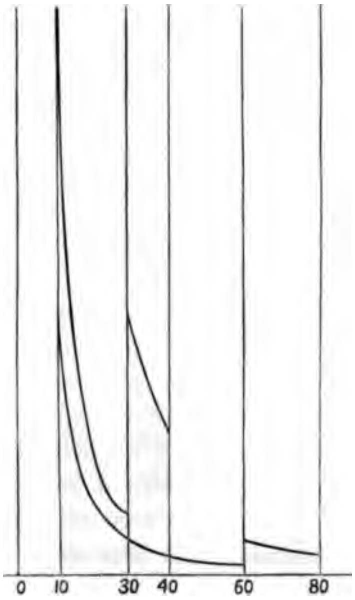

FIG. 3.

the rotation by the ordinates of the line which is broken at $r = 30$. The dilatation in the part $r < 60$ and the rotation in the part $r < 30$ are those which belong to the ultimate state.

10. *Momentary Force.*—We may pass to the example of a force which lasts for a short time by superposing upon the effects due to a force which gradually increases to a certain positive value and then remains constant those due to a force which begins to act a little later than the first, gradually diminishes from zero to a numerically equal negative value, and then remains constant. The dilatation will be confined to a thin pulse of which the front and rear move with velocity a, and the rotation to a thin pulse of which the front and rear move with velocity b. In front of the pulse of dilatation and behind the pulse of rotation the medium is at rest and without displacement or strain. Between the two pulses there is motion without dilatation or rotation, and therefore also without acceleration; so that this motion is of the type of steady irrotational motion in an incompressible fluid, as was observed by Stokes (p. 279). This motion is that which would be due to a certain doublet at the point of application of the force; the initiation of it involves a relatively intense but momentary state of compression, and the termination of it involves a relatively intense but momentary state of rotational strain.

We may regard a force which acts continually as equivalent to an indefinite succession of small momentary forces. The disturbance produced at any point O at an instant t will consist of (i.) a dilatation or compression corresponding with the initiation of motion due to the force which was operative at the latest instant at which its action can affect the state of the medium at the point O at the instant t; (ii.) a rotation corresponding with the termination of motion due to the force which was operative at the earliest instant at which its action can affect the state of the medium at the point O at the instant t; (iii.) superposed states of steady motion due to the action of the force at intermediate instants. The combined effect of these latter states may be of equal importance with the dilatation and the rotation (cf. Stokes, p. 279). The result that the magnitude of the force at the intermediate instants affects the state of the medium at a definite point and instant shows that, in some sense, there is propagation of effects through the medium with velocity intermediate between the velocities a and b. The recognition of the existence of such effects constitutes the interpretation of the terms of (14) which contain $\int_{r/a}^{r/b} t'\chi(t-t')\,dt'$, and hence it happens that in the case of periodic forces of simple harmonic type these effects are entirely masked, for when

$\chi(t)$ has the form $Ae^{\iota pt}$ the value of the integral is

$$A\left\{e^{\iota p(t-r/b)}p^{-2}\left(1+\frac{\iota rp}{b}\right)-e^{\iota p(t-r/a)}p^{-2}\left(1+\frac{\iota rp}{a}\right)\right\};$$

so that the whole effect can be analysed into two waves propagated with the velocities b and a.

11. *Double Forces.*—Various types of singular points may be obtained by supposing two or more singularities of the type of points at which forces act to coalesce. The following are the most important types :—

(i.) A force $h^{-1}\chi(t)$ acts at the origin in the positive direction of the axis of x, and an equal and opposite force acts at the point $(h, 0, 0)$. In the limit, when h is diminished indefinitely while $\chi(t)$ remains the same, the resulting displacement is, in the notation of Article 6,

$$\left(\frac{\partial u_1}{\partial x},\ \frac{\partial v_1}{\partial x},\ \frac{\partial w_1}{\partial x}\right).$$

We may describe the singularity as a "double force without moment." It is related to an axis, in this case the axis of x.

(ii.) We may combine three double forces without moment having their axes in the directions of the axes of coordinates. The resulting displacement is

$$\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z},\ \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z},\ \frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}+\frac{\partial w_3}{\partial z}\right),$$

and by equations (16) this is the same as $(\Delta_1, \Delta_2, \Delta_3)$, or the displacement is

$$\frac{1}{4\pi a^2\rho}\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\left\{r^{-1}\chi\left(t-\frac{r}{a}\right)\right\}. \tag{32}$$

The waves that are generated are dilatational waves of a well known type. The singular point may be described as a "centre of dilatation." It has no special relation to any axis. The forces applied in the neighbourhood of this centre have no statical resultant or moment. The state of the medium outside a spherical cavity with its centre at the centre of dilatation could be produced by pressure uniform over the cavity. I find the magnitude of this pressure to be expressed in terms of χ by the formula

$$\frac{1}{4\pi}\left(\nabla^2-\frac{4\mu}{\lambda+2\mu}\,\frac{1}{r}\,\frac{\partial}{\partial r}\right)\frac{\chi(t-r/a)}{r}. \tag{33}$$

(iii.) We may suppose a force $h^{-1}\chi(t)$ to act at the origin in the positive direction of the axis of x, and an equal and opposite force to act at the point (0, h, 0), and we may pass to a limit as before. The resulting displacement is

$$\left(\frac{\partial u_1}{\partial y}, \frac{\partial v_1}{\partial y}, \frac{\partial w_1}{\partial y}\right).$$

We may describe the singularity as a "double force with moment." The forces applied in the neighbourhood of the singularity are statically equivalent to a couple of moment $\chi(t)$ about the axis of z; the singularity is related also to the directions of the forces, *i.e.*, in this case to the axis of x.

(iv.) We may combine two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being at right angles to each other. We take the forces to be $h^{-1}\chi(t)$ and $-h^{-1}\chi(t)$ parallel to the axes of x and y at the origin, $-h^{-1}\chi(t)$ parallel to the axis of x at the point (0, h, 0), and $h^{-1}\chi(t)$ parallel to the axis of y at the point (h, 0, 0), and we pass to a limit as before. The resulting displacement is

$$\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}, \frac{\partial v_1}{\partial y}-\frac{\partial v_2}{\partial x}, \frac{\partial w_1}{\partial y}-\frac{\partial w_2}{\partial x}\right),$$

which is the same as $-2(\varpi_1''', \varpi_2''', \varpi_3''')$, or it is

$$\frac{1}{4\pi b^2\rho}\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}, 0\right)\left\{r^{-1}\chi\left(t-\frac{r}{b}\right)\right\}. \tag{34}$$

The waves that are generated are rotational waves of a well-known type. The forces applied near the singular point are statically equivalent to a couple of moment $2\chi(t)$ about the axis of z. A singular point of this kind may be described as a "centre of rotation about the axis of z"; it is not related to any other axis. The motion expressed by (34) has been discussed with numerical illustrations by Lord Kelvin.*

12. *Generalized Double Force.*—Other modes of combining double forces are reducible to those already considered, so long as no additional limiting operation is involved. A very general type may be constructed as follows:—Let ξ be any specified component of the traction across a given plane, and let ξ_1, ξ_2, ξ_3 be its values when the state of stress is due to force $\chi(t)$ in the directions of x, y, z respectively. Then (ξ_1, ξ_2, ξ_3) is the displacement due to a certain combination of double forces at the origin.

Denote in general by X_x, Y_y, Z_z, Y_z, Z_x, X_y the six components of stress at a point, and by X_ν, Y_ν, Z_ν the components of traction across the plane of which the normal is in the direction ν. Let l, m, n be the direction cosines of ν, so that

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* *Phil. Mag.* (Ser. 5), Vol. XLVI., 1898.

$$X_\nu = lX_x + mX_y + nZ_x, \qquad Y_\nu = lX_y + mY_y + nY_z, \qquad \dots;$$

also let ν' be the direction in which the traction is to be estimated, and l', m', n' the direction cosines of ν'; then we have

$$\xi = l'X_\nu + m'Y_\nu + n'Z_\nu,$$

$$\text{or } \xi = ll'X_x + mm'Y_y + nn'Z_z + (mn' + m'n)Y_z + (nl' + n'l)Z_x + (lm' + l'm)X_y; \qquad (35)$$

so that ξ is related symmetrically to the directions ν, ν'. When the state of stress is due to the action of a force $\chi(t)$ at the origin in the direction of the axis of x, the dependence of ξ upon this direction, the directions ν, ν', and the function $\chi(t)$ may be expressed by writing it $\xi_1(\nu, \nu', \chi)$, and in the same way $\xi_2(\nu, \nu', \chi)$, $\xi_3(\nu, \nu', \chi)$ will denote what ξ becomes when the forces are parallel to the axes of y and z, the directions ν, ν' and the function χ being the same as before.

Now the stress components are expressed in terms of the strains by equations of the types

$$X_x = \lambda\Delta + 2\mu\frac{\partial u}{\partial x}, \qquad Y_z = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right).$$

The terms of ξ_1, ξ_2, ξ_3 that involve λ are proportional to Δ_1, Δ_2, Δ_3, and are therefore displacements due to a suitable centre of dilatation at the origin. The terms that involve $\partial u_1/\partial x$, $\partial u_2/\partial x$, $\partial u_3/\partial x$ may, by equations (16), be replaced by like terms involving $\partial u_1/\partial x$, $\partial v_1/\partial x$, $\partial w_1/\partial x$, and they are therefore displacements due to a double force without moment, having for its axis the axis of x. The terms that involve $\partial w_1/\partial y$, $\partial w_2/\partial y$, $\partial w_3/\partial y$ may be replaced by like terms involving $\partial u_3/\partial y$, $\partial v_3/\partial y$, $\partial w_3/\partial y$, and they are therefore displacements due to a double force with moment about the axis of x and having its forces parallel to the axis of z. All the remaining terms of ξ_1, ξ_2, ξ_3 may be treated in the same way, and thus (ξ_1, ξ_2, ξ_3) is the displacement due to a certain combination of double forces at the origin.* A singularity of this kind may be described as a "generalized double force." It is specially and symmetrically related to the pair of directions ν, ν'.

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* The result holds of course in the case of equilibrium, *i.e.*, when $\chi(t)$ is replaced by a constant. The result that the quantities expressed by ξ_1, ξ_2, ξ_3 when χ is constant are solutions of the equations which determine the displacement in a body held in equilibrium was given by C. Somigliana, *Ann. di mat.* (Ser. 2), t. XVII., 1889.

13. *Multiple Forces.*—More complex types of waves of dilatation and rotation may be obtained by repeated differentiations from the formulæ that correspond with centres of dilatation and centres of rotation. For example, we may put*

$$(u,\ v,\ w) = \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{\beta} \left(\frac{\partial}{\partial z}\right)^{\gamma} (\Delta_1,\ \Delta_2,\ \Delta_3), \tag{36}$$

or, again,

$$(u,\ v,\ w) = \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{\beta} \left(\frac{\partial}{\partial z}\right)^{\gamma} (\varpi_1',\ \varpi_2',\ \varpi_3'), \tag{37}$$

and in the latter ϖ' may be replaced by ϖ'' or ϖ'''. All such solutions of the equations of motion can be regarded as derived by suitable limiting operations from combinations of centres of dilatation or of rotation.

An interesting special case is found by combining two pairs of centres of rotation in the same way as two pairs of forces are combined in one such centre. Let there be at the origin centres of rotation with moments $2h^{-1}\chi(t)$ and $-2h^{-1}\chi(t)$ about the axes of x and y, at the point $(0,\ h,\ 0)$ a centre of rotation with moment $-2h^{-1}\chi(t)$ about an axis parallel to the axis of x, and at the point $(h,\ 0,\ 0)$ a centre of rotation with moment $2h^{-1}\chi(t)$ about an axis parallel to the axis of y, and let us pass to a limit by diminishing h indefinitely. The resulting displacement is

$$-2\frac{\partial}{\partial y}(\varpi_1',\ \varpi_2',\ \varpi_3') + 2\frac{\partial}{\partial x}(\varpi_1'',\ \varpi_2'',\ \varpi_3''),$$

or it is proportional to

$$\left(\frac{\partial^2}{\partial x\,\partial z},\ \frac{\partial^2}{\partial y\,\partial z},\ -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \left\{\frac{1}{r}\chi\left(t - \frac{r}{b}\right)\right\}. \tag{38}$$

These are of the same form as the expressions given by Hertz† to represent the electric force in the field around his oscillator. It appears that there has been some discussion‡ concerning the nature of the forces required to produce in an elastic solid displacement expressed by the same formulæ.

We may combine two centres of dilatation of opposite signs and very great strength and placed close together on an axis, obtaining a formula of the type

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* These are the types of waves considered by Lord Kelvin in *Phil. Mag.* (Ser. 5), Vols. XLVII. and XLVIII., 1899.

† *Ann. Phys. Chem.* (Wiedemann), Bd. XXXVI., 1888, or *Electric Waves*, English edition, p. 140.

‡ W. König, *Ann. Phys. Chem.* (Wiedemann), Bd. XXXVII., 1889; and Lord Rayleigh, *Theory of Sound*, 2nd edition, Vol. II., p. 427 (London, 1896), also *Phil. Mag.* (Ser. 6), Vol. VI., 1903, p. 385.

$$(u, v, w) = \left(\frac{\partial^2}{\partial x\,\partial z}, \; \frac{\partial^2}{\partial y\,\partial z}, \; \frac{\partial^2}{\partial z^2}\right) \left\{\frac{1}{r}\,\chi\left(t - \frac{r}{a}\right)\right\}. \tag{39}$$

The states of the medium expressed by (38) and (39) may be supposed to be produced by tractions over the surface of a spherical cavity having its centre at the singular point. In both cases the traction is statically equivalent to a single force in the direction of the axis of z. Lord Kelvin has discussed these tractions (*loc. cit.*, p. 309), and has shown that the arbitrary functions in (38) and (39) may be adjusted so that the result of superposing the two systems of displacement can represent the effects due to the oscillatory motion of a rigid sphere.

14. *Extension of Kirchhoff's Theorem.*—In the case of motion determined by a single characteristic equation of the form $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$, Kirchhoff[*] has given a formula of the same kind as Green's formula in the theory of potential, for representing the value of ϕ at a point on one side of a closed surface as due to singularities distributed over the surface. By the help of the formulæ for displacement due to force at a point, we can obtain an analogous theorem for the more general case of waves in an elastic solid medium. The method of procedure need not be described in great detail, as the work is very similar to that which I have given elsewhere.[†] We set out from the reciprocal theorem first given by Betti[‡] and expressible in the form

$$\begin{aligned}\iiint \rho \left\{ \left(X - \frac{\partial^2 u}{\partial t^2}\right) u' + \left(Y - \frac{\partial^2 v}{\partial t^2}\right) v' + \left(Z - \frac{\partial^2 w}{\partial t^2}\right) w' \right\} dx\,dy\,dz \\ + \iint (X_\nu u' + Y_\nu v' + Z_\nu w')\,dS \\ = \iiint \rho \left\{ \left(X' - \frac{\partial^2 u'}{\partial t^2}\right) u + \left(Y' - \frac{\partial^2 v'}{\partial t^2}\right) v + \left(Z' - \frac{\partial^2 w'}{\partial t^2}\right) w \right\} dx\,dy\,dz \\ + \iint (X'_\nu u + Y'_\nu v + Z'_\nu w)\,dS,\end{aligned} \tag{40}$$

in which the surface integration extends over a closed surface S, and the volume integration extends through the volume within this surface, and the accented and unaccented letters denote corresponding quantities in two states of the medium. Every term in the equation (40) represents the work that would be done by a force of one type acting over a displacement due to force of another type; reversed kinetic reactions are counted as forces in this statement.

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[*] *Ann. Phys. Chem.* (Wiedemann), Bd. XVIII., 1883; or *Vorlesungen ü. math. Optik* (Leipzig), 1891.

[†] *Phil. Trans. R. S.* (Ser. A), Vol. CXCVII., 1901.

[‡] *Il nuovo Cimento* (Ser. 2), t. VII., 1872.

We shall suppose that the state (u, v, w) is not due to body force within S, and we shall choose for u', v', w' the displacement obtained from the u_1, v_1, w_1 of Article 6, by replacing $\chi(t-t')$ by $\chi(t+t')$ for all values of t', the singular point for u_1, v_1, w_1 being (x_0, y_0, z_0). Then Y' and Z' vanish, and X' vanishes except in the immediate neighbourhood of the point (x_0, y_0, z_0), and we may put

$$\iiint \rho (X'u + Y'v + Z'w)\, dx\, dy\, dz = u(x_0, y_0, z_0, t)\, \chi(t).$$

We choose for $\chi(t)$ a function which vanishes outside a very small range of values including the value t_0, and there is so great that the integral $\int \chi(t)\, dt$ over this range is equal to unity. Then we have

$$\int dt \iiint \rho (X'u + Y'v + Z'w)\, dx\, dy\, dz = u(x_0, y_0, z_0, t_0), \tag{41}$$

provided that the value t_0 is included within the range of integration with respect to t. We shall write $u^{(0)}$ for $u(x_0, y_0, z_0, t_0)$.

We integrate both members of equation (40) with respect to t over a range of values including t_0. We may choose the limits of integration so that the integrals of all such terms as $\iiint \rho \left(u \frac{\partial^2 u'}{\partial t^2} - u' \frac{\partial^2 u}{\partial t^2}\right) dx\, dy\, dz$ with respect to t shall vanish, and thus we obtain the formula

$$u^{(0)} = \iint dS \int_{t_1}^{t_2} dt\, (X_\nu u' + Y_\nu v' + Z_\nu w' - X'_\nu u - Y'_\nu v - Z'_\nu w), \tag{42}$$

where t_1 and t_2 are suitable extreme values for t, and t_0 lies between them. We may extend the result to space outside a closed surface, or to space bounded by two or more closed surfaces, provided that u, v, w have no singularities in the region of space bounded by the surfaces and containing the point (x_0, y_0, z_0), and provided that the normal ν is always drawn out of this region.

15. The integrations with respect to t in the right-hand member of (42) can be performed. We have

$$u' = \frac{1}{4\pi\rho} \frac{\partial^2 r^{-1}}{\partial x^2} \int_{r/a}^{r/b} t' \chi(t+t')\, dt'$$
$$+ \frac{1}{4\pi\rho} \frac{1}{r} \left(\frac{\partial r}{\partial x}\right)^2 \left\{ \frac{1}{a^2} \chi\left(t+\frac{r}{a}\right) - \frac{1}{b^2} \chi\left(t+\frac{r}{b}\right) \right\} + \frac{1}{4\pi\rho b^2 r} \chi\left(t+\frac{r}{b}\right), \tag{43}$$

and thus integrals of the type $\int_{t_1}^{t_2} X_\nu u'\,dt$ will consist of two classes of terms, viz., those containing $\int_{r/a}^{r/b} t'\chi(t+t')\,dt'$ and those containing χ. Now $\int_{r/a}^{r/b} t'\chi(t+t')\,dt'$ vanishes unless t_0 lies between $t+r/a$ and $t+r/b$, and then its value is (t_0-t). It follows that, for any function $F(t)$,

$$\int_{t_1}^{t_2} dt F(t) \int_{r/a}^{r/b} t'\chi(t+t')\,dt' = \int_{t_0-r/b}^{t_0-r/a} (t_0-t)F(t)\,dt = \int_{r/a}^{r/b} t'F(t_0-t')\,dt'. \tag{44}$$

Again, we have

$$\int_{t_1}^{t_2} F(t)\chi\left(t+\frac{r}{a}\right)dt = F\left(t_0-\frac{r}{a}\right), \qquad \int_{t_1}^{t_2} F(t)\chi\left(t+\frac{r}{b}\right)dt = F\left(t_0-\frac{r}{b}\right). \tag{45}$$

Further, it is clear that integrals of the type $\int_{t_1}^{t_2} X'_\nu u\,dt$ will contain, in addition to terms of the classes already considered, other terms involving $\frac{\partial}{\partial r}\chi\left(t+\frac{r}{a}\right)$ and $\frac{\partial}{\partial r}\chi\left(t+\frac{r}{b}\right)$. Now we have

$$\int_{t_1}^{t_2} F(t)\frac{\partial}{\partial r}\chi\left(t+\frac{r}{a}\right)dt = \frac{1}{a}\int_{t_1}^{t_2} F(t)\frac{\partial}{\partial t}\chi\left(t+\frac{r}{a}\right)dt,$$

and this is equal to $-\frac{1}{a}\int_{t_1}^{t_2}\chi\left(t+\frac{r}{a}\right)F'(t)\,dt$, or to the value of $-a^{-1}\partial F/\partial t$, when $t=t_0-r/a$, and we have therefore

$$\int_{t_1}^{t_2} F(t)\frac{\partial}{\partial r}\chi\left(t+\frac{r}{a}\right)dt = \frac{\partial}{\partial r}F\left(t_0-\frac{r}{a}\right). \tag{46}$$

It follows that the effect of all the integrations with respect to t is to discard $\chi(t+t')$ and replace $F(t)$ by $F(t_0-t')$ for all values of t'.

16. The effect of the integration with respect to t in the term $\int_{t_1}^{t_2} dt \iint X_\nu u'\,dS$ is to produce a term $\iint u_1(X_\nu)\,dS$ where $u_1(X_\nu)$ means an expression of the form of u_1 with X_ν in place of χ. Since u_1 is not altered when the point (x, y, z) and the origin of r are interchanged, the expression $u_1(X_\nu)\,dS$ represents the displacement parallel to the axis of x that would be produced at the point (x_0, y_0, z_0) by force $X_\nu dS$ acting at (x, y, z) in the

direction of the axis of x. Thus we may write

$$\int_{t_1}^{t_2} dt \iint (X_\nu u' + Y_\nu v' + Z_\nu w') \, dS = \iint \{u_1(X_\nu) + u_2(Y_\nu) + u_3(Z_\nu)\} \, dS, \tag{47}$$

and we may interpret this expression as the displacement that would be produced in an unlimited medium by forces acting at the points of S, the force at each element of area being equal to the traction exerted across the element in the actual motion.

The effect of the integration with respect to t in the term $-\int_{t_1}^{t_2} dt \iint X'_\nu u \, dS$ is to produce a term $\iint \xi_1(x, \nu, u) \, dS$, where $\xi_1(x, \nu, u)$ denotes the traction parallel to x across a plane normal to ν that would be produced at the point (x_0, y_0, z_0) by a force at (x, y, z), acting in the direction of the axis of x, and specified as regards magnitude and dependence on time by $u\,dS$. The change of sign arises from the observation that, when (x, y, z) and the origin of r are interchanged, the expressions for strain components and stress components that correspond with such displacements as (u_1, v_1, w_1) are, in fact, changed in sign. Thus we may write

$$-\int_{t_1}^{t_2} dt \iint (X'_\nu u + Y'_\nu v + Z'_\nu w) \, dS = \iint \{\xi_1(x, \nu, u) + \xi_1(y, \nu, v) + \xi_1(z, \nu, w)\} \, dS. \tag{48}$$

If the direction of (u, v, w) at any point of S were constant as regards time, the right-hand member would take the simpler form $\iint \xi_1(h, \nu, q) \, dS$, in which q denotes the resultant and h the direction of (u, v, w); but this simplification cannot be made when the ratios $u : v : w$ depend upon t, as they generally do.

The final expression for $u^{(0)}$ is

$$u^{(0)} = \iint \{u_1(X_\nu) + u_2(Y_\nu) + u_3(Z_\nu) + \xi_1(x, \nu, u) + \xi_2(y, \nu, v) + \xi_3(z, \nu, w)\} \, dS, \tag{49}$$

and corresponding expressions for $v^{(0)}$ and $w^{(0)}$ can be written down by symmetry.

17. The result expresses the displacement at a point as due to singular points distributed over a surface. The singular points are of two sorts— points at which forces act, and points where there are "generalized double forces." The forces are the actual tractions across the elements of area of the surface. There are three generalized double forces at each point;

they are related to the three pairs of directions (x, ν), (y, ν), (z, ν), and their mode of dependence upon time is specified by the values of u, v, w at the point.

In Kirchhoff's theorem, of which an extension has been given here, the motion is represented as due to sources and doublets. It is not surprising that the singularities which correspond, in the present case, with the sources of the simple theory should be points at which forces are operative, nor that these forces are the actual tractions exerted across the elements of area. It has here been made out that the singularities which, in the present case, correspond with the doublets of the simple theory are the generalized double forces which were investigated in Article 12.

The results are also of interest as emphasizing the result which was found in Article 10, viz., that, in some sense, there are effects which are propagated through the medium with velocities intermediate between a and b, as well as others which are propagated with those definite velocities. The effect produced at a point O, and at an instant t, has been expressed in terms of what was going on at points P of an arbitrary surface, and at instants previous to t by the intervals required to travel from P to O with the velocity a, the velocity b, and with all intermediate velocities.

18. *Disturbance due to an Initial State.*—We now take up the problem of determining the effects produced by a state of disturbance given at some instant. At this initial instant we suppose that the displacement and velocity are given at all points within a finite volume T, which is bounded by a closed surface, or by two or more closed surfaces. These surfaces will be denoted collectively by Σ_0. We suppose that the displacement and velocity at every point outside T vanish at the initial instant. The degree of arbitrariness in the initial values of the displacement (u_0, v_0, w_0) and velocity $(\dot{u}_0, \dot{v}_0, \dot{w}_0)$ is restricted by the conditions which are necessary to secure that there is no rupture of the material. It is therefore clearly necessary that u_0, v_0, w_0 vanish at Σ_0. As in Article 3, it follows that, at all points of Σ_0, we must have

$$\frac{\partial u_0/\partial x'}{\cos(n, x)} = \frac{\partial u_0/\partial y'}{\cos(n, y)} = \frac{\partial u_0/\partial z'}{\cos(n, z)} = \frac{\partial u_0}{\partial n}, \tag{50}$$

where (x', y', z') denotes in general a point within T, and n denotes the normal to Σ_0 drawn away from T. Similar equations containing v_0 and w_0 in place of u_0 hold also at Σ_0. It will appear later that, to avoid a tendency to rupture of the material, certain relations must connect the components of velocity at points of Σ_0 with the dilatation and the components of rotation at the same points. For the present, however, we shall proceed as if the initial velocity could be given arbitrarily.

It is convenient to restrict the arbitrariness of the initial displacement and velocity by the conditions that the components of these two vector quantities are expressed by functions of x', y', z' which are analytic and without singularities in the region T. We may then apply ordinary rules of transformation to definite integrals taken through volumes partly bounded by Σ_0, or over surfaces having edges on Σ_0, without having to pay attention to any exceptional points other than the points of Σ_0.

Greater generality could be attained, at the expense of greater complication, by supposing that the initial displacement and velocity are expressed by different analytic functions in different parts of T. At the surfaces separating such parts the displacement is necessarily continuous, but the velocities and strains are not necessarily continuous. When the velocities and strains are discontinuous in crossing such surfaces, the discontinuities travel through the medium, and the adjustments of velocity and strain described in Article 3 are made in the subsequent motion; difficulty can arise if they are not made initially. An example in which they are made initially would be afforded by the state of the medium produced by a single force, which gradually increases from zero to a definite value and then diminishes again to zero. Any instant after that at which the force ceases to act may be selected as initial instant, and then discontinuities of the kind here described will exist initially, but no tendency to rupture will arise. When the displacements and velocities are expressed by different analytic functions in different parts of T, but the velocities and strains are continuous in crossing the surfaces that separate such parts, no tendency to rupture can arise from the initial existence of such surfaces. Wave-motion which involves the continuous existence of such analytic discontinuities will be generated. This class of motions in the case of a fluid medium has been specially studied by H. Hugoniot.* We shall, in general, as already stated, exclude such motions from our theory, but it may be stated in advance that the existence of such analytic discontinuities cannot affect the formulæ (64) which will be obtained for the displacement, though they may affect the formulæ for the dilatation and the rotation.

19. *Potentials of Disturbance due to Initial Velocity.*—From the effect of force operative at one point we may deduce that due to given

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* *Jour. de Math.* (Liouville), (Ser. 4), tt. 3, 4, 1887, 1888. See also P. Duhem, *Recherches sur l'Hydrodynamique*, Paris, 1903.

initial disturbance, in so far as it depends upon initial velocities. We consider, in the first place, an initial velocity $\dot{u}_0$, parallel to the axis of x, at points of the medium within the volume T. The velocity $\dot{u}_0$ may be supposed to be communicated to the mass-element $\rho\,dx'\,dy'\,dz'$ by the action, during a very short interval of time (δt_0) of a force X, directed parallel to the axis of x, so that

$$\rho\dot{u}_0\,dx'\,dy'\,dz' = X\delta t_0, \tag{51}$$

and we may take the action of X to be confined to the interval between $t = 0$ and $t = \delta t_0$; so that X vanishes unless t lies between these values. Now, in equations (13) of Article 5, the integral $\int_0^{r/a} t'\chi(t-t')\,dt'$ vanishes if $r < at$; but, if $r > at$, it may be replaced by $tX\delta t_0$ or by $\rho\dot{u}_0 t\,dx'\,dy'\,dz'$. It follows that $\dot{u}_0$ contributes to ϕ the term

$$(4\pi)^{-1}\iiint t\dot{u}_0\frac{\partial r^{-1}}{\partial x}\,dV_1, \tag{52}$$

in which dV_1 denotes an element of the volume which is within T, and is also outside a sphere with centre at (x, y, z) and radius equal to at. We denote this sphere by S_1, and in like manner we denote by S_2 the sphere with centre at (x, y, z) and radius equal to bt, and by dV_2 an element of volume outside this sphere and within T.

We can now write down the potentials due to initial velocity $(\dot{u}_0, \dot{v}_0, \dot{w}_0)$ in the forms

$$\left.\begin{aligned} \phi &= \frac{t}{4\pi}\iiint\left(\dot{u}_0\frac{\partial r^{-1}}{\partial x}+\dot{v}_0\frac{\partial r^{-1}}{\partial y}+\dot{w}_0\frac{\partial r^{-1}}{\partial z}\right)dV_1 \\ F &= \frac{t}{4\pi}\iiint\left(\dot{w}_0\frac{\partial r^{-1}}{\partial y}-\dot{v}_0\frac{\partial r^{-1}}{\partial z}\right)dV_2 \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}\right\}. \tag{53}$$

The potentials due to initial displacement are to be obtained by replacing the $(\dot{u}_0, \dot{v}_0, \dot{w}_0)$ by (u_0, v_0, w_0) and differentiating with respect to t. The displacements are derived from the potentials by differentiating with respect to x, y, z, and forming certain sums of differential coefficients. In differentiating we have to take account of the circumstances that the volumes V_1 and V_2 are bounded partly by a fixed surface Σ_0, the boundary of T, and partly by spheres with centres at (x, y, z) and radii proportional to t. We require rules of differentiation adapted to these circumstances.*

20. *Rules of Differentiation.* — Consider first a volume integral $\iiint \psi\, dV_1$, where ψ involves x, y, z, t, x', y', z'. If the boundary S_1 were independent of x, y, z and t, we should have

$$\frac{\partial}{\partial x}\iiint \psi\, dV_1 = \iiint \frac{d\psi}{dx}\, dV_1, \quad \ldots, \quad \frac{\partial}{\partial t}\iiint \psi\, dV_1 = \iiint \frac{\partial\psi}{\partial t}\,\partial V_1;$$

but these require correction because the surface S_1 depends on x, y, z, t. When (x, y, z) is changed into $(x+\delta x, y, z)$, the sphere is shifted from the position S_1 without altering its radius, and the effect of the shifting is to include in V_1 an additional volume $-\delta x \iint \frac{x'-x}{r}\, dS_1$, where the integration extends over that part of S_1 which is included in T; for $\{(x'-x)/r\}\,\delta x$ is the displacement of a point on the sphere along the normal outwards, and when this is positive at (x', y', z') an element of volume which was included in V_1 comes to be excluded. We have therefore

$$\frac{\partial}{\partial x}\iiint \psi\, dV_1 = \iiint \frac{d\psi}{dx}\, dV_1 - \iint \frac{x'-x}{r}\,\psi\, dS_1. \tag{54}$$

Again, when t is changed into $t+\delta t$, the volume V_1 is diminished by $\iint a\,\delta t\, dS_1$, and, since $r = at$ on S_1, we have

$$\frac{\partial}{\partial t}\iiint \psi\, dV_1 = \iiint \frac{\partial\psi}{\partial t}\, dV_1 - \iint \frac{r}{t}\,\psi\, dS_1. \tag{55}$$

Exactly similar formulæ hold with V_2 and S_2 substituted for V_1 and S_1, the surface integration extending over the portion of S_2 that is within T.

21. *Differentiation of Surface Integrals.*—The corrections of such formulæ as

$$\frac{\partial}{\partial x}\iint \psi\,\partial S_1 = \iint \frac{\partial\psi}{\partial x}\, dS_1 \quad \text{and} \quad \frac{\partial}{\partial t}\iint \psi\, dS_1 = \iint \frac{\partial\psi}{\partial t}\, dS_1$$

are more complicated. When the centre of the sphere of radius at is shifted through δx, the value of ψ has to be integrated over the surface derived from S_1 by the shifting, and not over the surface S_1. Besides this, some additional elements of area are included in T. These elements adjoin the curve s_1 in which the surface S_1 cuts the boundary Σ_0 of T. To determine the increment of $\iint \psi\, dS_1$ due to the shifting of the sphere, we express the integral as $\iint \psi r^2 d\sigma$, where $d\sigma$ is an element of the solid angle subtended at (x, y, z) by any surface having s_1 for edge. The element $d\sigma$ is unaltered by the shifting, and any point P, associated with a definite $d\sigma$, is replaced by a new point Q on the same radius vector drawn from (x, y, z), which is the point O in Fig. 4. Now $PQ = \{(x'-x)/r\}\,\delta x$, and thus the increment of $\iint \psi\, dS_1$ on account of the shifting of the sphere is

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* The expression for ϕ is equivalent to that obtained by Stokes, and apparently the values of F, G, H are equivalent to those which he used, but the rules of differentiation are different from those employed by him.

$$\delta x \iint \frac{x'-x}{r} \frac{\partial(\psi r^2)}{\partial r}\, d\sigma,$$

or it is
$$\delta x \iint \frac{x'-x}{r^3} \frac{\partial(\psi r^2)}{\partial r}\, dS_1.$$

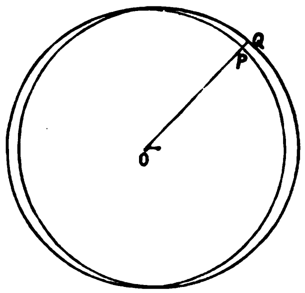

FIG. 4.

Again, when the sphere is shifted through δx a strip of a certain width along the curve s_1 comes to be included in the part of S_1 which lies within T. Let ϵ denote the width of this strip, n the direction of the outward drawn normal to the boundary Σ_0 of T, and let χ be the angle at which the surfaces S_1 and Σ_0 cut. The quantity ϵ would not be altered if we supposed S_1 to remain fixed and Σ_0 to be shifted through $-\delta x$. The new position of s_1 could then be obtained from the old by a displacement of magnitude ϵ directed at right angles to r and to the tangent of s_1, and the component of this displacement which is in the direction of n can be expressed either as $\epsilon \sin\chi$ or as $-\delta x \cos(x, n)$. Hence the additional area in question is $-\delta x \int \cos(x, n) \operatorname{cosec}\chi\, ds_1$, and we obtain the rule of differentiation

$$\frac{\partial}{\partial x}\iint \psi\, dS_1 = \iint \left\{\frac{\partial\psi}{\partial x} + \frac{x'-x}{r^3}\frac{\partial}{\partial r}(\psi r^2)\right\} dS_1 - \int \frac{\cos(x, n)}{\sin\chi}\psi\, ds_1. \quad (56)$$

The rules for differentiation of $\iint \psi\, dS_1$ with respect to t are of the same kind as the above. When t is changed into $t+\delta t$, we have a correction depending on the change of position of the part of S_1 that is within Σ_0, and another correction depending on the inclusion of additional elements of area within Σ_0. The first correction is found by taking $\iint \psi\, dS_1$ in the form $\iint \psi r^2 d\sigma$ to be $\iint a r^{-2} \{\partial(\psi r^2)/\partial r\}\, dS_1$. To determine the other correction, let P be any point on s_1, and let the radius vector OP meet in Q the sphere of radius $a(t+\delta t)$ described about O as centre. In the plane containing P, Q and the normal n to the surface Σ_0 at P, the sphere of radius $a(t+\delta t)$ will cut Σ_0 in a point R near to Q, so that $QR = -a\delta t \cot(r, n)$, and the increment of $\iint \psi\, dS_1$ on account of the inclusion of elements of area of breadths specified in general by QR is $-\delta t \iint a\psi \cot(r, n)\, ds_1$. Observing that $\sin(r, n)$ is the same as $\sin\chi$, we have the formula

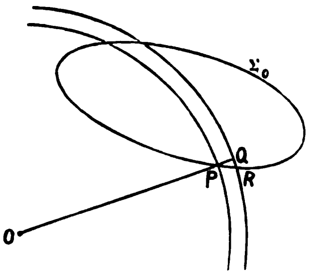

FIG. 5.

$$\frac{\partial}{\partial t}\iint \psi\, dS_1 = \iint \left\{\frac{\partial\psi}{\partial t} + \frac{1}{rt}\frac{\partial}{\partial r}(\psi r^2)\right\} dS_1 - \int a\frac{\cos(r, n)}{\sin\chi}\psi\, ds_1. \tag{57}$$

Exactly similar rules to those expressed in (56) and (57) hold for the differentiation of surface integrals, such as $\iint \psi\, dS_2$ taken over the part of S_2 that is within Σ_0. In formulating them the curve of intersection of S_2 with Σ_0 may be denoted by s_2.

22. *Displacement due to Initial Velocity.*—We have now to form expressions for u, v, w from the ϕ, F, G, H given in (53), by the formula

$$(u, v, w) = -(\text{gradient of } \phi) + \text{curl}\,(F, G, H),$$

and the rules of differentiation. The contribution of $\dot{u}_0$ to u is

$$-\frac{t}{4\pi}\iiint \dot{u}_0 \frac{\partial^2 r^{-1}}{\partial x^2}\, dV_1 - \frac{t}{4\pi}\iiint \dot{u}_0 \left(\frac{\partial^2 r^{-1}}{\partial y^2} + \frac{\partial^2 r^{-1}}{\partial z^2}\right) dV_2$$

$$+\frac{t}{4\pi}\iint \dot{u}_0 \frac{x'-x}{r}\,\frac{\partial r^{-1}}{\partial x}\, dS + \frac{t}{4\pi}\iint \dot{u}_0 \left(\frac{y'-y}{r}\,\frac{\partial r^{-1}}{\partial y} + \frac{z'-z}{r}\,\frac{\partial r^{-1}}{\partial z}\right) dS_2,$$

and this is the same as

$$\frac{t}{4\pi}\iiint \dot{u}_0 \frac{\partial^2 r^{-1}}{\partial x^2}\, dV + \frac{t}{4\pi}\iint \dot{u}_0 \frac{x'-x}{r}\,\frac{\partial r^{-1}}{\partial x}\, dS_1$$

$$+\frac{t}{4\pi}\iint \dot{u}_0 \left(\frac{1}{r^2} - \frac{x'-x}{r}\,\frac{\partial r^{-1}}{\partial x}\right) dS_2,$$

where V is that part of the volume between S_2 and S_1 which is within T, and the surface integrations extend over those parts of S_1 and S_2 which are within T. The contributions of $\dot{v}_0$, $\dot{w}_0$ can be expressed in the same way, and we have therefore, for the part of u which depends on initial velocities, the expression

$$\frac{t}{4\pi}\iiint\left(\dot{u}_0\frac{\partial^2 r^{-1}}{\partial x^2}+\dot{v}_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+\dot{w}_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)dV$$
$$+\frac{t}{4\pi}\iint\dot{q}_0\frac{\partial r^{-1}}{\partial x}dS_1+\frac{t}{4\pi}\iint\left(\dot{u}_0\frac{1}{r^2}-\dot{q}_0\frac{\partial r^{-1}}{\partial x}\right)dS_2, \quad (58)$$

where $\dot{q}_0$ denotes the component of $(\dot{u}_0, \dot{v}_0, \dot{w}_0)$ in the direction of the radius r drawn from the point (x, y, z) to the point (x', y', z'), so that

$$r\dot{q}_0=(x'-x)\dot{u}_0+(y'-y)\dot{v}_0+(z'-z)\dot{w}_0. \quad (59)$$

The expression (58) is equivalent to that obtained by Stokes (p. 268); it has been obtained here, as by him, from the expressions (53) for the potentials due to initial velocity. The expressions of type (58) may also be obtained from the expressions of type (14) for force operative at a point. In this mode of procedure we should have to replace $\chi(t-r/a)$ by $\rho\dot{u}_0 dx'\,dy'\,dz'/\delta t_0$ as in Article 19, and further to replace $dx'\,dy'\,dz'$ by $a\,dS_1\,\delta t_0$; we should have to make similar substitutions in the case of $\chi(t-r/b)$. The process is the reverse of that employed by Lord Rayleigh in his *Theory of Sound*, Vol. II., § 276.

23. *Poisson's Integral Formula.*—It may be observed that the process which has been used to obtain the displacement due to initial velocity may be applied to obtain Poisson's integral of the characteristic equation of wave-motion, viz., the equation $\partial^2 f/\partial t^2=a^2\nabla^2 f$. We may treat f as a "generalized displacement," and consider it as due to the operation of "generalized forces" $F_1(t)$. At points where the forces are acting we have

$$\frac{\partial^2 f}{\partial t^2}-a^2\nabla^2 f=F_1(t),$$

and we obtain a particular integral in the form

$$(4\pi a^2)^{-1}\iiint r^{-1}F_1\left(t-\frac{r}{a}\right)dx'\,dy'\,dz'.$$

Initial generalized velocity $\dot{f}_0$ may be thought of as due to generalized forces acting for very short intervals containing the initial instant, and then, by the method sketched in the last Article, we obtain for the part of f due to $\dot{f}_0$ the expression $(4\pi)^{-1}\iint t\dot{f}_0 r^{-2}\,dS_1$. The effect of f_0 may be found by Stokes's rule, and it follows that, if f can always be traced to the operation of generalized forces which produce velocity $\dot{f}$, we must have

$$f = \frac{t}{4\pi}\iint \dot{f}_0(at)\,d\sigma + \frac{1}{4\pi}\frac{\partial}{\partial t}\left\{ t \iint f_0(at)\,d\sigma \right\}, \tag{A bis}$$

as in the formula (A) of Article 1. According to the rule of differentiation (57), the value of f is given by the equation

$$f = \frac{1}{4\pi}\iint \frac{1}{r^2}\left(t\dot{f}_0 + f_0 + r\frac{\partial f_0}{\partial r}\right) dS_1 - \frac{1}{4\pi}\int \frac{\cos(r, n)}{r \sin\chi} f_0\, ds_1. \tag{60}$$

The line integral in the right-hand member of (60) constitutes a correction of the formula (B), and it appears that the formulæ (A) and (B) of Article 1 are not always equivalent.

One consequence of this result is that Poisson's integral formula (A) *can* represent a solution of $\partial^2 f/\partial t^2 = a^2\nabla^2 f$, which is discontinuous at a wave-boundary, although the formula (B), *i.e.* the right-hand member of (60) with the line integral omitted, cannot represent any such solution. Nevertheless it remains true that the integral formula in question cannot represent all solutions of the equation, and, in particular, that it does not always yield correct expressions for the dilatation and the components of rotation of an elastic solid medium. We may see this most simply by thinking of the case where initially there is no displacement but some velocity within the region T, and the initial velocity within T would be possible in an incompressible fluid moving irrotationally. In this case there is initially neither dilatation nor velocity of dilatation, but there is initial velocity within T, of which the normal component does not vanish at the surface Σ_0. It is clear that in the wave-motion that ensues there will be dilatation or compression due to the normal component of the initial velocity at the initial boundary. We shall find later on that the correction of formula (60), *i.e.*, of formula (A), when applied to the dilatation depends upon the normal component of the initial velocity at Σ_0, and that the correction of it when applied to the components of rotation depends upon the components of the initial velocity that are tangential to Σ_0.

When a component of *displacement* is determined by the characteristic equation, its value is expressed by Poisson's integral formula and the forms (A) and (B) are equivalent; for, if f_0 is a genuine displacement, it necessarily vanishes at all points of Σ_0. Thus the formula is available for expressing the components of displacement when the whole motion is either irrotational or equivoluminal—a well-known result.

24. *Displacement due to Initial Disturbance in general.*—The part of the displacement u which depends upon initial displacements can be

written down at once in the form

$$\frac{\partial}{\partial t}\left\{\frac{t}{4\pi}\iiint\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)\partial V\right.$$
$$\left.+\frac{t}{4\pi}\iint q_0\frac{\partial r^{-1}}{\partial x}\,dS_1+\frac{t}{4\pi}\iint\left(u_0\frac{1}{r^2}-q_0\frac{\partial r^{-1}}{\partial x}\right)dS_2\right\},$$

in which q_0 denotes the initial displacement in the direction r; so that

$$rq_0=(x'-x)\,u_0+(y'-y)\,v_0+(z'-z)\,w_0. \tag{61}$$

On carrying out the differentiations in accordance with the rules given in Articles 20 and 21, we find for the part of the displacement u due to initial displacements the expression

$$\frac{1}{4\pi}\iiint\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)dV$$
$$+\frac{1}{4\pi}\iint\left\{r\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)+\frac{\partial r^{-1}}{\partial x}\left(q_0+r\frac{\partial q_0}{\partial r}\right)\right\}dS_1$$
$$-\frac{1}{4\pi}\iint\left\{r\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)+\frac{\partial r^{-1}}{\partial x}\left(q_0+r\frac{\partial q_0}{\partial r}\right)\right.$$
$$\left.-\frac{1}{r^2}\left(u_0+r\frac{\partial u_0}{\partial r}\right)\right\}dS_2, \tag{62}$$

wherein $$r\frac{\partial u_0}{\partial r}=(x'-x)\frac{\partial u_0}{\partial x'}+(y'-y)\frac{\partial u_0}{\partial y'}+(z'-z)\frac{\partial u_0}{\partial z'}, \tag{63}$$

with a like formula for $\partial q_0/\partial r$. No line integrals occur in (62) because u_0, v_0, w_0 vanish at all points of Σ_0. The complete expression for u is obtained by adding the expressions (58) and (62) and is the same as that given by Stokes (p. 268). It is convenient to record it in the form

$$u=\frac{1}{4\pi}\iiint\left\{(t\dot{u}_0+u_0)\frac{\partial^2 r^{-1}}{\partial x^2}+(t\dot{v}_0+v_0)\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+(t\dot{w}_0+w_0)\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right\}dV$$
$$+\frac{1}{4\pi}\iint\left\{r\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)+\frac{\partial r^{-1}}{\partial x}\left(t\dot{q}_0+q_0+r\frac{\partial q_0}{\partial r}\right)\right\}dS_1$$
$$-\frac{1}{4\pi}\iint\left\{r\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\right)+\frac{\partial r^{-1}}{\partial x}\left(t\dot{q}_0+q_0+r\frac{\partial q_0}{\partial r}\right)\right.$$
$$\left.-\frac{1}{r^2}\left(t\dot{u}_0+u_0+r\frac{\partial u_0}{\partial r}\right)\right\}dS_2. \tag{64}$$

The formulæ for v and w can be written down by symmetry.

25. Expressions for the displacement equivalent, except as regards the independence of a and b, to those found by Stokes had been obtained previously by Poisson[*] and Ostrogradsky[†] by methods involving the synthesis of particular solutions of simple harmonic type, and the nature of the motion had been the subject of considerable discussion.[‡] The chief point of debate was concerned with the meaning and degree of importance of the terms expressed by volume integrals. Poisson showed that, at a distance from T which is so great that T may be regarded as little more than a point, the motion breaks up into two wave-motions. In the first of these, propagated with the velocity a, the displacement is almost purely longitudinal, *i.e.*, directed along the radius drawn from T; in the second, propagated with the velocity b, the motion is almost purely transverse. Poisson showed also that, in the same case, the displacement between the two waves is small in comparison with that in either wave. Stokes, taking the dilatation and the components of the rotation to be expressed by particular formulæ, pointed out that the first wave involves dilatation without rotation, and the second rotation without dilatation. The motion between them can involve neither dilatation nor rotation, and is of the character of steady irrotational motion in an incompressible fluid. This last result can be verified at once by means of the formulæ (64). It appears in fact that between the two waves, *i.e.*, when the whole of the region T lies between the spheres S_1 and S_2, we may write

$$(u,\ v,\ w) = \text{gradient of } \frac{1}{4\pi}\iiint\left\{(u_0+\dot{u}_0 t)\frac{\partial r^{-1}}{\partial x}+(v_0+\dot{v}_0 t)\frac{\partial r^{-1}}{\partial y}\right.$$
$$\left.+(w_0+\dot{w}_0 t)\frac{\partial r^{-1}}{\partial z}\right\}dV, \qquad (65)$$

where the integration extends through the volume T; so that (u, v, w) is derived from a potential which satisfies Laplace's equation and involves t linearly.

In general, it is clear that the terms of (64) which contain volume integrals correspond with the terms of (14) which contain $\int_{r/a}^{r/b} t'\chi(t-t')dt'$; so that they represent the effects which have been described in Articles 10 and 17 as being propagated with velocities intermediate between a and b.

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[*] Paris, *Mém. de l'Acad.*, t. x. (1831). An account of Poisson's memoir written by himself is contained in *Annales de Chimie et de Physique*, t. xliv. (1830).

[†] St. Petersburg, *Mém. de l'Acad.*, t. i. (1831). Ostrogradsky's results are recorded also in a note in Poisson's memoir above cited.

[‡] Cf. Todhunter and Pearson, *History of the Theory of Elasticity*, Vol. i., pp. 627–634 (Cambridge, 1886).

26. *The Dilatation and the Rotation.*—Stokes's doctrine of the irrotational and equivoluminal character of the two waves depends upon his assumed formulæ for the dilatation and rotation, but its correctness can be seen at once without assuming these formulæ. We observe in fact that the right-hand members of (58) would be harmonic functions if the boundaries of the volumes of integration were invariable, and it follows that the expressions for $-\nabla^2\phi$, $-\nabla^2 F$, ..., *i.e.*, for Δ, $2\varpi'$, ... can contain no volume integral terms. Further, all the terms that can occur in $-\nabla^2\phi$ will arise from the variability of the boundary of V_1, and hence we see that the dilatation will be expressed by integrals taken over S_1 and s_1 only. In like manner the rotation will be expressed by integrals taken over S_2 and s_2 only. It follows that the dilatation will be confined to the wave that travels with velocity a, and the rotation to the wave that travels with velocity b.

In forming an expression for Δ by help of the rules given in Articles 20 and 21, we attend in the first instance to the term of ϕ that contains $\dot{u}_0$. It contributes to $-\partial\phi/\partial x$ the terms

$$-\frac{t}{4\pi}\iiint \dot{u}_0 \frac{\partial^2 r^{-1}}{\partial x^2} dV_1 + \frac{t}{4\pi}\iint \frac{(x'-x)^2}{r^4} \dot{u}_0 dS_1,$$

and to $-\partial\phi/\partial y$ the terms

$$-\frac{t}{4\pi}\iiint \dot{u}_0 \frac{\partial^2 r^{-1}}{\partial x\,\partial y} dV_1 + \frac{t}{4\pi}\iint \frac{(x'-x)(y'-y)}{r^4} \dot{u}_0 dS_1,$$

and it contributes similar terms to $-\partial\phi/\partial z$. The terms contributed to $-\partial^2\phi/\partial x^2$ are

$$\begin{aligned}
&-\frac{t}{4\pi}\iiint \dot{u}_0 \frac{\partial^3 r^{-1}}{\partial x^3} dV_1 \\
&+\frac{t}{4\pi}\iint \left[\dot{u}_0 \left\{\frac{x'-x}{r}\,\frac{\partial^2 r^{-1}}{\partial x^2} + \frac{\partial}{\partial x}\,\frac{(x'-x)^2}{r^4}\right\} + \frac{\partial \dot{u}_0}{\partial r}\,\frac{(x'-x)^3}{r^5}\right] dS_1 \\
&-\frac{t}{4\pi}\int \frac{(x'-x)^2}{r^4}\,\frac{\cos(x, n)}{\sin\chi} \dot{u}_0 ds_1.
\end{aligned}$$

The terms contributed to $-\partial^2\phi/\partial y^2$ are

$$\begin{aligned}
&-\frac{t}{4\pi}\iiint \dot{u}_0 \frac{\partial^3 r^{-1}}{\partial x\,\partial y^2} dV_1 \\
&+\frac{t}{4\pi}\iint \left[\dot{u}_0 \left\{\frac{y'-y}{r}\,\frac{\partial^2 r^{-1}}{\partial x\,\partial y} + \frac{\partial}{\partial y}\,\frac{(x'-x)(y'-y)}{r^4}\right\} + \frac{\partial \dot{u}_0}{\partial r}\,\frac{(x'-x)(y'-y)^2}{r^5}\right] dS_1 \\
&-\frac{t}{4\pi}\int \frac{(x'-x)(y'-y)}{r^4}\,\frac{\cos(y, n)}{\sin\chi} \dot{u}_0 ds_1,
\end{aligned}$$

and similar terms are contributed to $-\partial^2\phi/\partial z^2$. Hence the contribution of $\dot{u}_0$ to $-\nabla^2\phi$ is found after a little reduction to be

$$\frac{t}{4\pi}\iint\frac{x'-x}{r^4}\left(2\dot{u}_0+r\frac{\partial\dot{u}_0}{\partial r}\right)dS_1-\frac{t}{4\pi}\int\frac{x'-x}{r^3}\,\frac{\cos(r,n)}{\sin\chi}\,\dot{u}_0\,ds_1.$$

It follows that the part of Δ which depends upon initial velocities is

$$\frac{t}{4\pi}\iint\left(\frac{2}{r^3}\,\dot{q}_0+\frac{1}{r^2}\,\frac{\partial\dot{q}_0}{\partial r}\right)dS_1-\frac{t}{4\pi}\int\frac{\cos(r,n)}{r^2\sin\chi}\,\dot{q}_0\,ds_1. \tag{66}$$

The part of Δ which depends upon initial displacements can be obtained by means of Stokes's rule, and it will be found that the complete expression for Δ is given by the equation

$$\Delta=\frac{t}{4\pi}\iint\left(\frac{2}{r^3}\,\dot{q}_0+\frac{1}{r^2}\,\frac{\partial\dot{q}_0}{\partial r}\right)dS_1+\frac{1}{4\pi}\iint\left(\frac{1}{r}\,\frac{\partial^2 q_0}{\partial r^2}+\frac{3}{r^2}\,\frac{\partial q_0}{\partial r}\right)dS_1$$
$$-\frac{1}{4\pi}\int\frac{\cos(r,n)}{r\sin\chi}\left(\frac{t\dot{q}_0}{r}+\frac{\partial q_0}{\partial r}\right)ds_1. \tag{67}$$

By precisely similar processes it may be shown that the components of rotation are expressed by equations of the type

$$2\varpi'=\frac{t}{4\pi}\iint\left\{\frac{y'-y}{r^3}\left(\frac{\partial\dot{w}_0}{\partial r}+\frac{2\dot{w}_0}{r}\right)-\frac{z'-z}{r^3}\left(\frac{\partial\dot{v}_0}{\partial r}+\frac{\dot{v}_0}{r}\right)\right\}dS_2$$
$$+\frac{1}{4\pi}\iint\left\{\frac{y'-y}{r^2}\left(\frac{\partial^2 w_0}{\partial r^2}+\frac{3}{r}\,\frac{\partial w_0}{\partial r}\right)-\frac{z'-z}{r^2}\left(\frac{\partial^2 v_0}{\partial r^2}+\frac{3}{r}\,\frac{\partial v_0}{\partial r}\right)\right\}dS_2$$
$$-\frac{1}{4\pi}\int\frac{\cos(r,n)}{r\sin\chi}\left\{\frac{y'-y}{r}\left(\frac{t\dot{w}_0}{r}+\frac{\partial w_0}{\partial r}\right)-\frac{z'-z}{r}\left(\frac{t\dot{v}_0}{r}+\frac{\partial v_0}{\partial r}\right)\right\}ds_2. \tag{68}$$

But the question of the reduction of the formulæ here obtained for the dilatation and the rotation to the forms assumed by Stokes requires a further examination.

27. *Dependence of the Dilatation and Rotation upon their Initial Values.*—To examine this question, we transform the line integrals in (67) and (68). For this purpose, we observe that the direction cosines dx'/ds_1, ...

of the tangent to s_1 are given by equations of the type

$$\frac{dx'}{ds_1} = \frac{1}{\sin\chi}\left\{\frac{y'-y}{r}\cos(z, n) - \frac{z'-z}{r}\cos(y, n)\right\}, \tag{69}$$

from which we obtain three equations of the type

$$\cos(r, n)\frac{x'-x}{r} - \cos(x, n) = \sin\chi\left(\frac{y'-y}{r}\frac{dz'}{ds_1} - \frac{z'-z}{r}\frac{dy'}{ds_1}\right). \tag{70}$$

From these we deduce the equation

$$\begin{aligned}\cos(r, n)\,\dot{q}_0 = {} & \dot{u}_0\cos(x, n) + \dot{v}_0\cos(y, n) + \dot{w}_0\cos(z, n) \\ & + \sin\chi\left\{\frac{dx'}{ds_1}\left(\dot{v}_0\frac{z'-z}{r} - \dot{w}_0\frac{y'-y}{r}\right) + \frac{dy'}{ds_1}\left(\dot{w}_0\frac{x'-x}{r} - \dot{u}_0\frac{z'-z}{r}\right)\right. \\ & \left. + \frac{dz'}{ds_1}\left(\dot{u}_0\frac{y'-y}{r} - \dot{v}_0\frac{x'-x}{r}\right)\right\},\end{aligned} \tag{71}$$

and hence we find, since r is constant along s_1 and over S_1,

$$\begin{aligned}\int\frac{\cos(r, n)}{r\sin\chi}\frac{t\dot{q}_0}{r}\,ds_1 = {} & t\int\{\dot{u}_0\cos(x, n) + \dot{v}_0\cos(y, n) + \dot{w}_0\cos(z, n)\}\frac{ds_1}{r^2\sin\chi} \\ & + t\iint\frac{1}{r^2}\left[\frac{x'-x}{r}\left\{\frac{\partial}{\partial y'}\left(\dot{u}_0\frac{y'-y}{r} - \dot{v}_0\frac{x'-x}{r}\right)\right.\right. \\ & \left.\left. - \frac{\partial}{\partial z'}\left(\dot{w}_0\frac{x'-x}{r} - \dot{u}_0\frac{z'-z}{r}\right)\right\} + \text{two similar terms}\right] dS_1,\end{aligned} \tag{72}$$

where the surface integration extends over the part of the surface S_1 which is within the curve s_1, *i.e.*, over the part which is within T.

The surface integral in (72) is found, after some algebraic rearrangement, to be equal to

$$\iint\left(\frac{2}{r^3}\dot{q}_0 + \frac{1}{r^2}\frac{\partial\dot{q}_0}{\partial r} - \frac{1}{r^2}\dot{\Delta}_0\right)dS_1.$$

The line integral $\int\frac{\cos(r, n)}{r\sin\chi}\frac{\partial q_0}{\partial r}ds_1$ in (67) may be transformed in the same way, and it will be found finally that the expression (67) for Δ can be written

$$\begin{aligned}\Delta = {} & \frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{\Delta}_0 + \Delta_0 + r\frac{\partial\Delta_0}{\partial r}\right)dS_1 \\ & - \frac{1}{4\pi}\int\left\{\left(\frac{t\dot{u}_0}{r} + \frac{\partial u_0}{\partial r}\right)\cos(x, n) + \left(\frac{t\dot{v}_0}{r} + \frac{\partial v_0}{\partial r}\right)\cos(y, n)\right. \\ & \left. + \left(\frac{t\dot{w}_0}{r} + \frac{\partial w_0}{\partial r}\right)\cos(z, n)\right\}\frac{ds_1}{r\sin\chi}.\end{aligned} \tag{73}$$

Exactly similar transformations can be effected in the expression (68) for $2\varpi'$, and the result will be found to be

$$2\varpi' = \frac{1}{2\pi}\iint\frac{1}{r^2}\left(t\dot{\varpi}_0'+\varpi_0'+r\frac{\partial\varpi_0'}{\partial r}\right)dS_2$$
$$-\frac{1}{4\pi}\int\left\{\left(\frac{t\dot{w}_0}{r}+\frac{\partial w_0}{\partial r}\right)\cos(y,\,n)-\left(\frac{t\dot{v}_0}{r}+\frac{\partial v_0}{\partial r}\right)\cos(z,\,n)\right\}\frac{ds_2}{r\sin\chi}. \qquad (74)$$

Similar expressions can be obtained for $2\varpi''$ and $2\varpi'''$. The terms of (73) and (74) which are expressed by line integrals admit of a further transformation by the aid of the relations (50). Taking the terms in the line integral in (73) which contain $\partial u_0/\partial r$, we observe that

$$\frac{\partial u_0}{\partial r}\cos(x,\,n) = \left(\frac{x'-x}{r}\frac{\partial u_0}{\partial x'}+\frac{y'-y}{r}\frac{\partial u_0}{\partial y'}+\frac{z'-z}{r}\frac{\partial u_0}{\partial z'}\right)\cos(x,\,n)$$
$$= \frac{\partial u_0}{\partial x'}\cos(r,\,n)+\frac{y'-y}{r}\left\{\frac{\partial u_0}{\partial y'}\cos(x,\,n)-\frac{\partial u_0}{\partial x'}\cos(y,\,n)\right\}$$
$$+\frac{z'-z}{r}\left\{\frac{\partial u_0}{\partial z'}\cos(x,\,n)-\frac{\partial u_0}{\partial x'}\cos(z,\,n)\right\},$$

and the last two terms in the right-hand member can be omitted, since they vanish at all points of Σ_0, and therefore at all points of s_1. Hence the terms of (73) which depend on the initial strain near to Σ_0 can be expressed in the form $-\dfrac{1}{4\pi}\displaystyle\int\Delta_0\cos(r,\,n)\frac{ds_1}{r\sin\chi}$. In like manner, taking the terms of the line integral in (74) which involve $\partial w_0/\partial r$ and $\partial v_0/\partial r$, we may prove that at all points of s_2

$$\frac{\partial w_0}{\partial r}\cos(y,\,n)-\frac{\partial v_0}{\partial r}\cos(z,\,n) = \left(\frac{\partial w_0}{\partial y'}-\frac{\partial v_0}{\partial z'}\right)\cos(r,\,n),$$

and therefore the terms of (74) which depend on the initial strain near to Σ_0 can be expressed in the form $-\dfrac{1}{2\pi}\displaystyle\int\varpi_0'\cos(r,\,n)\frac{ds_2}{r\sin\chi}$.

We may now write the formulæ for Δ and $2\varpi'$ in the forms

$$\Delta = \frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{\Delta}_0+\Delta_0+r\frac{\partial\Delta_0}{\partial r}\right)dS_1-\frac{1}{4\pi}\int\frac{\cos(r,\,n)}{r\sin\chi}\Delta_0\,ds_1$$
$$-\frac{t}{4\pi}\int\{\dot{u}_0\cos(x,\,n)+\dot{v}\cos(y,\,n)+\dot{w}_0\cos(z,\,n)\}\frac{ds_1}{r^2\sin\chi}, \qquad (75)$$

$$2\varpi' = \frac{1}{2\pi}\iint\frac{1}{r^2}\left(t\dot{\varpi}_0'+\varpi_0'+r\frac{\partial\varpi_0'}{\partial r}\right)dS_2-\frac{1}{2\pi}\int\frac{\cos(r,\,n)}{r\sin\chi}\varpi_0'\,ds_2$$
$$-\frac{t}{4\pi}\int\{\dot{w}_0\cos(y,\,n)-\dot{v}_0\cos(z,\,n)\}\frac{ds_2}{r^2\sin\chi}. \qquad (76)$$

It appears from these results that the values assumed by Stokes for the dilatation and the components of rotation do, in fact, require correction whenever there is initial velocity at the surface Σ_0. The first line of (75) is the value that would be given for the dilatation by using Poisson's integral formula; for it is precisely of the same form as equation (60), and and the like holds for (76). The corrections that must be made in Stokes's values are expressed by the second lines of (75) and (76). The correction of the expression for the dilatation depends exclusively upon the normal component of the initial velocity at Σ_0, and the correction of the expression for the rotation depends exclusively upon the components of the initial velocity tangential to Σ_0, as was stated by anticipation in Article 23.

28. *Dependence of the Displacement upon Initial Dilatation.*—The dependence of the motion in the waves of dilatation and rotation upon initial conditions can be shown still more explicitly by transforming the expressions for the components of displacement. The volume integral term of (58) may be written

$$\frac{t}{4\pi}\iiint\left(\dot{u}_0\frac{\partial^2 r^{-1}}{\partial x'^2}+\dot{v}_0\frac{\partial^2 r^{-1}}{\partial x'\partial y'}+\dot{w}_0\frac{\partial^2 r^{-1}}{\partial x'\partial z'}\right)dV,$$

and this is equal to

$$-\frac{t}{4\pi}\iiint\left\{\frac{\partial}{\partial x'}\left(\dot{u}_0\frac{x'-x}{r^3}\right)+\frac{\partial}{\partial y'}\left(\dot{v}_0\frac{x'-x}{r^3}\right)+\frac{\partial}{\partial z'}\left(\dot{w}_0\frac{x'-x}{r^3}\right)-\frac{x'-x}{r^3}\dot{\Delta}_0\right\}dV,$$

which may be transformed into

$$\begin{aligned}&\frac{t}{4\pi}\iiint\frac{x'-x}{r^3}\dot{\Delta}_0\,dV\\&-\frac{t}{4\pi}\iint\frac{x'-x}{r^3}\left\{\dot{u}_0\cos(x,n)+\dot{v}_0\cos(y,n)+\dot{w}_0\cos(z,n)\right\}d\Sigma_0\\&-\frac{t}{4\pi}\iint\frac{x'-x}{r^3}\dot{q}_0\,dS_1+\frac{t}{4\pi}\iint\frac{x'-x}{r^3}\dot{q}_0\,dS_2,\end{aligned}\tag{77}$$

where the integration expressed by $\iint\ldots d\Sigma_0$ extends over that part of Σ_0 which lies between S_1 and S_2. The volume integral in (62) may in like manner be transformed into

$$\frac{1}{4\pi}\iiint\frac{x'-x}{r^3}\Delta_0\,dV-\frac{1}{4\pi}\iint\frac{x'-x}{r^3}q_0\,dS_1+\frac{1}{4\pi}\iint\frac{x'-x}{r^3}q_0\,dS_2,\tag{78}$$

where there is no $\iint\ldots d\Sigma_0$ because u_0, v_0, w_0 vanish at Σ_0.

The surface integrals in (62) also admit of transformation. We have

$$u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}+\frac{\partial r^{-1}}{\partial x}\,\frac{\partial q_0}{\partial r}$$
$$=-\frac{\partial}{\partial x'}\left(\frac{q_0}{r^2}\right)+\frac{x'-x}{r^3}\,\frac{\partial q_0}{\partial r}+\frac{x'-x}{r^3}\,\frac{\partial u_0}{\partial x'}+\frac{y'-y}{r^3}\,\frac{\partial v_0}{\partial x'}+\frac{z'-z}{r^3}\,\frac{\partial w_0}{\partial x'},$$

which is equal to

$$-\frac{\partial}{\partial x'}\left(\frac{q_0}{r^2}\right)+\frac{x'-x}{r^3}\,\frac{\partial q_0}{\partial r}+\frac{x'-x}{r^3}\,\Delta_0+\left(\frac{y'-y}{r^3}\,\frac{\partial v_0}{\partial x'}-\frac{x'-x}{r^3}\,\frac{\partial v_0}{\partial y'}\right)$$
$$+\left(\frac{z'-z}{r^3}\,\frac{\partial w_0}{\partial x'}-\frac{x'-x}{r^3}\,\frac{\partial w_0}{\partial z'}\right),$$

and this becomes after some reduction

$$\frac{x'-x}{r^3}\,\Delta_0+\frac{1}{r^2}\left\{-\frac{x'-x}{r}\left(\frac{\partial v_0}{\partial y'}+\frac{\partial w_0}{\partial z'}\right)+\frac{y'-y}{r}\,\frac{\partial v_0}{\partial x'}+\frac{z'-z}{r}\,\frac{\partial w_0}{\partial x'}\right\}$$
$$+\frac{x'-x}{r}\left\{\frac{\partial}{\partial y'}\left(\frac{y'-y}{r^3}\,q_0\right)+\frac{\partial}{\partial z'}\left(\frac{z-z'}{r^3}\,q_0\right)\right\}$$
$$-\frac{y'-y}{r}\,\frac{\partial}{\partial x'}\left(\frac{y'-y}{r^3}\,q_0\right)-\frac{z'-z}{r}\,\frac{\partial}{\partial x'}\left(\frac{z'-z}{r^3}\,q_0\right). \tag{79}$$

Now the integral

$$\iint\frac{1}{r}\left\{-\frac{x'-x}{r}\left(\frac{\partial v_0}{\partial y'}+\frac{\partial w_0}{\partial z'}\right)+\frac{y'-y}{r}\,\frac{\partial v_0}{\partial x'}+\frac{z'-z}{r}\,\frac{\partial w_0}{\partial x'}\right\}dS_1$$
$$=\frac{1}{at}\int\left(w_0\frac{dy'}{ds_1}-v_0\frac{dz'}{ds_1}\right)ds_1,$$

which vanishes because (u_0, v_0, w_0) vanishes along s_1. In like manner the surface integral of the same quantity taken over S_2 vanishes, and the integral of the last three terms of (79) when taken over S_1 or S_2 also vanishes. Collecting the results, we obtain the equation

$$u=\frac{1}{4\pi}\iiint\frac{x'-x}{r^3}(t\dot{\Delta}_0+\Delta_0)\,dV+\frac{1}{4\pi}\iint\frac{x'-x}{r^2}\,\Delta_0\,dS_1-\frac{1}{4\pi}\iint\frac{x'-x}{r^2}\,\Delta_0\,dS_2$$
$$-\frac{t}{4\pi}\iint\frac{x'-x}{r^3}\left\{\dot{u}_0\cos(x,n)+\dot{v}_0\cos(y,n)+\dot{w}_0\cos(z,n)\right\}d\Sigma_0$$
$$+\frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{u}+u_0+r\frac{\partial u_0}{\partial r}\right)dS_2. \tag{80}$$

This equation shows the dependence of the motion in the dilatational wave, and between the two waves, upon the initial dilatation and velocity of dilatation, and upon the normal component of the initial velocity at the boundary Σ_0 of T. The equation shows also that when there is initially no dilatation or velocity of dilatation, and when the normal component of the initial velocity at the surface Σ_0 vanishes, the rotational wave alone is propagated, and the displacement in it is expressed in terms of the initial displacement and velocity by means of Poisson's integral formula, as we know it ought to be. In this case the velocity with which the boundary of the disturbed portion of the medium advances is the velocity b.

29. *Dependence of the Displacement upon Initial Rotation.*—To obtain like results involving the initial rotation, we write

$$\dot{u}_0\frac{\partial^2 r^{-1}}{\partial x^2}+\dot{v}_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+\dot{w}_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}$$

in the form $-\dfrac{\partial}{\partial x'}\left(\dfrac{\dot{q}_0}{r^2}\right)+\dfrac{x'-x}{r^3}\;\dfrac{\partial\dot{u}_0}{\partial x'}+\dfrac{y'-y}{r^3}\;\dfrac{\partial\dot{v}_0}{\partial x'}+\dfrac{z'-z}{r^3}\;\dfrac{\partial\dot{w}_0}{\partial x'}$,

and observe that this is equal to

$$-\frac{\partial}{\partial x'}\left(\frac{\dot{q}_0}{r^2}\right)+\frac{\partial}{\partial x'}\left(\frac{x'-x}{r^3}\,\dot{u}_0\right)+\frac{\partial}{\partial y'}\left(\frac{y'-y}{r^3}\,\dot{u}_0\right)+\frac{\partial}{\partial z'}\left(\frac{z'-z}{r^3}\,\dot{u}_0\right)$$
$$+2\,\frac{y'-y}{r^3}\,\dot{\varpi}_0'''-2\,\frac{z'-z}{r^3}\,\dot{\varpi}_0''.$$

A similar result holds when the dots are omitted, and we may therefore transform the volume integral terms of (64) into

$$\frac{1}{2\pi}\iiint\left\{\frac{y'-y}{r^3}\,(t\dot{\varpi}_0'''+\varpi_0''')-\frac{z'-z}{r^3}\,(t\dot{\varpi}_0''+\varpi_0'')\right\}dV$$
$$+\frac{1}{4\pi}\iint\left\{\frac{\dot{u}_0t+u_0}{r^2}-\frac{x'-x}{r^3}\,(\dot{q}_0t+q_0)\right\}dS_1$$
$$-\frac{1}{4\pi}\iint\left\{\frac{\dot{u}_0t+u_0}{r^2}-\frac{x'-x}{r^3}\,(\dot{q}_0t+q_0)\right\}dS_2$$
$$+\frac{t}{4\pi}\iint\left[\frac{y'-y}{r^3}\left\{\dot{u}_0\cos(y,n)-\dot{v}_0\cos(x,n)\right\}\right.$$
$$\left.+\frac{z'-z}{r^3}\left\{\dot{u}_0\cos(z,n)-\dot{w}_0\cos(x,n)\right\}\right]d\Sigma_0, \quad (81)$$

where the integration expressed by $\iint\ldots d\Sigma_0$ has the same meaning as before.

Also we found

$$\iint r\left(u_0\frac{\partial^2 r^{-1}}{\partial x^2}+v_0\frac{\partial^2 r^{-1}}{\partial x\,\partial y}+w_0\frac{\partial^2 r^{-1}}{\partial x\,\partial z}+\frac{\partial r^{-1}}{\partial x}\,\frac{\partial q_0}{\partial r}\right)dS_1=\iint\frac{x'-x}{r^2}\Delta_0\,dS_1, \tag{82}$$

and a like formula holds when the integration is taken over the part of S_2 that is within T. Further we have

$$\frac{x'-x}{r^2}\Delta_0-\frac{1}{r}\,\frac{\partial u_0}{\partial r}$$

$$=2\left(\frac{y'-y}{r^2}\varpi_0'''-\frac{z'-z}{r^2}\varpi_0''\right)+\left[\frac{x'-x}{r^2}\left(\frac{\partial v_0}{\partial y'}+\frac{\partial w_0}{\partial z'}\right)-\frac{y'-y}{r^2}\,\frac{\partial v_0}{\partial x'}-\frac{z'-z}{r^2}\,\frac{\partial w_0}{\partial x'}\right], \tag{83}$$

and the integral of the terms in square brackets over the part of S_1 or S_2 that is within T vanishes. We may therefore write the expression for u in the form

$$\begin{aligned}u=\;&\frac{1}{2\pi}\iiint\left\{\frac{y'-y}{r^3}(t\dot{\varpi}_0'''+\varpi_0''')-\frac{z'-z}{r^3}(t\dot{\varpi}_0''+\varpi_0'')\right\}dV\\&+\frac{1}{2\pi}\iint\left(\frac{y'-y}{r^2}\varpi_0'''-\frac{z'-z}{r^2}\varpi_0''\right)dS_1-\frac{1}{2\pi}\iint\left(\frac{y'-y}{r^2}\varpi_0'''-\frac{z'-z}{r^2}\varpi_0''\right)dS_2\\&+\frac{t}{4\pi}\iint\left[\frac{y'-y}{r^3}\left\{\dot{u}_0\cos(y,n)-\dot{v}_0\cos(x,n)\right\}\right.\\&\qquad\qquad\left.+\frac{z'-z}{r^3}\left\{\dot{u}_0\cos(z,n)-\dot{w}_0\cos(x,n)\right\}\right]d\Sigma_0\\&+\frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{u}_0+u_0+r\frac{\partial u_0}{\partial r}\right)dS_1.\end{aligned} \tag{84}$$

This equation shows the dependence of the motion in the rotational wave, and between the two waves, upon the initial rotation and velocity of rotation, and upon the tangential components of the initial velocity at the boundary Σ_0 of T. It shows also that, if there is no initial rotation or velocity of rotation, and if the tangential components of the initial velocity at the surface Σ_0 vanish, the dilatational wave alone is propagated, and the displacement in it is expressed in terms of the initial displacement and velocity by means of Poisson's integral formula, as we know it ought to be. In this case the velocity with which the boundary of the disturbed portion of the medium advances is the velocity a.

30. *Discontinuities at Wave-Fronts.*—Expressions for the components of velocity, strain, and stress at any point may be formed in the same

manner as the expressions for the dilatation and the components of the rotation, by differentiating the expressions such as (64) for the components of displacement. The possibility of non-vanishing values for the velocity, dilatation, rotation, &c., at the boundary of an advancing wave is bound up with the occurrence of line integrals in the expressions for these quantities, but the magnitudes of these quantities at such wave-fronts cannot be determined by the formulæ already obtained, because, when the point (x, y, z) is on a wave-front at time t, one of the spheres S_1, S_2 touches the surface Σ_0, and thus the formulæ (56) and (57) obtained in Article 21 become nugatory. We have to determine the forms of the differential coefficients of $\iint \psi\, dS_1$, say, at an instant when S_1 touches Σ_0.

We take as the standard case that in which the point O or (x, y, z) is outside T, and the sphere of radius at with this point as centre touches Σ_0 at the point P, which is the nearest point of Σ_0 to (x, y, z). [Cf. Fig. 6 in Article 32.] Then the sphere of radius $a(t+\delta t)$ with its centre at the same point will cut Σ_0 in a small closed curve lying near to the point P. We take temporary axes of x_1, y_1, z_1, so that the origin is at the point P, and the axis of z_1 in the positive sense is drawn from P to O. The equation of Σ_0 near to P may be written

$$2z_1 = \frac{x_1^2}{\rho_1} + \frac{y_1^2}{\rho_2}, \tag{85}$$

where ρ_1 and ρ_2 are the principal radii of curvature of Σ_0 at P, and these are estimated as positive when the z_1-coordinates of the corresponding centres of curvature are positive. In the figure ρ_1 and ρ_2 are negative. The equation of S_1 is $x_1^2+y_1^2+(z_1-r)^2 = r^2$; and that of the sphere of radius $a(t+\delta t)$ is given, with sufficient approximation, by changing r^2 on the right-hand side to $r^2+2ar\,\delta t$. This sphere cuts Σ_0 in a curve of which the projection on the plane of (x_1, y_1) is given, with sufficient approximation, by the equation

$$x_1^2+y_1^2-r\left(\frac{x_1^2}{\rho_1}+\frac{y_1^2}{\rho_2}\right) = 2ar\,\delta t; \tag{86}$$

and the area of the part of the sphere of radius $a(t+\delta t)$ which is included within this curve is, with sufficient approximation, equal to

$$\left\{\left(1-\frac{r}{\rho_1}\right)\left(1-\frac{r}{\rho_2}\right)\right\}^{-\frac{1}{2}} 2\pi ar\,\delta t.$$

The quantity is real in the standard case, and thus we find the progressive differential coefficient of $\iint \psi\, dS_1$ with respect to t at the instant of contact in the form

$$\frac{\partial}{\partial t}\iint \psi\, dS_1 = \frac{2\pi a r \psi(P)}{\{(1-r/\rho_1)(1-r/\rho_2)\}^{\frac{1}{2}}}. \tag{87}$$

This formula may be extended to any case in which the quantity in the denominator is real: for example, it may be extended to the case where P is the most distant point of Σ_0 from (x, y, z). In that case the sphere of radius $a(t+\delta t)$ would cut Σ_0 if δt were negative, and the differential coefficient is then regressive. A similar formula may be obtained in cases where S_2 touches Σ_0.

Slight changes in the details of the argument are required in determining the effect of contact on the spatial differential coefficients of the surface integrals. Taking, for example, the standard case of contact of S_1 and Σ_0, the shifting of the sphere S_1 through a positive δx may result in the surfaces cutting one another, and then there is a progressive differential coefficient of $\iint \psi\, dS_1$ with respect to x, and it may be proved that

$$\frac{\partial}{\partial x}\iint \psi\, dS_1 = -\frac{2\pi r \cos(x, n)\psi(P)}{\{(1-r/\rho_1)(1-r/\rho_2)\}^{\frac{1}{2}}}, \tag{88}$$

and a like formula may be obtained for $\iint \psi\, dS_2$.

By means of the formulæ which have now been obtained, we may form expressions for the components of velocity, strain, and stress, at points on the fronts and rears of the waves of dilatation and rotation, and we may verify that equations of the types of (1) and (2) in Article 3 are satisfied at the front of the wave of dilatation and at the rear of the wave of rotation, and that equations of the types of (3) and (4) are satisfied at the rear of the wave of dilatation and at the front of the wave of rotation.

31. *Tendency to the Formation of Surfaces of Discontinuity.*—The reality of the denominator in the two expressions (87) and (88) is secured if the point of contact is the nearest point of Σ_0 to (x, y, z) or the most distant point of Σ_0 from (x, y, z); for in these two cases the sphere (S_1 or S_2) cannot cut Σ_0, but merely touches it. In these cases r is less than the less or greater than the greater of ρ_1 and ρ_2, when these are positive. Here r denotes the distance of a point (x, y, z) from a point on the surface Σ_0, and the line joining the points is normal to the surface. If we observe the variations in the formulæ (87) and (88) when the point (x, y, z) moves along a normal to the surface Σ_0, we see that, as (x, y, z) approaches a centre of principal

curvature of Σ_0, the right-hand members of (87) and (88) tend to become infinite, and that, in passing from one side to the other of the centre of curvature, a discontinuity is encountered. This tendency to discontinuity, leading to infinite velocities and infinite strains, and therefore to rupture of the material, is confined to the immediate neighbourhood of the centro-surface of Σ_0. It is of the same nature as the tendency to infinite pressure at the centre of spherical waves of sound. In the latter case we know that the tendency may be counteracted by a proper adjustment of initial conditions, and it is important to ascertain the conditions which are necessary in the present problem in order that discontinuity of this kind may be avoided.

32. *Conditions to be satisfied at the Initial Boundary.*—To avoid the tendency to discontinuity which has just been explained, it is sufficient that the quantity which has to be integrated over the small part of S_1 or S_2, near its point of contact with Σ_0, should vanish when the centre of S_1 and S_2 is one of the centres of principal curvature of Σ_0 at the point of contact. Now we are concerned with the differential coefficients of u, v, w with respect to x, y, z, t, and the quantity which has to be integrated over S_1 or S_2 may be simplified by omitting the terms that contain u_0, v_0, w_0, q_0, since these quantities vanish at Σ_0. The following conditions are sufficient, and are also in general necessary, to secure that no discontinuity shall arise :—

(i.) At a point of contact of Σ_0 with a sphere S_1 having its centre at a centre of principal curvature of Σ_0 the quantity $t\dot{q}_0+r\,\partial q_0/\partial r$ vanishes.

(ii.) At a point of contact of Σ_0 with a sphere S_2 having its centre at a centre of principal curvature of Σ_0 the three quantities of the type

$$\left(t\dot{q}_0+r\frac{\partial q_0}{\partial r}\right)\frac{\partial r^{-1}}{\partial x}-\frac{1}{r^2}\left(t\dot{u}_0+r\frac{\partial u_0}{\partial r}\right)$$

vanish.

These conditions hold, of course, for different values of t.

Now, we have, when $r = at$,

$$t\dot{q}_0+r\frac{\partial q_0}{\partial r} = r\left\{\left(\frac{\dot{u}_0}{a}+\frac{\partial u_0}{\partial r}\right)\cos(r,x)+\left(\frac{\dot{v}_0}{a}+\frac{\partial v_0}{\partial r}\right)\cos(r,y)\right.$$

$$\left.+\left(\frac{\dot{w}_0}{a}+\frac{\partial w_0}{\partial r}\right)\cos(r,z)\right\};$$

and, when (x, y, z) is a centre of curvature of Σ_0 at (x', y', z'), the direction of r is either the same as that of n or opposite to it. To fix ideas, we may take Σ_0 to be a single closed surface which is everywhere convex; then, at any contact of S_1 with Σ_0 in which at can be equal to ρ_1 or ρ_2, the direction of r is necessarily that of n, and we must have

$$\left(\frac{\dot{u}_0}{a}+\frac{\partial u_0}{\partial n}\right)\cos(n, x)+\left(\frac{\dot{v}_0}{a}+\frac{\partial v_0}{\partial n}\right)\cos(n, y)$$
$$+\left(\frac{\dot{w}_0}{a}+\frac{\partial w_0}{\partial n}\right)\cos(n, z)=0.$$

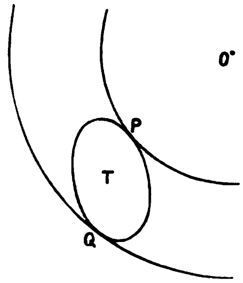

FIG. 6.

On introducing the results (50) we find that this equation becomes

$$\dot{u}_0\cos(x, n)+\dot{v}_0\cos(y, n)+\dot{w}_0\cos(z, n)=-a\Delta_0; \tag{89}$$

so that the initial velocity in the direction of the normal at any point of Σ_0 must have the value here given in terms of the initial dilatation at the point.

Again we have, when $r=bt$,

$$\left[\left(t\dot{q}+r\frac{\partial q_0}{\partial r}\right)\frac{\partial r^{-1}}{\partial x}-\frac{1}{r^2}\left(t\dot{u}_0+r\frac{\partial u_0}{\partial r}\right)\right]$$
$$=\frac{(x'-x)^2}{r^3}\left(\frac{\dot{u}_0}{b}+\frac{\partial u_0}{\partial r}\right)+\frac{(x'-x)(y'-y)}{r^3}\left(\frac{\dot{v}_0}{b}+\frac{\partial v_0}{\partial r}\right)$$
$$+\frac{(x'-x)(z'-z)}{r^3}\left(\frac{\dot{w}_0}{b}+\frac{\partial w_0}{\partial r}\right)-\left(\frac{\dot{u}_0}{b}+\frac{\partial u_0}{\partial r}\right)\left(\frac{(x'-x)^2}{r^3}+\frac{(y'-y)^2}{r^3}+\frac{(z'-z)^2}{r^3}\right)$$
$$=\frac{y'-y}{r^2}\left\{\frac{x'-x}{r}\left(\frac{\dot{v}_0}{b}+\frac{\partial v_0}{\partial r}\right)-\frac{y'-y}{r}\left(\frac{\dot{u}_0}{b}+\frac{\partial u_0}{\partial r}\right)\right\}$$
$$+\frac{z'-z}{r^2}\left\{\frac{x'-x}{r}\left(\frac{\dot{w}_0}{b}+\frac{\partial w_0}{\partial r}\right)-\frac{z'-z}{r}\left(\frac{\dot{u}_0}{b}+\frac{\partial u_0}{\partial r}\right)\right\};$$

and, to make all the expressions of this type vanish, when n and r have either the same direction or the opposite direction, we must have

$$\frac{\dot{u}_0}{b}+\frac{\partial u_0}{\partial r}:\frac{\dot{v}_0}{b}+\frac{\partial v_0}{\partial r}:\frac{\dot{w}_0}{b}+\frac{\partial w_0}{\partial r}=\cos(n, x):\cos(n, y):\cos(n, z).$$

On introducing the results (50), and taking the directions of n and r to be the same, we obtain three equations of the type

$$\dot{v}_0\cos(n, z)-\dot{w}_0\cos(n, y)=2b\varpi'_0; \tag{90}$$

so that the tangential components of the initial velocity at points on Σ_0 must be connected with the initial rotation by relations of the type (90).

The equations (89) and (90) hold when r and n are in the same direction; when they are opposite in direction the signs of the left-hand members must be changed. The geometrical relations involved may be summed up as follows:—Take any point O on the centro-surface of Σ_0, and draw from it the normal to Σ_0 at that point (P) of Σ_0 for which O is a centre of principal curvature: when this normal, in this sense, passes at P from the region T within Σ_0 to the region outside Σ_0, r and n have the same sense; otherwise they have the opposite sense.

It follows that, if Σ_0 is synclastic near a point P, one or other of the two relations

$$\pm\,\{\dot{u}_0\cos(x,n)+\dot{v}_0\cos(y,n)+\dot{w}_0\cos(z,n)\} = -a\Delta_0 \qquad (91)$$

holds at P, and also one or other of the two sets of relations of the type

$$\pm\,\{\dot{v}_0\cos(z,n)-\dot{w}_0\cos(y,n)\} = 2b\varpi'_0, \qquad (92)$$

and the ambiguous sign is the same for all these relations, viz., it is + when the normal drawn from the centre of curvature passes from within T to outside T. But when the surface Σ_0 is anticlastic near P the relations of type (91) and those of type (92) must in general hold with both signs. In the latter case (anticlastic) the initial velocity must vanish at P and likewise the initial dilatation and rotation must vanish at P. In the case where Σ_0 is synclastic at P the initial velocity, dilatation, and rotation need not vanish at P; but the normal component of the velocity must be "compensated" by the initial dilatation, and the tangential components of the velocity must be "compensated" by the initial rotation.

33. *Application of these conditions to the case of Spherical Waves of Sound.*—It seems worth while to illustrate the necessity of the conditions of type (91) and (92) by reference to the problem of spherical waves of sound. In the simplest case which can serve as an illustration, the motion is confined initially to the volume within a sphere Σ_0, and the motion is symmetrical about the centre of the sphere. Let C be the centre of Σ_0 and c its radius, and let ϕ be the velocity potential of the motion and a the velocity of sound. Then ϕ is a function of r, the distance of any point from C, and of t; and we must have

$$\phi = r^{-1}\{f(at-r)+F(at+r)\}, \qquad (93)$$

where f and F are functions which, for positive arguments, are connected by the equation

$$f(\zeta)+F(\zeta) = 0; \qquad (94)$$

for this is the condition that the centre C may not act as a source.

In the initial state the velocity, $\partial\phi/\partial r$, and the condensation, $-a^{-2}\partial\phi/\partial t$, are given functions of r in the region $r \leqslant c$, and vanish in the region $r > c$. Let ϕ_0 be the initial value of ϕ and $\dot{\phi}_0$ that of $\partial\phi/\partial t$. From the nature of the problem $\partial\phi_0/\partial r$ must vanish at the centre C, and, since ϕ_0 is indeterminate to the extent of an additive arbitrary constant when $\partial\phi_0/\partial r$ is given, we may take ϕ_0 to vanish at $r = c$. To simplify the problem as much as possible we shall suppose that in the region $r \leqslant c$ the functions $\partial\phi_0/\partial r$ and $\dot{\phi}_0$ are analytic and without singularities. We may take ϕ_0 to be determined for all positive values of r as a function $\psi(r)$ which has the following properties:—

(i.) $\psi(r)$ is analytic and without singularities in the region $r \leqslant c$,

(ii.) $\psi(r) = 0$ in the region $r > c$,

(iii.) $\psi'(r) = 0$ at $r = 0$,

(iv.) $\psi(r) = 0$ at $r = c$.

In like manner we may take $\dot{\phi}_0$ to be determined for all positive values of r as a function $\Psi_1(r)$ which has the following properties:—

(i.) $\Psi_1(r)$ is analytic and without singularities in the region $r \leqslant c$.
(ii.) $\Psi_1(r) = 0$ in the region of $r > c$.

It is convenient to introduce a new function $\Psi(r)$ by the definitions

$$\text{(i.)}\quad \Psi(r) = -\int_r^c \zeta\Psi_1(\zeta)\,d\zeta \quad \text{in the region } r \leqslant c,$$

$$\text{(ii.)}\quad \Psi(r) = 0 \qquad\qquad \text{in the region } r > c.$$

Then $\dot{\phi}_0 = r^{-1}\Psi'(r)$, and $\Psi'(r)$ vanishes at $r = 0$. At $r = c$ the derivatives $\psi'(c)$ and $\Psi'(c)$ must be understood to be *regressive*, and at $r = 0$ the derivatives $\psi'(0)$ and $\Psi'(0)$ must be understood to be *progressive*.

The functions f and F are now to be determined for all values of their arguments which can come into consideration by the equation (94) and the equations

$$F(r)+f(-r) = r\psi(r), \quad F'(r)+f'(-r) = a^{-1}\Psi'(r),$$

the first of which gives $F'(r)-f'(-r) = \psi(r)+r\psi'(r)$. For all positive values of ζ the functions $F'(\zeta)$ and $f'(\zeta)$ are given by the equations

$$F'(\zeta) = \tfrac{1}{2}[\psi(\zeta)+\zeta\psi'(\zeta)+a^{-1}\Psi'(\zeta)], \tag{95}$$

$$f'(\zeta) = -F'(\zeta), \tag{96}$$

of which the second is derived from (94).

Consider the value of the condensation at a point O near to C, and at an instant after the inwards travelling wave has reached the point O but before the motion at the point O has ceased. At such an instant we have $c+r > at > c-r$, and we may put, therefore, $at = c+r-2r\vartheta$, where ϑ is a positive number less than unity. At any point O within Σ_0, other than C, the interval in question is finite, and ϑ runs through all values from 1 to 0, although the interval diminishes indefinitely as O moves towards C. The condensation at O is $-(ar)^{-1}\{F''(at+r)+f''(at-r)\}$. Now $at+r = c+2r(1-\vartheta)$, which is $> c$, and thus $F'(at+r)$ vanishes by (95) and the definitions of ψ and Ψ. Also $at-r = c-2r\vartheta$, and thus (95) and (96) give

$$f'(at-r) = -\tfrac{1}{2}[\psi(c-2r\vartheta)+(c-2r\vartheta\psi'(c-2r\vartheta)+a^{-1}\Psi'(c-2r\vartheta)]. \quad (97)$$

Since $\psi(c) = 0$, this expression tends to the limit $-\tfrac{1}{2}\{c\psi'(c)+a^{-1}\Psi'(c)\}$ as r tends to the limit zero. Hence it follows that the condensation near C tends to become infinite* in the order $\tfrac{1}{2}(ar)^{-1}\{c\psi'(c)+a^{-1}\Psi'(c)\}$ unless

$$c\psi'(c)+a^{-1}\Psi'(c) = 0. \quad (98)$$

This equation is equivalent to the equation

$$a\left(\frac{\partial\phi_0}{\partial r}\right)_{r=c}+(\dot{\phi}_0)_{r=c} = 0, \quad (99)$$

which is the analogue of (91) and (92) for the problem in hand. In terms of initial velocity u_0 and condensation s_0 it is the same as $u_0 = as_0$ at $r = c$. I have verified that the same result would be arrived at by evaluating ϕ in terms of ϕ_0 and $\dot{\phi}_0$ by means of Poisson's integral formula.† This method shows also that the condition (99) is sufficient to secure that neither the pressure nor the velocity shall tend to become infinite in the neighbourhood of C.

The problem of spherical waves of sound symmetrical about the centre of the sphere has been treated by many writers, including Poisson,‡ Stokes § and Lord Rayleigh,‖ but the necessity of the condition here investigated does not appear to have been noticed.

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* At the point C the condensation is of course finite, the infinite limit not being "attained." This peculiarity is of constant occurrence in the the theory of non-analytical functions. See, for example, *Ency. Brit.* (*Supplement*), Vol. XXVIII., Article "Functions of Real Variables." It may be held that infinite limits are never attained.

† Cf. my paper, *supra*, p. 44, where, however, the interpretation of the results for points near the centre of the sphere is imperfect.

‡ *Traité de Mécanique*, 2nd edition (Paris, 1833), t. II., pp. 699-708.

§ *Phil. Mag.* (Ser. 3), Vol. XXXIV. (1849), or *Math. and Phys. Papers*, Vol. II., p. 83.

‖ *Theory of Sound*, Vol. II., pp. 109-111.

34. *Simplification of the Results by the Conditions at the Initial Boundary.*—The relations between initial velocities and initial dilatations and rotations at the boundary Σ_0 of T, which have now been shown to be necessary in general in order to avoid dangerous discontinuities and rupture of the material, may be used to simplify the expressions obtained in Articles 27 to 29. For example, when Σ_0 is a single closed surface which is everywhere convex, we have for the dilatation

$$\Delta = \frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{\Delta}_0+\Delta_0+r\,\frac{\partial\Delta_0}{\partial r}\right)dS_1+\frac{1}{4\pi}\int\frac{1-\cos(r,\,n)}{r\sin\chi}\,\Delta_0\,ds_1, \quad (100)$$

and for the rotation we have such equations as

$$2\varpi' = \frac{1}{2\pi}\iint\frac{1}{r^2}\left(t\dot{\varpi}'_0+\varpi'_0+r\,\frac{\partial\varpi'_0}{\partial r}\right)dS_2,+\frac{1}{2\pi}\int\frac{1-\cos(r,\,n)}{r\sin\chi}\,\varpi'_0\,ds_2. \quad (101)$$

For the displacement we have alternative formulæ of the types

$$\begin{aligned} u = {} & \frac{1}{4\pi}\iint\frac{x'-x}{r^3}\,(t\dot{\Delta}_0+\Delta_0)\,dV+\frac{1}{4\pi}\iint\frac{x'-x}{r^2}\,\Delta_0\,dS_1-\frac{1}{4\pi}\iint\frac{x'-x}{r^2}\,\Delta_0\,dS_2 \\ & +\frac{at}{4\pi}\iint\frac{x'-x}{r^3}\,\Delta_0\,d\Sigma_0+\frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{u}_0+u_0+r\,\frac{\partial u_0}{\partial r}\right)dS_2 \qquad (102) \end{aligned}$$

and

$$\begin{aligned} u = {} & \frac{1}{2\pi}\iiint\left\{\frac{y'-y}{r^3}\,(t\dot{\varpi}'''_0+\varpi'''_0)-\frac{z'-z}{r^3}\,(t\dot{\varpi}''_0+\varpi''_0)\right\}dV \\ & +\frac{1}{2\pi}\iint\left(\frac{y'-y}{r^2}\,\varpi'''_0-\frac{z'-z}{r^2}\,\varpi''_0\right)dS_1-\frac{1}{2\pi}\iint\left(\frac{y'-y}{r^2}\,\varpi'''_0-\frac{z'-z}{r^2}\,\varpi''_0\right)dS_2 \\ & +\frac{bt}{2\pi}\iint\left(\frac{y'-y}{r^3}\,\varpi'''_0-\frac{z'-z}{r^3}\,\varpi''_0\right)d\Sigma_0+\frac{1}{4\pi}\iint\frac{1}{r^2}\left(t\dot{u}_0+u_0+r\,\frac{\partial u_0}{\partial r}\right)dS_1. \end{aligned}$$

$$(103)$$

Like results with suitable modifications hold also when Σ_0 is not of the above simple character; in all cases the normal and tangential components of the initial velocity at points on Σ_0 can be replaced by quantities of the form $\pm a\Delta_0$, $\pm 2b\varpi'_0$, ... It appears that, in all cases, the dilatation can be expressed in terms of the initial dilatation and velocity of dilatation, and the rotation can be expressed in terms of the initial rotation and velocity of rotation; but the expressions are not identical with those given by Poisson's integral formula unless the dilatation, in the one case, and the rotation, in the other, vanish at Σ_0. It has been shown further that the displacement can be resolved into two constituent displacements in two alternative ways. In one way, one of the constituents depends entirely on the initial dilatation and velocity of dilatation, and the other constituent is expressed, in terms of initial displacement and velocity, by the formulæ that would express the displacement if a rotational wave alone were propagated. In the other mode of resolution, one of the constituents depends entirely on the initial rotation and velocity of rotation, and the other is expressed, in terms of initial displacement and velocity, by the formulæ that would express the displacement if a dilatational wave alone were propagated. At places which are so distant from T that the rear of the wave of dilatation has passed before the front of the wave of rotation arrives, the motion in the wave of dilatation depends on the initial dilatation and velocity of dilatation only, and the motion in the wave of rotation depends on the initial rotation and velocity of rotation only.

The initial displacement and velocity may involve dilatation without rotation or velocity of rotation throughout the region T. In this case the conditions that no discontinuity leading to rupture of the material may arise are that the displacement vanishes at the boundary Σ_0 of T, and that, at the same surface, the initial velocity is directed along the normal to Σ_0, and is equal to the value of $a\Delta_0$ at each point of Σ_0. In this case a wave of dilatation only will be propagated, and the displacement in it will be determined by Poisson's integral formula. Again the initial displacement and velocity may involve rotation without dilatation or velocity of dilatation throughout the region T, and the components of the displacement and the normal component of the initial velocity may vanish at all points of Σ_0. In this case, if the tangential components of velocity are adjusted in accordance with the conditions (92), a rotational wave only will be propagated, and the displacement in it will be determined by Poisson's integral formula. A more general type of initial conditions can be arrived at by superposing the displacements and velocities in two initial states of the two kinds just described. But this type is not the most general possible type of initial conditions. It is true that any initial displacement and velocity may be resolved into two systems of displacement and velocity, of which one involves no rotation and the other no dilatation. This resolution would be effected by the intervention of scalar potentials and vector potentials. But, in general, the resolution of the displacement cannot be effected so that the displacements of both systems vanish at Σ_0; and the resolution of the velocity cannot be effected so that the velocity of one system shall be normal to Σ_0, and have a given value at each point on Σ_0, while the velocity of the other system is tangential to Σ_0, and has a given magnitude and direction at each point on Σ_0.

35. *Dilatation and Rotation at the Fronts and Rears of Waves.*— We take the case in which Σ_0 is a single closed surface which is everywhere convex, and take any point O outside it which is not a centre of principal curvature at any point of it. Let P and Q be the points of Σ_0 which are respectively nearest to O and furthest from O, as shown in Fig. 6 of Article 32, and let ρ_1', ρ_2' be the numerical values of the principal radii of curvature at P, and ρ_1'', ρ_2'' those of the principal radii of curvature at Q. A sphere with centre at O and radius less than OP or greater than OQ does not cut Σ_0. When the sphere is of radius OP, the denominator in the formulæ (87) and (88) must be replaced by $\{(1+r/\rho_1')(1+r/\rho_2')\}^{\frac{1}{2}}$; when its radius is OQ, the denominator in these formulæ becomes $\{(r/\rho_1''-1)(r/\rho_2''-1)\}^{\frac{1}{2}}$, and this is necessarily real.

When $at = OP$, the sphere S_1 touches Σ_0 at P, and the dilatation at O when the wave reaches it is obtained from the formulæ of type (64) by the rules of differentiation such as (88). Now n and r have opposite signs, and the expression $\dot{q}_0/a+\partial q_0/\partial r$ may be replaced by

$$-a^{-1}\{\dot{u}_0\cos(x, n)+\dot{v}_0\cos(y, n)+\dot{w}_0\cos(z, n)\}+\Delta_0,$$

which is $2\Delta_0(P)$, and it follows that the dilatation at O when the front of the wave of dilatation reaches O is given by the equation

$$\Delta = \left\{\left(1+\frac{r}{\rho_1'}\right)\left(1+\frac{r}{\rho_2'}\right)\right\}^{-\frac{1}{2}}\Delta_0(P), \tag{104}$$

in which $r = at$.

When $at = OQ$, the sphere S_1 touches Σ_0 at Q, and the dilatation at O just before the rear of the wave of dilatation passes over O is found, by the above process and by paying attention to the signs, to be zero.

It appears in like manner that, when $bt = OP$, so that the front of the wave of rotation has just reached O, the values of the components of rotation at O are given by such formulæ as

$$\varpi' = \left\{\left(1+\frac{r}{\rho_1'}\right)\left(1+\frac{r}{\rho_2'}\right)\right\}^{-\frac{1}{2}}\varpi_0'(P); \tag{105}$$

but, when the rear of the wave of rotation is just about to pass over O, the rotation at O is reduced to zero.

It may be observed that, if O is so near to Σ_0 that the waves are practically plane, the dilatation at the front of the wave of dilatation, and the rotation at the front of the wave of rotation, are practically the same as the initial dilatation and rotation at the surface Σ_0.

36. *Summary.*—The theory of this paper rests upon certain integrals of the equations of motion of an elastic solid medium under the action of body forces. These integrals have in effect been known for a long time, and the proof of their correctness has here been recapitulated briefly. The results which can be obtained by regarding the forces as operative at a single point have been considered in detail. Besides the direct proof, various verifications have been adduced, and the exact nature of the motion specified by the integrals in particular cases has been discussed with graphic illustrations. From the results in the case of a single force have been deduced those that belong to various kinds of double and multiple forces, and it is found that various well-known types of wave-motion can be regarded as due to singularities of these kinds. Among the types of wave-motion in an elastic solid which have been expressed in terms of singularities are the simplest type of dilatational waves, the simplest type of rotational waves and the type of rotational waves in which the displacement is expressed by the same formulæ as were given by Hertz to represent the electric field around his oscillator. From a knowledge of the effects produced by force at a point and of certain types of double forces it was possible to obtain a theorem which is a generalization of Kirchhoff's version of Huygens' principle. In Kirchhoff's theorem, the function which represents the disturbance in wave-motion, determined by a single partial differential equation, is expressed in terms of sources and doublets distributed over a surface. The sources depend on the normal gradient of the function, and the doublets on the value of the function itself, at points on the surface, and at instants determined by the condition that the effects travel from the surface to any point with the appropriate wave-velocity. In the present case the two sorts of singularities are points at which forces act and points at which there are double forces of a certain general type. The forces are the actual tractions across the elements of the surface, and the double forces depend upon the displacements at points of the surface, but the instants at which the forces and double forces must be calculated are not determined by a single wave-velocity. It is as if some effects travelled through the medium with velocities intermediate between that of rotational waves and that of dilatational waves, while other particular effects travel with those definite velocities. Various results enforce this conclusion.

The problem of the propagation of waves due to given initial conditions is taken up, and at the outset it is assumed that the initial disturbance is confined to a region of space occupying a finite volume. Within this region it is assumed that the components of the initial displacement and velocity are expressed by functions which are analytic and without singularities. The condition of continuity of the displacement at the boundary of this region is supposed to be satisfied by the initial displacement, but no other restrictions are imposed upon the initial state. The potentials of the disturbance due to initial velocity are obtained from the results found for force at a point, and the displacement due to initial velocity is deduced. The complete expression for the displacement is then found by means of Stokes's rule. The process involves the determination of rules for the differentiation of integrals taken over portions of moving spherical surfaces which are bounded by the intersections of the spheres with a fixed surface. It is pointed out that the application of these rules of differentiation to Poisson's integral formula brings to light a difference between two forms of it which are sometimes treated as equivalent. The expressions for the displacement which are obtained by the method above described are equivalent to those which were found by Stokes, and previously by Poisson and Ostrogradsky; but the method is different, inasmuch as it takes account at every step of the existence of a boundary to the portion of the medium which is disturbed initially. The importance of this is seen at the next stage of the investigation, when the dilatation and rotation are calculated from the displacement; for the values of these quantities, when the initial velocity does not vanish at the boundary of the initially disturbed portion of the medium, are not the same as those assumed as a basis of calculation by Stokes. His doctrine, however, of the irrotational and equivoluminal character of the two waves is confirmed. Correct values for the dilatation and rotation are obtained, and some new formulæ are found for the displacement. By the aid of these formulæ the parts played in the motion by the initial dilatation and rotation can be distinguished. The corrections to be made in Stokes's values for the dilatation and rotation depend respectively upon the normal and tangential components of the initial velocity at the boundary of the initially disturbed portion of the medium.

An important question arises as to whether the motion determined by initial conditions, when no restrictions except that of continuity of displacement are imposed at the boundary of the initially disturbed portion of the medium, will or will not result in accumulation of strain at some point, leading to rupture of the material. The question is analogous to one which occurs in the theory of sound. Spherical sound waves may involve a tendency to infinite pressure in the neighbourhood of the centre of the sphere. It is shown that in the present case arbitrary initial conditions will result in such accumulations of strain, and these will occur in the neighbourhood of the points which are centres of principal curvature of the bounding surface of that portion of the medium to which the initial disturbance is confined. To avoid such accumulations of strain, conditions must be imposed on the initial state. These conditions take the form of relations between initial velocity and strain at the boundary of the initially disturbed portion of the medium. The normal component of the initial velocity is equal to the product of the velocity of dilatational waves and the initial dilatation. The tangential components are similarly related to the velocity of rotational waves and the initial rotation. These conditions are illustrated by a problem concerning spherical waves of sound. It is shown that, if the initial disturbance in a gaseous medium is confined to the volume within a spherical surface and is symmetrical about the centre of the sphere, the tendency to infinite pressure at the centre is obviated if the initial velocity and condensation at the surface of the sphere are connected by the same relation as holds in the case of plane waves of sound. In the problem of the elastic solid medium, when the like conditions are taken into account it appears that the corrections which must be made in Stokes's formulæ for the dilatation and rotation can be expressed in terms of the dilatation and rotation at the initial boundary. When this boundary is a single closed surface which is everywhere convex, it appears further that the dilatation always vanishes at the rear (though not at the front) of the wave of dilatation, and the rotation always vanishes at the rear of the wave of rotation.

ON THE UNIQUE EXPRESSION OF BINARY AND TERNARY FORMS

By P. W. WOOD (*communicated by* J. H. GRACE).

[Received and Read November 12th, 1903.]

The present paper furnishes simple proofs of the two following theorems:—

> I. If $x_1+x_2+x_3 = 0$, any rational integral homogeneous function of degree n in any two of x_1, x_2, x_3 is a linear function of terms each involving two of x_1, x_2, x_3, and in each term one of the two letters involved has an index $\geqslant \frac{2}{3}n$ (Jordan).
>
> II. If $x_1+x_2+x_3+x_4 = 0$, any rational integral homogeneous function of degree n in any three of x_1, x_2, x_3, x_4 is a linear function of terms each involving three of x_1, x_2, x_3, x_4, and in each term one of the three letters involved has an index $\geqslant \frac{1}{2}n$.

The first theorem is due to Camille Jordan*: his proof depends on the evaluation of a determinant whose elements are binomial coefficients. The second theorem is new, and to prove it in the same manner as Jordan proved the first would require the evaluation of a determinant whose elements are trinomial coefficients. Another proof of the first theorem has been given by Stroh†, and an extension of his method is open to the same objections. Neither Jordan's nor Stroh's proof is inductive.

Mr. J. H. Grace‡ has given an inductive proof of a more general result, from which Stroh's form of the first theorem may be deduced as a special case. This result is:—

> Any binary form of order n can be expressed uniquely in the form $a_x^{n-a}A+b_x^{n-\beta}B+c_x^{n-\gamma}C+\dots$, where a_x, b_x, c_x, ... are r distinct linear forms, A, B, C, ... are of orders a, β, γ, ... respectively, and
>
> $$a+\beta+\gamma+\dots = n-r+1.$$

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* *Liouville's Journal*, 1876.

† *Math. Ann.*, Bd. XXXI.

‡ *Algebra of Invariants*, Grace and Young, Camb. Univ. Press, 1903.

The present paper consists of an essential modification of this proof with an obvious extension to the case of four variables—the proof of the second theorem being simpler than that of the first.

Just as the Jordan lemma owes its importance to its application to the fundamental identity $(bc)\,a_x+(ca)\,b_x+(ab)\,c_x\equiv 0$, arising in the symbolical treatment of covariants of binary forms (especially in perpetuants* where this identity may be used freely), so it seems probable that the second theorem may be of value in the symbolical treatment of ternary forms, where the fundamental identity is

$$(bcd)\,a_x+(cad)\,b_x+(abd)\,c_x\equiv(abc)\,d_x.$$

It will be seen that the method of this paper does not admit of direct application to the case of five or more variables.

I. *The Jordan Lemma.*

1. We shall first establish the result:—

> If λ_x, μ_x, ν_x be three distinct linear binary forms, we cannot find binary quantics p_x, q_x, r_x satisfying any one of the three following relations for any value of m:—
>
> total order $3m+2$,
> $$\lambda_x^{2m+2}p_x^m+\mu_x^{2m+2}q_x^m+\nu_x^{2m+2}r_x^m\equiv 0, \tag{i.}$$
> total order $3m+1$,
> $$\lambda_x^{2m+2}p_x^{m-1}+\mu_x^{2m+1}q_x^m+\nu_x^{2m+1}r_x^m\equiv 0, \tag{ii.}$$
> total order $3m$,
> $$\lambda_x^{2m+1}p_x^{m-1}+\mu_x^{2m+1}q_x^{m-1}+\nu_x^{2m}r_x^m\equiv 0. \tag{iii.}$$

The proof will be inductive: assuming that the relation of total order n requires each of the quantics involved to be identically zero, then the same will be proved true for the relation of total order $(n+1)$. Three cases must be considered, according as n is of the form $3m+1$, or $3m$, or $3m-1$.

(i.) Operate on (i.) with $(\lambda_1\,\partial/\partial x_2-\lambda_2\,\partial/\partial x_1)$: the resulting relation is of the form $$\lambda_x^{2m+2}p_x'^{m-1}+\mu_x^{2m+1}q_x'^m+\nu_x^{2m+1}r_x'^m\equiv 0$$ and therefore by hypothesis $p_x'^{m-1}\equiv q_x'^m\equiv r_x'^m\equiv 0$. Now $q_x'^m$ requires $(\lambda_2\,\partial/\partial x_1-\lambda_1\,\partial/\partial x_2)\,[\mu_x^{2m+2}q_x^m]\equiv 0$, and therefore $q_x^m\equiv 0$, since λ_x and μ_x are distinct; and similarly $r_x'^m\equiv 0$ requires $r_x^m\equiv 0$, since λ_x and ν_x are distinct: but, if $q_x^m\equiv r_x^m\equiv 0$ in the relation (i.), then $p_x^m\equiv 0$ also.

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* Grace: *Proc. London Math. Soc.*, Vol. xxxv.

(ii.) Now operate on (ii.) with $(\mu_2 \partial/\partial x_1 - \mu_1 \partial/\partial x_2)$; we get a relation of the same form as (iii.), and the quantics in this relation are by hypothesis identically zero. We may show in the same way that the quantics p_x^{m-1}, r_x^m in (ii.) are each identically zero, and therefore $q_x^m \equiv 0$ also.

(iii.) Finally operate on (iii.) with $(\nu_2 \partial/\partial x_1 - \nu_1 \partial/\partial x_2)$: it gives a relation of the same form as (i.) when $(m-1)$ is written in that relation for m—in other words, it leads to a relation of total order $3m-1$, wherein by hypothesis each quantic involved is identically zero. As before, this requires that each of the quantics p_x^{m-1}, q_x^{m-1} of (iii.) is identically zero, and therefore $r_x^m \equiv 0$ also.

Hence for all values of n, if the relation of total order n is impossible, so also is the relation of total order $(n+1)$, and it is therefore sufficient to show that the relation of total order unity requires each of the quantics (constants) involved to be identically zero. (Where a quantic of negative order appears the corresponding term is of course absent.)

Putting $m = 0$ in (ii.), we have $(\mu_1 x_1 + \mu_2 x_2) Q + (\nu_1 x_1 + \nu_2 x_2) R \equiv 0$, and therefore $\mu_1 Q + \nu_1 R = 0$, $\mu_2 Q + \nu_2 R = 0$, and hence $Q \equiv R \equiv 0$, since $\mu_1/\mu_2 \neq \nu_1/\nu_2$.

The result is therefore universally true.

2. From this result we can deduce the theorem :—

Any binary form F_x^n of order n in any two of the variables x_1, x_2, x_3 is *uniquely* expressible in one of the following ways :—

$$n = 3m+2: \quad F_x^{3m+2} \equiv y_1^{2m+2} a_{1_y}^{m} + y_2^{2m+2} a_{2_y}^{m} + y_3^{2m+2} a_{3_y}^{m}, \tag{iv.}$$

$$n = 3m+1: \quad F_x^{3m+1} \equiv y_1^{2m+2} a_{1_y}^{m-1} + y_2^{2m+1} a_{2_y}^{m} + y_3^{2m+1} a_{3_y}^{m}, \tag{v.}$$

$$n = 3m: \quad F_x^{3m} \equiv y_1^{2m+1} a_{1_y}^{m-1} + y_2^{2m+1} a_{2_y}^{m-1} + y_3^{2m} a_{3_y}^{m}, \tag{vi.}$$

where y_1, y_2, y_3 are x_1, x_2, x_3 in some definite order and

$$y_1 + y_2 + y_3 \equiv x_1 + x_2 + x_3 \equiv 0.$$

There is a single expression for every way of taking y_1, y_2, y_3 as x_1, x_2, x_3.

For any binary form of order n may be expressed linearly in terms of $(n+1)$ linearly independent binary forms of the same order, and the number of arbitrary constants on the dexter sides of (iv.), (v.), (vi.) are $3m+3$, $3m+2$, $3m+1$ respectively: moreover, putting $\lambda_x \equiv y_1$, $\mu_x \equiv y_2$, $\nu_x \equiv y_3$, it follows from the preceding theorem that the forms on the dexter

For the number of arbitrary constants involved in the quantics on the dexter side of (i.) is

$$\frac{3(m-1)(m+2)}{2}+\frac{m(m+1)}{2}=\frac{(m+1)}{2}[4m+6]=\frac{(2m+2)(2m+3)}{2}$$

and the number of arbitrary constants involved in the quantics on the dexter side of (ii.) is

$$\frac{3m(m+1)}{2}+\frac{(m+1)(m+2)}{2}=\frac{m+1}{2}[4m+2]=\frac{(2m+1)(2m+2)}{2}.$$

Now any ternary quantic of order n is expressible in terms of $\frac{1}{2}\{(n+1)(n+2)\}$ linearly independent ternary quantics of the same order; and from the preceding theorem it follows that the forms on the dexter sides of (iii.), (iv.) respectively are linearly independent, and that the expression is in each case unique; for otherwise there would arise relations of the forms (i.), (ii.) respectively, which have been proved to be impossible.

Express a_{1_y}, a_{2_y}, a_{3_y}, a_{4_y} in terms of $y_1y_2y_3$, $y_2y_3y_4$, $y_3y_4y_1$, $y_4y_1y_2$ respectively. Then every term on the dexter side of (i.) involves three letters, and in every term one of these letters has an index $\geqslant (m+1)$, *i.e.* $\geqslant \frac{1}{2}n$; and every term on the dexter side of (ii.) involves three letters, and in every term one of these letters has an index $\geqslant m$, *i.e.* $\geqslant \frac{1}{2}n$.

This proves the second theorem stated at the beginning of this paper.

FORMS OF MACLAURIN'S THEOREM

By F. H. JACKSON.

[Received September 5th, 1903.—Read November 12th, 1903.—Revised January 14th, 1904.]

1. Suppose that $f(x)$ is capable of expansion in a series of the form

$$A_0+A_1x^{[1]}+A_2x^{[2]}+\dots+A_mx^{[m]}+\dots, \tag{1}$$

in which $[1]=p_1$, $[2]=p_1+p_2$, $[3]=p_1+p_2+p_3$, and in general

$$[n]=p_1+p_2+p_3+\dots+p_n,$$

namely, the sum of the first n elements in an infinite sequence $(p_1, p_2, p_3, \dots)$; then

$$f(x)=f(0)+f^{(1)}(0)x^{[1]}+f^{(2)}(0)\,\frac{p_2!}{(2)!}\,x^{[2]}+\dots+f^{(m)}(0)\,\frac{p_m!}{(m)!}\,x^{[m]}+\dots. \tag{2}$$

The notation is as follows:

$$\begin{aligned}(n) &= p_1+p_2+p_3+\dots+p_n,\\ (n-1) &= p_2+p_3+\dots+p_n = [n]-[1],\\ (n-2) &= p_3+p_4+\dots+p_n = [n]-[2],\end{aligned}$$

$$(n)! = \{p_1+p_2+\dots+p_n\}\,\{p_2+p_3+\dots+p_n\}\dots\{p_{n-1}-p_n\}\,\{p_n\}, \tag{3}$$

$$p_n! = p_1p_2p_3\dots p_n, \qquad f^{(n)}(x) = D^{(n)}\{f(x)\},$$

$D^{(n)}$ denotes the operator

$$\frac{d}{d(x^{p_n})}\left\{\frac{d}{d(x^{p_{n-1}})}\left\{\dots\left\{\frac{d}{d(x^{p_2})}\left\{\frac{d}{d(x^{p_1})}\right\}\right\}\right\}\right\}, \tag{4}$$

which reduces when $p_1=p_2=p_3=\dots=1$ to the rator $(d/dx)^r$, and when the sequence of elements is $(1, p, p^2, p^3, \dots$ the operator $D^{[r]}$ (*supra*, pp. 84 *et seq*.). Examples of the theorem afforded by Gauss's series

$$1+\sum_{n=1}^{n=\infty} x^{\frac{1}{2}[n(n+1)]} \quad (x<1), \qquad (p_1, p_2, p_3, \dots \equiv (1, 2, 3, 4, \dots),$$

and Euler's celebrated series

$$\sum_{n=-\infty}^{n=+\infty}(-1)^n x^{\frac{1}{2}(3n^2+n)}\,(x<1), \quad (p_1, p_2, p_3, \dots) = (1,, 3, 2, 5, 3, \dots, 2r-1, r, \dots).$$

2. If we operate on $x^{[m]}$ with $\dfrac{d}{d(x^{p_1})}$, we obtain

$$\frac{p_1+p_2+p_3+\ldots+p_m}{p_1}\,x^{p_2+p_3+\ldots+p_m}.$$

Operating on this with $\dfrac{d}{d(x^{p_2})}$, we obtain

$$\frac{p_1+p_2+p_3+\ldots+p_m}{p_1}\,\frac{p_2+p_3+\ldots+p_m}{p_2}\,x^{p_3+p_4+\ldots+p_m};$$

and in general $D^{(r)}\{x^{[m]}\}$ gives us $\dfrac{(m)(m-1)(m-2)\ldots(m-r+1)}{p_1\,p_2\,p_3\cdots p_r}\,x^{(m-r)}$;

so that $D^{(m)}\{x^{[m]}\}$ gives $(m)!/p_m!$.

Suppose $f(x)$ capable of expansion in a convergent series

$$f(x) = A_0+A_1x^{[1]}+A_2x^{[2]}+\ldots+A_mx^{[m]}+\ldots.$$

If we now operate in succession with $D^{(1)}, D^{(2)}, D^{(3)}, \ldots, D^{(r)}, \ldots$, we obtain

$$\left.\begin{aligned}
f^{(1)}(x) &= A_1+A_2\frac{(2)}{p_1}\,x^{p_2}+A_3\frac{(3)}{p_1}\,x^{p_2+p_3}+\ldots+A_m\frac{(m)}{p_1}\,x^{p_2+p_3+\ldots+p_m}+\ldots\\
f^{(2)}(x) &= A_2\frac{(2)(1)}{p_1p_2}+\ldots+A_m\frac{(m)(m-1)}{p_1p_2}\,x^{p_3+\ldots+p_m}+\ldots\\
f^{(3)}(x) &= A_3\frac{(3)!}{p_3!}+\ldots+A_m\frac{(m)(m-1)(m-2)}{p_1\,p_2\,p_3}\,x^{p_4+\ldots+p_m}+\ldots\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
f^{(r)}(x) &= A_r\frac{(r)!}{p_r!}+\ldots+A_m\frac{(m)(m-1)(m-2)\ldots(m-r+1)}{p_1\,p_2\cdots p_r}\,x^{p_{r+1}+\ldots+p_m}
\end{aligned}\right\} \quad (5)$$

Putti , we have

so that $f(0),\quad A_1 = \dfrac{p_1!}{(1)!}f^{(1)}(0),\quad \ldots,\quad A^{(r)}\dfrac{p_r!}{(r)!}f^{(r)}(0)$;

$$f(x)\;\;0)+\frac{p_1!}{(1)!}f^{(1)}(0)\,x^{[1]}+\ldots+\frac{p_m!}{(m)!}f^{(m)}(0)\,x^{[m]}+\ldots. \quad (6)$$

3. In the spe case when $p_1=p_2=p_3=\ldots=1$ we see that $p_m! = p_1p_2\ldots p_m =$ $(m)! = m(m-1)(m-2)\ldots 3.2.1 = m!$ and the expression (6) reduc to the normal form of Maclaurin's series. If, however, $(p_1, p_2, p_3, \ldots_n, \ldots) = (1, 2, 3, \ldots, n, \ldots)$, then

$$_n! = 1.2.3\ldots m = m!$$

and

$$(m)! = \{1+2+3+\ldots+m\}\{2+3+4+\ldots+m\}\ldots\{m-1+m\}\{m\}$$

$$= \frac{m(m+1)}{2}\,\frac{(m-1)(m+2)}{2}\ldots\frac{1\,.\,2m}{2} = \frac{2m!}{2^m},$$

and the theorem is

$$f(x) = f(0)+f^{(1)}(0)\,x+f^{(2)}(0)\frac{2!\;2^2}{4!}x^3+\ldots+f^{(m)}(0)\frac{m!\;2^m}{2m!}x^{\frac{1}{2}(m\,.\,m+1)}+\ldots \tag{7}$$

The function due to Gauss

$$\frac{1-x^2\,.\,1-x^4\,.\,1-x^6\ldots\text{ ad inf.}}{1-x\,.\,1-x^3\,.\,1-x^5\ldots\text{ ad inf.}} = 1+\sum_{m=1}^{m=\infty} x^{\frac{1}{2}[m(m+1)]} \quad (x<1) \tag{8}$$

comes under form (7), and, if we consider the series

$$1+x^{(1)}+x^{(2)}+\ldots+x^{(m)}+\ldots \tag{9}$$

as a fundamental form to which the series

$$1+x+x^2+\ldots+x^n+\ldots = \frac{1}{1-x} \tag{10}$$

and $$1+x+x^3+\ldots+x^{\frac{1}{2}(n\,.\,n+1)}+\ldots = \frac{1-x^2\,.\,1-x^3\ldots\text{ ad inf.}}{1-x\,.\,1-x^3\ldots\text{ ad inf.}} \tag{11}$$

belong, we may look upon the functions (10) and (11) as analogues for the sequences (1, 1, 1, 1, ...) and (1, 2, 3, 4, ...) respectively.

4. Consider the sequence

$$(p_1, p_2, p_3, \ldots, p_n, \ldots) = (1, 3, 5, \ldots, 2n-1, \ldots);$$

then the general coefficient $f^{(m)}(0)\,\frac{p_m!}{(m)!}$ becomes

$$f^{(m)}(0)\,\frac{1\,.\,3\,.\,5\,.\,7\ldots 2m-1}{m^2\,.\,m^2-1^2\,.\,m^2-2^2\ldots 2m-1} = \frac{f^{(m)}(0)}{2^{m-1}\,.\,m!}$$

$$f(x) = f(0)+f^{(1)}(0)x+f^{(2)}(0)\,\frac{x^4}{2!\,2^2}+\ldots+f^{(m)}(0)\,\frac{x^{m^2}}{m!\,2^{m-1}}+\ldots. \tag{12}$$

I have not been able to discover any Π function capable of expansion in this form, but it would seem natural to expect that the ordinary functions of analysis such as e^x, $\sin x$, $\log(1-x)$, &c., whose expansions are for the simple sequence (1, 1, 1, ...), would have their analogues in other sequences such as (1, 3, 5, ..., $2n-1$).

Consider the sequence $(p_1, p_2, p_3, \ldots, p_n, \ldots) = (1, p, p^2, \ldots, p^{n-1}, \ldots)$. Here

$$(m) = 1+p+p^2+\ldots+p^{m-1} = \frac{p^m-1}{p-1} = [m],$$

$$(m-1) = p+p^2+\ldots+p^{m-1} = \frac{p^{m-1}-1}{p-1} = p\,[m-1],$$

$$f(x) = f(0)+f^{(1)}(0)\,x^{[1]}+\ldots+f^{(m)}(0)\,\frac{x^{[m]}}{[m]!}+\ldots. \tag{13}$$

I have discussed series and functions of this type in the papers noted below.* Euler's theorem

$$(1-x)(1-x^2)(1-x^3)\ldots \text{ ad inf.} = \sum_{n=-\infty}^{n=+\infty}(-1)^n x^{\frac{1}{2}(3n^2+n)} \tag{14}$$

is an example of the general theorem for the sequence of odd integers and all integers respectively in order, viz.:

$$(p_1, p_2, \ldots, p_r, \ldots) = (1, 1, 3, 2, 5, 3, \ldots, 2r-1, r, \ldots). \tag{15}$$

The form of the theorem then is

$$\begin{aligned} f(x) = f(0)+f^{(1)}(0)\,x+f^{(2)}(0)\,\frac{x^2}{2}+f^{(3)}(0)\,\frac{x^5}{20}+\ldots \\ +f^{(2r-1)}(0)\,\frac{2^r}{6r-4\,.\,6r-7\,\ldots\,5\cdot 2}\,x^{\frac{1}{2}(3r^2-r)} \\ +f^{(2r)}(0)\,\frac{2^r}{6r-2\,.\,6r-5\,\ldots\,4\cdot 1}\,x^{\frac{1}{2}(3r^2+r)}+\ldots. \end{aligned} \tag{16}$$

The forms for the sequences (2, 2, 2, ...) and (1, 2, 2, 2, ...) are respectively

$$f(x) = f(0)+f^{(1)}(0)\,x^2+f^{(2)}(0)\,\frac{x^4}{2!}+\ldots+f^{(m)}(0)\,\frac{x^{2m}}{m!}+\ldots, \tag{17}$$

$$f(x) = f(0)+f^{(1)}(0)\,x+f^{(2)}(0)\,\frac{x^3}{3}+\ldots+f^{(m)}(0)\,\frac{x^{2m-1}}{(m-1)!\,(2m-1)}+\ldots. \tag{18}$$

These are the forms of the theorem for even functions and odd functions respectively. Comparing the coefficients of these series with the coefficients of the normal form of Maclaurin's series, we see that for even functions

$$f^{(m)}(0) = \frac{m!}{2m!}\,f^{2m}(0), \tag{19}$$

while for odd functions

$$f^{(m)}(0) = \frac{(m-1)!}{(2m-2)!}\,f^{2m-1}(0). \tag{20}$$

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* "Series connected with the Enumeration of Partitions," *supra*, pp. 84 *et seq.*; *Trans. R.S. Edin.*, "Generalized Functions of Legendre and Bessel," Vol. XLI.; "Extension of Neumann's Theorem," *infra*, pp. 361 *et seq.*; *Proc. R.S. Edin.*, Vol. XXIV., pp. 440–447.

In (17) $f^{(m)}(x) = \left(\frac{d}{d(x^2)}\right)^m f(x)$, but in (18) $f^{(m)}(x) = \left(\frac{d}{d(x^2)}\right)^{m-1} \left(\frac{d}{dx}\right) f(x)$.

The fundamental hypergeometric series and its differential equation are discussed in a paper (*Trans. R. S. Edin.*, Vol. XLI., p. 29). This series is of type (1), being

$$F\left[(\alpha_1)(\alpha_2)(\alpha_3) \dots (\alpha_n)(\beta_1)(\beta_2) \dots (\beta_m)\,x\right] = \Sigma A_r x^{p_1+p_2+\dots+p_r} = \Sigma A_r x^{(r)},$$

in which

$$\begin{aligned} A_r = p_1^r \Bigg[\Big\{ &[(\alpha_1)(\alpha_1+p_1)(\alpha_1+p_1+p_2) \dots (\alpha_1+p_1+p_2+\dots+p_{r-1})] \\ &\quad \times [(n-1) \text{ similar products for } \alpha_2, \alpha_3, \dots, \alpha_n] \Big\} \\ \div \Big\{ &[(p_1)(p_1+p_2)(p_1+p_2+p_3) \dots (p_1+p_2+\dots+p_r)] \\ &\quad \times [(\beta_1)(\beta_1+p_2)(\beta_1+p_2+p_3) \dots (\beta_1+p_2+p_3+\dots+p_r)] \\ &\qquad \times [(m-1) \text{ similar products for } \beta_2, \dots, \beta_m] \Big\} \Bigg]. \end{aligned} \tag{21}$$

For the sequence (1, 1, 1, ...) this reduces to the hypergeometric series with n elements in the numerator, and m elements in the denominator of each term.

ON THE DISTRIBUTION OF THE POINTS OF UNIFORM CONVERGENCE OF A SERIES OF FUNCTIONS

By W. H. YOUNG.

[Received 6th December, 1903.—Read 10th December, 1903.]

1. In my paper on "Non-uniform Convergence," *supra*, p. 89, I showed that, in the general case when the functions $f_1(x), f_2(x), \ldots$ are pointwise discontinuous functions, the points of non-uniform convergence of the series $F(x) = f_1(x)+f_2(x)+\ldots$ can be divided into two classes, which, for reasons there given, I called *visible* and *invisible* points of non-uniform convergence, and that, while the visible points (which cannot present themselves in the case treated by Prof. Osgood in the *American Journal of Mathematics*, Vol. XIX., when all the functions concerned are continuous) may fill up whole intervals, the invisible points obey Osgood's law of distribution, viz., they form a set which is the limit G of a series of closed sets $G_1, G_2, \ldots$, each of which is dense nowhere, and contains all the preceding sets: it was further shown that, when only a finite number of the functions are discontinuous, the whole set of points of non-uniform convergence, visible and invisible, form such a limiting set G.

From the investigation of the paper first referred to it was not evident whether *any* restriction could be laid on the distribution of the points of non-uniform convergence in the general case, or, which comes to the same thing, on the distribution of the points of uniform convergence. In the present paper I give the restriction in question, and show that *the most general distribution of the points of uniform convergence of a series of pointwise discontinuous functions is when they form what I have elsewhere called an* inner limiting set.

2. An *inner limiting set* was defined as the set of points consisting of all the points internal to intervals of every set of a series of sets of intervals, and it was shown* that, any inner limiting set being given, the defining intervals could be so chosen that (i.) the intervals in each individual set did not overlap, and (ii.) each interval of any set was contained

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* "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipziger Bericht*, Aug. 1, 1903.

in an interval of the preceding set; such defining intervals I called *normal intervals*.

It is at once evident that Osgood's set G is a special case of the complementary set of an inner limiting set, viz., when the normal intervals of each set are dense everywhere; these normal intervals may in fact be taken to be the black intervals* of the closed sets $G_1, G_2, \ldots$.

3. THEOREM 1.—*The points of uniform convergence of a series of pointwise discontinuous functions form an inner limiting set.*

By definition, a series of functions of x is said to be *uniformly convergent* at a point ξ if, given any positive quantity A, however small, an interval δ can be described, having ξ as internal point, so that, for all points x within the interval δ, the residue $R_n(x) < A$, for all values of $n \geqslant m$, where m is an integer, independent of x, which can always be determined.†

Hence, assuming any sequence of small positive quantities, having zero as limit, $A_1, A_2, \ldots$, and describing intervals corresponding, $\delta_{\xi,1}, \delta_{\xi,2}, \ldots$, for each point ξ of uniform convergence of a given series of functions, we get a series of sets of intervals (in general overlapping intervals) whose inner limiting set certainly contains all the points ξ of uniform convergence, Further, if x be any point of this inner limiting set, we can assign one interval from each successive set of intervals, such that x is internal to that interval, so that, corresponding to any assigned A, however small, we have an interval such as is required in order that, by the definition, x should be a point of uniform convergence.

Thus the inner limiting set of the series of sets of intervals so constructed consists of all the points of uniform convergence of the series of functions. Q. E. D.

4. Before proceeding to prove the converse of Theorem 1, a few preliminary remarks will not be out of place.

Let E be any given inner limiting set, and let the sets of defining intervals be denoted by $I_1, I_2, \ldots$.

If the intervals in each of these sets are dense everywhere, Osgood has shown‡ how to form a series of *continuous* functions whose points of uniform convergence form the set E. If, on the other hand, the above is not the case, we can assign a definite set (and we may without any loss of generality take it to be I_1), in which the intervals are not dense everywhere, so that we can find an interval which is entirely external to every one of the intervals of I_1. In this interval, so found, there are then no points of E; it must therefore lie inside one of the black intervals of the set got by closing E. Thus we see that, if we add to each set I_n the set of black intervals of the set got by closing E, the intervals of each of the new sets, which we may denote by $J_1, J_2, \ldots$, are dense everywhere. We can therefore find an Osgood series having as points of uniform convergence the points of the inner limiting set of $J_1, J_2, \ldots$. It is at once evident that this inner limiting set will consist of the set E together with all the internal points of the black intervals of the set got by closing E. In this way we have succeeded in constructing a series of continuous functions uniformly convergent at all points of E, and at all points internal to the black intervals of the set got by closing E, and non-uniformly convergent at all the remaining points (which are, of course, limiting points of E).

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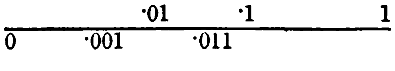
* That is, the intervals which have no points of the closed set internal to them, but whose end points are points of the closed set.

† W. H. Y., *loc. cit.*

‡ *Loc. cit.*, p. 171, Theorem 3. A ξ-point is Osgood's notation for a point of non-uniform convergence.

5. From the definition of non-uniform convergence at a point it is evident that, if we add together two series each of which is uniformly convergent at a point P, the resulting series will be uniformly convergent at P; and that, if one of the series is non-uniformly convergent at P and the other uniformly convergent, the resulting series will be non-uniformly convergent at P.

From this we see that, if we can solve the problem of constructing a series non-uniformly convergent at every internal point of a set of non-overlapping intervals and uniformly convergent at all the remaining points of the segment under discussion, this, in conjunction with § 4, will enable us to prove the converse of Theorem 1. As a preliminary we require a solution of the following problem :—



Consider the interval (0, 1), and let us, for convenience, use the binary notation.

Let all the functions $f_1, f_2, \ldots$ be zero always, except at the points to be specified; further, let f_{i+1} have values equal and opposite to those of f_i at all points where f_i is positive; finally, let

f_1	have the value	$\cdot 1$	at the point	$\cdot 1$,
f_2	,,	$\cdot 01$	,,	$\cdot 01$,
		1	,,	$\cdot 11$,
f_3	,,	$\cdot 0^2 1$	,,	$\cdot 001$,
		$\cdot 01$	,,	$\cdot 011$,
		1	,,	$\cdot 101$ and $\cdot 111$,
f_4	,,	$\cdot 0^3 1$	,,	$\cdot 0^3 1$,
		$\cdot 0^2 1$	,,	$\cdot 0^2 11$,
		$\cdot 01$	,,	$\cdot 0101$ and $\cdot 0111$,
		1	,,	$\cdot 1001$, $\cdot 1011$, and $\cdot 1111$,

and so on, the general rule being as follows:—Let $\cdot N$ denote any binary fraction with n figures; then f_n has at the point $\cdot 0^{n-1} 1$ the value $\cdot 0^{n-1} 1$, and at any other point $\cdot N$ to the left of the point $\cdot 1$ the value of f_n is to be that of f_{n-1} at the nearest point on the left of $\cdot N$ where f_{n-1} was positive; while, if N lie to the right of $\cdot 1$, f_n is to have there the value 1.

To show that this function is uniformly convergent at the origin only, and non-uniformly convergent at every other point of the segment (0, 1), we have to consider that between P and the origin there is a definite point $\cdot N$ with least number n of figures; in any neighbourhood of P on the left there are then, by construction, discontinuities equal to $\cdot N$ of an infinite number of the functions $s_i(x)$, and therefore, since F is continuous, of $R_i(x)$; thus P is a point of non-uniform convergence, whose measure of non-uniformity is $\cdot N$. That the origin is a point of right-handed uniform convergence is evident, since the above mentioned measure of non-uniformity decreases without limit as we approach the origin.

If instead of the segment (0, 1) we have any segment (P, M), we proceed precisely on the same lines. We divide this segment by continued bisection as before, and denote the successive points of division by P_1, P_{01}, P_{11}, P_{001}, ... (as we formerly denoted them by $\cdot 1$, $\cdot 01$, $\cdot 11$, $\cdot 001$, ...). The value of each function f_i at P_N is then taken to be the same as formerly at the point $\cdot N$.

6. The preceding construction enables us to solve the required problem:—*To construct a series, non-uniformly convergent at the internal points of a set of non-overlapping intervals, and uniformly convergent at all the remaining points.*

If (P, Q) be any one of the given intervals, we bisect it at M, and apply the construction just given to the segments (P, M) and (Q, M) separately, P and Q being the points of uniform convergence, and we multiply all the functions f_i by the length of (P, Q).

Doing this with all the intervals, and making all the functions f_i zero at all the remaining points, we have the desired series. It is, in fact, evident that at all the limiting points of the intervals the series is uniformly convergent, since the measure of non-uniformity in the neighbourhood of such a point decreases without limit, as the lengths of the intervals do so.

7. THEOREM 2.—*Given any inner limiting set E a series of pointwise discontinuous functions can be constructed having the points of E for its points of uniform convergence, and non-uniformly convergent at every other point of the segment under consideration.*

We have, in fact, only to take as our set of intervals the black intervals of the set got by closing E, and apply the construction of the preceding article to form a series of pointwise discontinuous functions non-uniformly convergent throughout those intervals, and uniformly convergent at all their end points and external points. This series we add term by term to the series formed as in § 4. By the reasoning at the beginning of § 5 the series so formed from the two series has the required properties.

A GENERALIZATION OF NEUMANN'S EXPANSION OF AN ARBITRARY FUNCTION IN A SERIES OF BESSEL'S FUNCTIONS

By F. H. JACKSON.

[Received November 26th, 1903.—Read December 10th, 1903.]

NEUMANN'S theorem is that, if $f(x)$ be a function capable of expansion in ascending powers of x, then $f(x)$ may be expanded in a series of Bessel functions $a_0J_0(x)+a_1J_1(x)+a_2J_2(x)+\dots$, where

$$a_n = 2\left\{f(0)+\frac{n^2}{2!}f''(0)+\frac{n^2(n^2-2^2)}{4!}f^{\text{iv}}(0)+\dots+2^{n-1}f^{(n)}(0)\right\} \quad (n \text{ even}),$$

$$a_n = 2\left\{nf'(0)+\frac{n(n^2-1^2)}{3!}f'''(0)+\frac{n(n^2-1^2)(n^2-3^2)}{5!}f^{\text{v}}(0)+\dots \right.$$
$$\left. +2^{n-1}f^{(n)}(0)\right\} \quad (n \text{ odd}).$$

In this paper the generalization of Bessel's function $J_n(x)$ is denoted

$$J_{[n]}(x) = \Sigma(-1)^r\frac{x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}}$$
$$[p>1, \text{ reducing to } J_n(x) \text{ when } p=1], \quad (1)$$

$[r]$ denotes $(p^r-1)/(p-1)$, $[r]!$ denotes $[1][2][3]\dots[r]$, $(2)_r$ denotes $(p+1)(p^2+1)\dots(p^r+1)$, from which we see that

$$[r]!\,(2)_r = [2][4][6]\dots[2r],$$

which, for convenience, will be denoted $\{2r\}!$.

In general $[n]!$ is $\Gamma_p([n+1])$, which is

$$\underset{\kappa=\infty}{\mathrm{L}}\frac{[1][2][3][4]\dots[\kappa]}{[n+1][n+2]\dots[n+\kappa]}p^{\frac{1}{2}[n(n+1)]}[\kappa]^n, \quad (2)$$

$(2)_n$ is $(p+1)^n\dfrac{\Gamma_{p^2}([n+1])}{\Gamma_p([n+1])}$. Some properties of this Gamma function and the functions $J_{[n]}(x)$, $P^{\nu}_{[n]}(x)$, ... I have investigated elsewhere (*Trans. Roy. Soc. Edin.*, Vol. XLI., pp. 1–28).

If $f(x)$ be capable of expansion in the form $c_0+c_1x^{[1]}+c_2x^{[2]}+\dots$, then $f(x)$ may be expanded as

$$f(0)+f^{(1)}(0)\frac{x^{[1]}}{[1]!}+\dots+f^{(n)}(0)\frac{x^{[n]}}{[n]!}+\dots.$$

If now we assume the possibility of the expansion of $f(x)$ in a series of generalized Bessel's functions $a_0 J_{[0]}(x)+a_1 J_{[1]}(x)+a_2 J_{[2]}(x)+\ldots$, we may determine the coefficients $a_0, a_1, a_2, \ldots$; which will be independent of x, being in fact

$$a_n = \frac{[2n]}{[n]}\left\{p^{\frac{1}{4}[n(n-2)]} f(0)+p^{\frac{1}{4}[(n-2)(n-4)]}\frac{[n]^2}{[2]!} f^{(2)}(0)\right.$$

$$\left. +p^{\frac{1}{4}[(n-4)(n-6)]}\frac{[n+2][n]^2[n-2]}{[4]!} f^{(\text{iv})}(0)+\ldots+(2)_{n-1}\right\} \quad (n \text{ even}), \qquad (3)$$

$$a_n = \frac{[2n]}{[n]}\left\{p^{\frac{1}{4}[(n-1)(n-3)]}[n] f'(0)+p^{\frac{1}{4}[(n-3)(n-5)]}\frac{[n+2][n]^2[n-2]}{[3]!} f^{(\prime\prime\prime)}(0)\right.$$

$$\left. +\ldots+(2)_{n-1}\right\} \quad (n \text{ odd}).$$

Writing then (*supra*, p. 354)

$$f(z) = f^{(0)}(0)+f^{(1)}(0)\frac{x^{[1]}}{[1]!}+f^{(2)}(0)\frac{x^{[2]}}{[2]!}+\ldots, \qquad (4)$$

and also $f(z) = a_0 J_{[0]}(x)+a_1 J_{[1]}(x)+a_2 J_{[2]}(x)+\ldots$

$$= a_0\left\{1-\frac{x^{[2]}}{\{2\}!\,\{2\}!}+\frac{x^{[4]}}{\{4\}!\,\{4\}!}-\ldots\right\}$$

$$+a_1\left\{\frac{x^{[1]}}{\{0\}!\,\{2\}!}-\frac{x^{[3]}}{\{2\}!\,\{4\}!}+\frac{x^{[5]}}{\{4\}!\,\{6\}!}-\ldots\right\}$$

$$+a_2\left\{\frac{x^{[2]}}{\{0\}!\,\{4\}!}-\frac{x^{[4]}}{\{2\}!\,\{6\}!}+\frac{x^{[6]}}{\{4\}!\,\{8\}!}-\ldots\right\}$$

$$+\ldots, \qquad (5)$$

by equating the coefficients of $x^{[0]}$, $x^{[2]}$, $x^{[4]}$, ... in (4) and (5), we obtain the following series of equations for determining the even coefficients $a_0, a_2, a_4, \ldots$:—

$$a_0 = f(0),$$

$$\frac{a_2}{\{0\}!\,\{4\}!}-\frac{a_0}{\{2\}!\,\{2\}!} = \frac{1}{[2]} f^{(2)}(0),$$

$$\frac{a_4}{\{0\}!\,\{8\}!}-\frac{a_2}{\{2\}!\,\{6\}!}+\frac{a_0}{\{4\}!\,\{4\}!} = \frac{1}{[4]} f^{(\text{iv})}(0),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{a_n}{\{0\}!\,\{n\}!}-\frac{a_{n-2}}{\{2\}!\,\{2n-2\}!}+\ldots+(-1)^{\frac{1}{2}n}\frac{a_0}{\{n\}!\,\{n\}!} = \frac{1}{[n]!} f^{(n)}(0).$$

From these we find

$$a_0 = f(0),$$

$$a_2 = \frac{\{4\}!}{\{2\}!\,\{2\}!} f(0) + \frac{\{4\}!}{[2]!} f^{(2)}(0) = \frac{[4]}{[2]} \left\{ f(0) + \frac{[2]^2}{[2]!} f^{(2)}(0) \right\}$$

$$(\{4\}! = [4][2]).$$

Similarly we obtain without difficulty

$$a_4 = \frac{[8]}{[4]} \left\{ p^2 f(0) + \frac{[4]^2}{[2]!} f^{(2)}(0) + \frac{[6][4]^2[2]}{[4]!} f^{(\text{iv})}(0) \right\},$$

$$a_6 = \frac{[12]}{[6]} \left\{ p^6 f(0) + p^2 \frac{[6]^2}{[2]!} f^{(2)}(0) + \ldots + (2)_5 f^{(6)}(0) \right\},$$

$$a_8 = \frac{[16]}{[8]} \left\{ p^{12} f(0) + \frac{[8]^2}{[2]!} p^6 f^{(2)}(0) + \frac{[10][8]^2[6]}{[4]!} p^2 f^{(4)}(0) + \ldots + (2)_7 f^{(8)}(0) \right\},$$

$$a_{10} = \frac{[20]}{[10]} \left\{ p^{20} f(0) + \frac{[10]^2}{[2]!} p^{12} f^{(2)}(0) + \ldots + (2)_9 f^{(10)}(0) \right\}.$$

These coefficients are easily found; for the series which occur in the calculation are analogous to the series of Euler,

$$1 - a_2 + a_1(1 - a_2) + a_1 a_2 (1 - a_3) + \ldots,$$

and may be summed in the same way term by term.

Consider the series

$$1 - \frac{[a][b]}{[1][b+a-1]} + p \frac{[a+2].[a-1][b][b-1]}{[1][2][b+a-1][b+a]}$$

$$+ (-)^s p^{\frac{1}{2}[s(s+1)]} [a+2s] \frac{[a-1][a]\ldots[a+s-2].[b][b-1]\ldots[b-s]}{[s+1]!\,[b+a-1][b+a]\ldots[b+a+s-1]}, \qquad (6)$$

which is typical of all the series which occur in calculating the coefficients. The first two terms combined give $-p\dfrac{[a-1][b-1]}{[1][b+a-1]}$. Taking this with the third term, we have $-p\dfrac{[a-1][b-1]}{[1][b+a-1]}\left\{1 - \dfrac{[a+2][b]}{[2][b+a]}\right\}$, which is $p^3\dfrac{[a-1][a][b-1][b-2]}{[1][2][b+a-1][b+a]}$. Combining with the fourth term, we obtain $-p^6\dfrac{[a-1][a][a+1][b-1][b-2][b-3]}{[3]!\,[b+a-1][b+a][b+a+1]}$. To $s+1$ terms the sum is

$$(-1)^s p^{\frac{1}{2}[s(s+1)]} \frac{[a-1][a]\ldots[a+s-2].[b-1][b-2]\ldots[b-s]}{[s]!\,[b+a-1][b+a]\ldots[b+a+s-2]}. \qquad (7)$$

We proceed to establish the form of a_{n+2} by induction, assuming that the coefficients $a_0, a_2, \ldots, a_n$ (even) are of the form

$$a_n = \frac{[2n]}{[n]} \left\{ p^{\frac{1}{2}[n(n-2)]} f(0) + p^{(n-2)(n-4)} \frac{[n]^2}{[2]} f^{(2)}(0) + \ldots \right.$$

$$\left. + p^{\frac{1}{2}[(n-r)(n-r-2)]} \frac{[n+r-2] \ldots [n]^2 \ldots [n-r+2]}{[r]!} f^{(r)}(0) + \ldots + (2)_{n-1} f^{(n)}(0) \right\}, \tag{7A}$$

in which both n and r are necessarily even integers.

Now

$$\frac{a_{n+2}}{\{0\}!\,\{2n+4\}!} - \frac{a_n}{\{2\}!\,\{2n+2\}!} + \frac{a_{n-2}}{\{4\}!\,\{2n\}!} - \ldots$$

$$+ (-1)^{\frac{1}{2}(n+2)} \frac{a_0}{\{n+2\}!\,\{n+2\}} = \frac{1}{[n+2]!} f^{(n+2)}(0), \tag{8}$$

from which, replacing $a_n, a_{n-2}, \ldots, a_0$ by expressions of form (7A), we have

$$a_{n+2} = \frac{\{2n+4\}!}{\{2\}!\,\{2n+2\}!} \left\{ p^{\frac{1}{2}[n(n-2)]} f(0) + \ldots \right.$$

$$+ p^{\frac{1}{2}[(n-r)(n-r-2)]} \frac{[n+r-2] \ldots [n]^2 \ldots [n-r+2]}{[r]!} f^{(r)}(0) + \ldots$$

$$\left. + (2)_{n-1} f^{(r)}(0) \right\} \frac{[2n]}{[n]} + \frac{\{2n+4\}!}{[n+2]!} f^{(n+2)}(0)$$

$$- \frac{\{2n+4\}!}{\{4\}!\,\{2n\}!} \left\{ p^{\frac{1}{2}[(n-2)(n-4)]} f(0) + \ldots \right.$$

$$+ p^{\frac{1}{2}[(n-r-2)(n-r-4)]} \frac{[n+r-4] \ldots [n-2]^2 \ldots [n-r]}{[r]!} f^{(r)}(0) + \ldots$$

$$\left. + (2)_{n-2} f^{(n-1)}(0) \right\} \frac{[2n-4]}{[n-2]} + \ldots$$

$$+ (-1)^{\frac{1}{2}(n-r)} \frac{\{2n+4\}!}{\{n-r+2\}!\,\{n+r+2\}!} \left\{ p^{\frac{1}{2}[r(r-2)]} f(0) + \ldots \right.$$

$$\left. + p^0 \frac{[2r-2] \ldots [r]^2 \ldots [2]}{[r]!} f^{(r)}(0) \right\} \frac{[2r]}{[r]} \ldots . \tag{9}$$

Here we see that

$$a_{n+2} = \lambda f(0) + \lambda_2 f^{(2)}(0) + \lambda_4 f^{(4)}(0) + \ldots + \lambda_{n+2} f^{(n+2)}(0),$$

and that λ_r, the coefficient of $f^{(r)}(0)$ (r even), is

$$\frac{\{2n+4\}!}{[r]!}\left\{\frac{[2r-2]\dots[r]^2\dots[2]}{\{n-r+2\}!\,\{n+r+2\}!}\,\frac{[2r]}{[r]}\right.$$

$$-\frac{[2r]\dots[r+2]^2\dots[4]}{\{n-r\}!\,\{n+r+4\}!}\,\frac{[2r+4]}{[r+2]}+\dots$$

$$\left.+(-1)^{\frac{1}{2}(n-r)}p^{\frac{1}{2}[(n-r)(n-r-2)]}\frac{[n+r-2]\dots[n]^2\dots[n-r+2]}{\{2\}!\,\{2n+2\}!}\,\frac{[2n]}{[n]}\right\}. \tag{10}$$

The first two and the last two terms only are given.

This expression for λ_r may be written

$$\frac{\{2n+4\}!}{\{n-r+2\}!\,\{n+r+2\}!}\,\frac{\{2r\}!}{[r]!}\left\{1-\frac{[2r+4]}{[2]}\,\frac{[n-r+2]}{[n-r+4]}\right.$$

$$\left.+p^2\frac{[2r+8].[2r+2][n-r][n-r+2]}{[2][4][n+r+4][n+r+6]}+\dots\text{ to }\frac{n-r}{2}+1\text{ terms}\right\}. \tag{11}$$

The series within the large bracket is the typical series (6), and its sum is

$$p^{\frac{1}{2}[(n+2-r)(n-r)}\frac{[n-r][n-r-2]\dots[2].[2r+2]\dots[n+r]}{[2][4]\dots[n-r].[n+r+4]\dots[2n+2]}$$

$$=p^{\frac{1}{2}[(n+2-r)(n-r)]}\frac{\{n-r\}!\,\{n+r\}!\,\{n+r+2\}!}{\{n-r\}!\,\{2r\}!\,\{2n+2\}!}$$

$$=p^{\frac{1}{2}[(n+2-r)(n-r)]}\frac{\{n+r\}!\,\{n+r+2\}!}{\{2r\}!\,\{2n+2\}!};$$

so that a_{n+2} is

$$\sum_{r=0}^{r=n+2}\frac{[2n+4]}{[n+2]}p^{\frac{1}{2}[(n-r)(n-r+2)]}\frac{[n+r]\dots[n+2]^2\dots[n-r+4]}{[r]!}f^{(r)}(0),$$

which establishes the form of the coefficients in general.

Similarly, by equating the coefficients of $x^{[1]}$, $x^{[3]}$, ..., we obtain the following set of equations for determining a_n (n odd) :—

$$\frac{a_1}{\{0\}!\,\{2\}!}=\frac{1}{[1]!}f^{(1)}(0),$$

$$\frac{a_3}{\{0\}!\,\{6\}!}-\frac{a_1}{\{2\}!\,\{4\}!}=\frac{1}{[3]!}f'''(0),$$

... ...

$$\frac{a_n}{\{0\}!\,\{2n\}!}-\frac{a_{n-2}}{\{2\}!\,\{2n-2\}!}+\dots+(-1)^{\frac{1}{2}(n-1)}\frac{a_1}{\{n-1\}!\,\{n+1\}!}=\frac{1}{[n]!}f^{(n)}(0),$$

... ...

from which we find that

$$a_1 = \frac{[2]}{[1]} \{[1] f^{(1)}(0)\},$$

$$a_3 = \frac{[6]}{[3]} \left\{ [3] f^{(1)}(0) + \frac{[5][3]^2[1]}{[3]!} f^{(3)}(0) \right\},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_n = \frac{[2n]}{[n]} \left\{ p^{\frac{1}{2}[(n-1)(n-3)]} [n] f^{(1)}(0) + p^{\frac{1}{2}[(n-3)(n-5)]} \frac{[n+1][n]^2[n-1]}{[3]!} f^{(3)}(0) + \dots \right.$$
$$\left. + (2)_{n-1} f^{(n)}(0) \right\}. \quad (13)$$

Consider $$f(x) = 1 - \frac{x^{[2]}}{[2]!} + \frac{x^{[4]}}{[4]!} - \dots \quad (p > 1),$$

analogous to $\cos x$; then $f(0) = 1$, $f^{(2)} = -1$, $f^{(4)}(0) = 1$, and

$$a_n = \frac{[2n]}{[n]} \left\{ p^{\frac{1}{2}[n(n-2)]} - p^{\frac{1}{2}[(n-2)(n-4)]} \frac{[n]^2}{[2]!} + \dots + (-1)^n \frac{[2n-2]\dots[n]^2\dots[2]}{[n]!} \right\}$$
$$= (-2)^{\frac{1}{2}n} p^{\frac{1}{2}[n(n-1)]} \frac{[2n]}{[n]},$$

$$f(x) = J_{[0]}(x) - p \frac{[4]}{[2]} J_{[2]}(x) + p^6 \frac{[8]}{[4]} J_{[4]}(x) - p^{15} \frac{[12]}{[6]} J_{[6]}(x) + \dots \quad (p \geqslant 1), \quad (14)$$

analogous to $\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \dots$.

There is a similar theorem for

$$\phi(x) = x^{[1]} - \frac{x^{[3]}}{[3]!} + \dots \quad (p \geqslant 1).$$

ON AN EXPRESSION OF THE ELECTROMAGNETIC FIELD DUE TO ELECTRONS BY MEANS OF TWO SCALAR POTENTIAL FUNCTIONS

By E. T. WHITTAKER.

[Received 11th November, 1903.—Read 12th November, 1903.]

1. *Object of Paper.*

The object of the present paper is to show that when any number of electrons are moving in any manner the functions which define the resulting electrodynamic field, namely, the three components of dielectric displacement in the æther and the three components of the magnetic force at every point of the field, can be expressed in terms of the derivates of two scalar potential functions. (Previous writers have expressed them in terms of a scalar potential function and a vector potential function, which are equivalent to *four* scalar potential functions.) These two scalar potential functions are explicitly evaluated in terms of the charges and co-ordinates of the electrons. It is then shown that from these results the general functional form of an electrodynamic disturbance due to electrons can be derived.

2. *Explanation of Notation, and Summary of previously known Results.*

The work of previous writers, so far as it concerns the present investigation and explains the notation used, may be briefly summarized as follows:—

Let ρ be the volume density of electricity at any place and time, and let v_x, v_y, v_z be the components of its velocity, and c the velocity of light in the æther. Let d_x, d_y, d_z be the three components of the dielectric displacement in the æther, and h_x, h_y, h_z the three components of the magnetic force. Then the fundamental equations of electrodynamics may be written in Lorentz's form (the units being suitably chosen):

$$\frac{\partial \rho}{\partial t}+\frac{\partial (\rho v_x)}{\partial x}+\frac{\partial (\rho v_y)}{\partial y}+\frac{\partial (\rho v_z)}{\partial z}=0,$$

$$\frac{\partial d_x}{\partial x}+\frac{\partial d_y}{\partial y}+\frac{\partial d_z}{\partial z}=\rho, \qquad \frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0,$$

$$c\left(\frac{\partial h_z}{\partial y}-\frac{\partial h_y}{\partial z}\right)=\frac{\partial d_x}{\partial t}+\rho v_x, \qquad c\left(\frac{\partial d_z}{\partial y}-\frac{\partial d_y}{\partial z}\right)=-\frac{\partial h_x}{\partial t},$$

$$c\left(\frac{\partial h_x}{\partial z}-\frac{\partial h_z}{\partial x}\right)=\frac{\partial d_y}{\partial t}+\rho v_y, \qquad c\left(\frac{\partial d_x}{\partial z}-\frac{\partial d_z}{\partial x}\right)=-\frac{\partial h_y}{\partial t},$$

$$c\left(\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}\right)=\frac{\partial d_z}{\partial t}+\rho v_z, \qquad c\left(\frac{\partial d_y}{\partial x}-\frac{\partial d_x}{\partial y}\right)=-\frac{\partial h_z}{\partial t}.$$

In place of d_x, d_y, d_z, h_x, h_y, h_z, we can define the field by a scalar potential function ϕ and three functions a_x, a_y, a_z, which are usually regarded as the three components of a vector potential. The quantities d_x, d_y, d_z, h_x, h_y, h_z are given in terms of ϕ, a_x, a_y, a_z by the equations

$$d_x=-\frac{1}{c}\frac{\partial a_x}{\partial t}-\frac{\partial \phi}{\partial x}, \qquad h_x=\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z},$$

and four similar equations for d_y, d_z, h_y, h_z.

The scalar potential and the three components of the vector potential satisfy the system of equations

$$c^2\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}=-c^2\rho, \qquad c^2\nabla^2 a_x-\frac{\partial^2 a_x}{\partial t^2}=-c\rho v_x,$$

$$c^2\nabla a_y-\frac{\partial^2 a_y}{\partial t^2}=-c\rho v_y, \qquad c^2\nabla^2 a_z-\frac{\partial^2 a_z}{\partial t^2}=-c\rho v_z,$$

$$\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}+\frac{1}{c}\frac{\partial \phi}{\partial t}=0.$$

For the fundamental case, namely, that in which the field is due to any number of electrons moving in any way, the scalar potential and the three components of the vector potential are given by the equations

$$\phi(x, y, z, t)=\Sigma\frac{ec}{4\pi}\cdot\frac{1}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})}, \qquad a_x(x, y, z, t)=\Sigma\frac{e}{4\pi}\frac{\bar{v}_x}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})},$$

and two similar equations for a_y and a_z, where e is the charge on a typical electron, r is its distance from the point (x, y, z), v is its velocity, (v_x, v_y, v_z) the components of v, (v, r) the angle between the direction of v and r, and the bars over the letters mean that the position of the electron considered is that which it occupied at a time $t-\bar{r}/c$; and the summation is taken over all the electrons. We shall assume throughout the paper that the velocities of all the electrons are less than the velocity of radiation.

3. *Introduction and Evaluation of the two Scalar Potentials.*

Now let $x'(t)$, $y'(t)$, $z'(t)$ denote the position of the electron e at time t; and let $\bar{x}'$ be used to denote $x'(t-\bar{r}/c)$, so that $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ are known functions of x, y, z, t, when the motions of the electrons are known; we have $\bar{r}^2 = (\bar{x}'-x)^2+(\bar{y}'-y)^2+(\bar{z}'-z)^2$, and therefore

$$\bar{r}\frac{\partial\bar{r}}{\partial x} = -(\bar{x}'-x)-\{(\bar{x}'-x)v_x+(\bar{y}'-y)v_y+(\bar{z}'-z)v_z\}\frac{1}{c}\frac{\partial\bar{r}}{\partial x}$$

or

$$\frac{\partial\bar{r}}{\partial x} = -\frac{c(\bar{x}'-x)}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})}.$$

More generally, if f be any function of the three quantities $\bar{x}'-x$, $\bar{y}'-y$, $\bar{z}'-z$, and if f_1, f_2, f_3 denote its derivates with respect to these three arguments respectively, we easily find that

$$\frac{\partial f}{\partial x} = -f_1+\frac{(\bar{x}'-x)(f_1\bar{v}_x+f_2\bar{v}_y+f_3\bar{v}_z)}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})}.$$

Similar equations hold for $\partial f/\partial y$ and $\partial f/\partial z$, while $\partial f/\partial t$ is given by the equation

$$\frac{\partial f}{\partial t} = \frac{c(f_1\bar{v}_x+f_2\bar{v}_y+f_3\bar{v}_z)}{c+\bar{v}\cos(\bar{v},\bar{r})}.$$

Now define functions F, G, ψ by the equations

$$F(x, y, z, t) = \Sigma\frac{e}{4\pi}\sinh^{-1}\frac{\bar{z}'-z}{\{(\bar{x}'-x)^2+(\bar{y}'-y)^2\}^{\frac{1}{2}}},$$

$$G(x, y, z, t) = \Sigma\frac{e}{4\pi}\tan^{-1}\frac{\bar{y}'-y}{\bar{x}'-x},$$

$$\psi(x, y, z, t) = \Sigma\frac{e}{4\pi}\log\{(\bar{x}'-x)^2+(\bar{y}'-y)^2\}^{\frac{1}{2}},$$

where the summation is taken over all the electrons. Using the formulæ just obtained for the derivates of a function of the kind f, we find that

$$\frac{\partial\psi}{\partial x}-\frac{\partial G}{\partial y} = \Sigma\frac{e}{4\pi}\frac{\bar{v}_x}{c\bar{r}+r\bar{v}\cos(\bar{v},\bar{r})},$$

$$\frac{\partial\psi}{\partial y}+\frac{\partial G}{\partial x} = \Sigma\frac{e}{4\pi}\frac{\bar{v}_y}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})},$$

$$\frac{\partial\psi}{\partial z}+\frac{1}{c}\frac{\partial F}{\partial t} = \Sigma\frac{e}{4\pi}\frac{\bar{v}_z}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})},$$

$$\frac{1}{c}\frac{\partial\psi}{\partial t}+\frac{\partial F}{\partial z} = -\Sigma\frac{ec}{4\pi}\frac{1}{c\bar{r}+\bar{r}\bar{v}\cos(\bar{v},\bar{r})}.$$

Combining these results with the expressions already found for ϕ, a_x, a_y, a_z, we have

$$\frac{\partial\psi}{\partial x} - \frac{\partial G}{\partial y} = a_x, \qquad \frac{\partial\psi}{\partial y} + \frac{\partial G}{\partial x} = a_y, \qquad \frac{\partial\psi}{\partial z} + \frac{1}{c}\,\frac{\partial F}{\partial t} = a_z,$$

$$\frac{1}{c}\,\frac{\partial\psi}{\partial t} + \frac{\partial F}{\partial z} = -\phi.$$

Substituting these results for ϕ, a_x, a_y, a_z in the equations of the type

$$d_x = -\frac{1}{c}\,\frac{\partial a_x}{\partial t} - \frac{\partial\phi}{\partial x}, \qquad h_x = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z},$$

which give the components of the dielectric displacement and the magnetic force, *we find that* ψ *disappears automatically*, and we obtain

$$d_x = \frac{\partial^2 F}{\partial x\,\partial z} + \frac{1}{c}\,\frac{\partial^2 G}{\partial y\,\partial t}, \qquad d_y = \frac{\partial^2 F}{\partial y\,\partial z} - \frac{1}{c}\,\frac{\partial^2 G}{\partial x\,\partial t}, \qquad d_z = \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2}\,\frac{\partial^2 F}{\partial t^2};$$

$$h_x = \frac{1}{c}\,\frac{\partial^2 F}{\partial y\,\partial t} - \frac{\partial^2 G}{\partial x\,\partial z}, \qquad h_y = -\frac{1}{c}\,\frac{\partial^2 F}{\partial x\,\partial t} - \frac{\partial^2 G}{\partial y\,\partial z}, \qquad h_z = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}.$$

These equations show that the six components of the dielectric displacement and the magnetic force can be expressed in terms of the derivates of two scalar potentials F and G, defined by the equations

$$F(x,\, y,\, z,\, t) = \Sigma\,\frac{e}{4\pi}\sinh^{-1}\frac{\bar{z}'-z}{\{(\bar{x}'-x)^2+(\bar{y}'-y)^2\}^{\frac{1}{2}}},$$

$$G(x,\, y,\, z,\, t) = \Sigma\,\frac{e}{4\pi}\tan^{-1}\frac{\bar{y}'-y}{\bar{x}'-x},$$

where the summation is taken over all the electrons in the field.

It can without difficulty be shown that, if any number of electrons whose total charge is zero are moving in any manner so as to remain always in the vicinity of a given point (*i.e.*, to be in *stationary* motion), then the electromagnetic field thus generated is of the type given by

$$F = \frac{1}{r}\,f\left(t - \frac{r}{c}\right), \qquad G = 0,$$

where r is the distance from the point and f is an arbitrary function; or, more generally, of a field of this type superposed on fields of the same type, but related to the axes of y and x in the same way as this is related to the axis of z. This is perhaps of some interest in connection with the view advocated by some physicists that the atoms of the chemical elements consist of sets of electrons, whose total charge is zero, in stationary motion.

4. *Discussion of the Apparent Asymmetry of the preceding Result, and its Vector Expression.*

The formulæ thus obtained are not symmetrical with respect to x, y, and z. In order to discuss their relation to symmetrical formulæ, we observe that they can be written in the form of vector equations

$$d = \operatorname{curl}\operatorname{curl} f + \operatorname{curl}\frac{1}{c}\dot{g}, \qquad h = \operatorname{curl}\frac{1}{c}\dot{f} - \operatorname{curl}\operatorname{curl} g,$$

where d and h are the electric and magnetic vectors, and f and g are vectors directed parallel to the axis of z, whose magnitudes are F and G respectively. These vector equations are quite symmetrical, and our result is that, if, instead of regarding the electromagnetic field as defined by the vectors d and h, we regard it as defined by vectors f and g, connected with d and h by the above vector equations, then f and g are simple functions of the coordinates of the electrons, whereas d and h are complicated functions of their velocities and accelerations; and we have also obtained the result that without loss of generality we can take f and g to be everywhere, and at all times, parallel to some fixed direction in space (*e.g.*, the axis of z), a fact which makes it possible to specify them by two scalar quantities only.

It might be asked whether vectors f and g exist which satisfy the above vector equations and which are perfectly symmetrical—the answer to this is in the negative; in fact, although the equations are themselves invariantive, and can therefore be expressed in the vector notation, yet they do not possess invariant solutions; just as the vector equation

$$\operatorname{grad}\left(\frac{1}{r}\right) = \operatorname{curl} a$$

(where r is the scalar distance from the origin and a is a vector to be determined) possesses an infinite number of solutions a, which can readily be found, but each of which is specially related to some line in space, so that no solution is symmetrical.

5. *Deduction of the General Functional Form of an Electrodynamic Disturbance in the Æther.*

Having now shown that an electrodynamic field due to electrons is completely characterized by two scalar potential functions F and G, we can proceed to deduce its general functional form.

The functions F and G have singularities at those points which are actually occupied by electrons; at all other points we find by direct differentiation, or by substituting in the original electrodynamic equations the values of the components of dielectric displacement and magnetic force in terms of F and G, that F and G satisfy the partial differential equations

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}=0,$$

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}+\frac{\partial^2 G}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}=0.$$

Writing down the general solution of these latter equations,* we obtain the result that *the most general type of electrodynamic disturbance at a place in the æther not occupied by an electron is that for which the components of the dielectric displacement and magnetic force are represented by the equations*

$$d_x=\frac{\partial^2 F}{\partial x\,\partial z}+\frac{1}{c}\frac{\partial^2 G}{\partial y\,\partial t},\qquad d_y=\frac{\partial^2 F}{\partial y\,\partial z}-\frac{1}{c}\frac{\partial^2 G}{\partial x\,\partial t},\qquad d_z=\frac{\partial^2 F}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2};$$

$$h_x=\frac{1}{c}\frac{\partial^2 F}{\partial y\,\partial t}-\frac{\partial^2 G}{\partial x\,\partial z},\qquad h_y=-\frac{1}{c}\frac{\partial^2 F}{\partial x\,\partial t}-\frac{\partial^2 G}{\partial y\,\partial z},\qquad h_z=\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2},$$

where

$$F=\int_0^\pi\int_0^{2\pi} f(x\sin u\cos v+y\sin u\sin v+z\cos u+ct,\ u,\ v)\,du\,dv,$$

$$G=\int_0^\pi\int_0^{2\pi} g(x\sin u\cos v+y\sin u\sin v+z\cos u+ct,\ u,\ v)\,du\,dv,$$

and f and g are arbitrary functions of their arguments.

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* Cf. a paper by the author in *Math. Ann.*, Vol. LVII., pp. 333–355, 1903.

ON MODES OF CONVERGENCE OF AN INFINITE SERIES OF FUNCTIONS OF A REAL VARIABLE

By E. W. HOBSON.

[Received and Read, December 10th, 1903.]

IT is well known that a series of functions of a real variable, each of which is continuous in a given interval, and such that for every point in the interval the sum of n terms of the series converges as n is increased indefinitely to a definite number, has a continuous function for its sum, provided the convergence of the series is uniform in the interval, but that this condition, though sufficient, is not necessary. A less stringent condition which is also sufficient, but not necessary, for the continuity of the sum-function, was introduced by Dini, and has been adopted by other writers: this condition is that the series must be simply uniformly convergent; in the first part of the present paper some remarks are made upon this mode of convergence. The necessary and sufficient conditions that the sum-function may be continuous throughout its domain have been given by Arzelà.* They consist in the necessity for a mode of convergence of the series of a less stringent character than either of the above modes: this mode is called "uniform convergence by intervals" and includes the above-mentioned modes as special cases. A principal object of the present communication is to show that the necessity and sufficiency of this mode of convergence may be established in a simple manner by means of an application of the Heine-Borel theorem concerning an infinite set of intervals. The latter part of the paper is concerned with the determination of the necessary and sufficient conditions that the sum-function may be an integrable function, the terms of the series being no longer necessarily continuous, but being assumed to be integrable; the main condition is expressed by saying that the series must have a mode of convergence called "uniform convergence by intervals in general"— a term due also to Arzelà. The conditions that an integrable sum-function is such that its integral is obtained as the sum of the integrals of the terms of the series are not dealt with here. Special cases have been treated by Osgood,† by myself,‡ and by W. H. Young,§ and the matter has been treated more generally by Arzelà.* A criticism is given of Arzelà's proof of the conditions of integrability of the sum-function; the proof given in the present paper depends upon an application of the Heine-Borel theorem.

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* See his memoir in two parts "Sulle serie di funzioni," *Memorie della R. Accad. degli Sci. di Bologna*, Series 5, Vol. VIII., 1900.

† *American Journal*, Vol. XIX.

‡ *Proc. London Math. Soc.*, Vol. XXXIV.

§ *Supra*, p. 89.

Simple Uniform Convergence.

1. Let us suppose that $u_1(x)$, $u_2(x)$, ..., $u_n(x)$, ... denote an unending sequence of functions each of which is defined for every point of the continuous interval (a, b) of the real variable x; moreover, suppose that the sum $s_n(x) \equiv u_1(x)+u_2(x)+\ldots+u_n(x)$, for each value of x in (a, b), converges as n is increased indefinitely to a definite number $s(x)$ which depends upon the value of x; the difference $s(x)-s_n(x) = R_n(x)$, we call the "remainder function."

The series $u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$ is said to converge uniformly in the interval (a, b), to the limiting sum $s(x)$, when corresponding to every arbitrarily chosen positive number σ an integer m independent of x can be found such that $|R_m(x)|$, $|R_{m+1}(x)|$, $|R_{m+2}(x)|$ ... are all less than σ, whatever value x may have. When the condition of uniform convergence in the interval is not satisfied the mode of convergence may be that which has been termed by Dini "simple uniform convergence." Dini's† definition is as follows:—The series is said to be simply uniformly convergent in the interval (a, b) when corresponding to every arbitrarily chosen positive number σ as small as we please and to every integer m', only one or several integers m exist which are not less than m', and are such that, for all values of x in the interval (a, b), the $|R_m(x)|$ are $<\sigma$.

The condition of simple uniform convergence is less stringent than that of uniform convergence, in that in the latter case all the remainders after a certain one are numerically less than σ, whereas in the former case one or several, but not all, the remainders are numerically less than σ.

As regards Dini's definition, it may in the first place be remarked that, if there is one value of m which satisfies the prescribed condition, there must be an infinite number of such values, because we have only to ascribe to m' a series of values increasing indefinitely, and for each of these there is a value of m; any one of this infinite series of values of m may be taken to correspond to the smallest value of m'.

Moreover the definition contains a redundancy; it will here be shown that, if, corresponding to every σ, a value of n can be found such that $|R_n(x)| < \sigma$, independently of x, then there must be an indefinitely great number of such values of n. Let us denote by $\overline{R}_s$ the upper limit* of $|R_s(x)|$ for the whole interval (a, b); $\overline{R}_s$ may be either finite or indefinitely great. If $|R_n(x)| < \sigma$, for all values of x, we have $\overline{R}_n \leqslant \sigma$: take a positive number ϵ_1 less than $\overline{R}_n$ and also less than the least of the numbers $\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_{n-1}$; then by hypothesis a number n_1 can be found such that for all values of x, $|R_{n_1}(x)| < \epsilon_1$. This number n_1 cannot be one of the numbers $1, 2, 3, \ldots, n$, for it is always possible to find a value of x for which $|R_s(x)|$ is arbitrarily near its upper limit $\overline{R}_s$, and is thus greater than ϵ_1; hence a number $n_1 > n$ has been shown to exist such that, for all the values of x, $|R_{n_1}(x)| < \sigma$. Similarly, it may be shown that a number $n_2 > n_1$ exists which has the same property; thus there are an indefinite series of values of n such that $|R_n(x)| < \sigma$.

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* See the paper already referred to.

† See *Grundlagen*, by Lüroth and Schepp, p. 137.

It thus appears that the definition may be more simply stated as follows:—The series is said to converge simply uniformly in the interval if, corresponding to every arbitrarily chosen number σ as small as we please, a number n can be found such that, independently of x, $|R_n(x)| < \sigma$.

If the series converges simply uniformly, but not uniformly, there must also be an indefinitely great number of values of n for which the condition $|R_n(x)| < \sigma$, for all values of x, is not satisfied; for, if there were only a finite number of such values of n, we could take a value of n greater than all of them, and, starting from this value of n, the condition of uniform convergence would be satisfied.

2. Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be a sequence of diminishing positive numbers which converges to zero: if the series is simply uniformly convergent, we can find n_1 so that $|R_{n_1}(x)| < \epsilon_1$, for all values of x; we can then find n_2 a number greater than n_1, so that $|R_{n_2}(x)| < \epsilon_2$; then $n_3 > n_2$ such that $|R_{n_3}(x)| < \epsilon_3$; and so on. If now we amalgamate the first n_1 terms into one, then those after the first n_1 up to and including $u_{n_2}(x)$ into one term, and so on, the series may be written

$$s_{n_1}(x) + [s_{n_2}(x) - s_{n_1}(x)] + [s_{n_3}(x) - s_{n_2}(x)] + \ldots;$$

in this form the series is uniformly convergent. It thus appears that a simply uniformly convergent series can be changed into a uniformly convergent series by bracketing the terms suitably, and amalgamating the functions within each bracket. Conversely, a uniformly convergent series may be replaced by one which is only simply uniformly convergent; if each term $u_n(x)$ of a uniformly convergent series be replaced by the sum of r_n functions, such that $u_n(x) = U_{n,1}(x) + U_{n,2}(x) + \ldots + U_{n,r_n}(x)$, the new series $U_{1,1}(x) + U_{1,2}(x) + \ldots + U_{1,r_1}(x) + U_{2,1}(x) + \ldots$ is not necessarily convergent, but, if the functions U are so chosen that the series converges for every value of x in (a, b), then the series converges at least simply uniformly. It thus appears[*] that the distinction between uniform convergence and simple uniform convergence is an unessential one.

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* The trivial case in which $R_n(x) = 0$ for every x is omitted from consideration.

Conditions for the Continuity of the Sum-Function at a Point.

3. Let us now suppose that all the functions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ are continuous throughout the interval (a, b); it is known that, if $s(x)$ is discontinuous at a point x, the series is non-uniformly convergent in any neighbourhood $(x-\delta,\ x+\delta')$ of the point x, but that, if it is so non-uniformly convergent, the function $s(x)$ is not necessarily discontinuous at the point x. It is frequently convenient to replace the remainder $R_n(x)$ by the transformed remainder-function[†] $R(x, y)$, obtained by taking $y = 1/n$, and considering $R(x, y)$ as a function of the two variables x, y; the function $R(x, 0)$ is defined to be zero for all values of x. At a point x of uniform continuity of the series the function $|R(x, y)|$ is continuous at the point $(x, 0)$ with regard to (x, y); at a point x of non-uniform continuity of the series the function $|R(x, y)|$ is discontinuous at $(x, 0)$ with regard to (x, y), the saltus of the function at the point giving the measure of non-uniform convergence at that point. To find the necessary and sufficient conditions that $x_1(P)$ is a point of continuity of $s(x)$, let us first assume this to be the case: then a neighbourhood of P can be found such that, if p be any point whatever, in that neighbourbood, $|s(p)-s(P)| < \frac{1}{3}\epsilon$, where ϵ is arbitrarily chosen. Now let n be so chosen that $|R_n(P)| < \frac{1}{3}\epsilon$: this is possible on account of the convergency condition of the series at the point P. Since $s_n(x)$ is continuous, a neighbourhood of P can be found such that, if p is any point in it, $|s_n(p)-s_n(P)| < \frac{1}{3}\epsilon$; take now that neighbourhood of P which is the common part of the two neighbourhoods which have been already chosen, denote this by $(x-h, x+k)$; then, if p is any point in $(x-h, x+k)$, we have both $|s(p)-s(P)| < \frac{1}{3}\epsilon$ and $|s_n(p)-s_n(P)| < \frac{1}{3}\epsilon$. Since

$$R_n(p)-R_n(P) = \{s(p)-s(P)\} - \{s_n(p)-s_n(P)\},$$

we have $\quad |R_n(p)-R_n(P)| \leqslant |s(p)-s(P)| + |s_n(p)-s_n(P)|$;

hence, if p is any point in $(x-h, y+k)$, we have $|R_n(p)-R_n(P)| < \frac{2}{3}\epsilon$, and, since $|R_n(P)| < \frac{1}{3}\epsilon$, we have $|R_n(p)| < \epsilon$. It has thus been shown that, if $s(x)$ is continuous at x, and n has a sufficiently great value, viz., one such that $| R_n(x) | < \frac{1}{3}\epsilon$, a neighbourhood $(x-h, x+k)$ of x can be found such that everywhere in it $| R_n(x) | < \epsilon$; this condition is necessary for the continuity of $s(x)$ at the point x. Conversely, assume this condition to be satisfied for every value of n sufficiently great; then it will be shown that x is a point of continuity of $s(x)$. A neighbourhood of x can be so chosen that for any point p in it both $| s_n(p)-s_n(P) | < \eta$, where η is an arbitrarily chosen positive number, and also

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[*] I find that the substance of Art. 2 is contained in a paper by Arzelà in the *Bologna Rendiconti* for 1899. However, I retain Art. 2 in the present paper for the sake of completeness.

[†] See *Proc. London Math. Soc.*, Vol. XXXIV.

$$| R_n(p)-R_n(P) | \leqslant | R_n(p) | + | R_n(P) | < \tfrac{4}{3}\epsilon ;$$

in this neighbourhood of P, $| s(p)-s(P) | < \frac{4}{3}\epsilon+\eta$, and thus, since ϵ and η are both at choice, the function $s(x)$ is continuous at x. It has now been shown that the necessary and sufficient conditions that $s(x)$ may be continuous at the point P are that corresponding to every fixed positive number ϵ chosen arbitrarily, and as small as we please, and for any value whatever of n which is greater than or equal to a fixed number dependent on ϵ, a neighbourhood of P can be found such that at every point in it $| R_n(x) | < \epsilon$.

4. Let the function $R(x, y)$ be represented on the plane of (x, y), the function being defined for all values of x in the interval (a, b) and for all values of y in the interval $(0, 1)$; the values of the function for those values of y which are not the reciprocal of an integer may be defined[*] so that $R(x, y)$ is a continuous function of y; moreover, $R(x, 0) = 0$. If a positive number ϵ is prescribed, then for a certain range of values of y from zero upwards the condition $| R(x, y) | < \epsilon$ is satisfied; the upper limit of these values of y may be denoted by $\phi_\epsilon(x)$; there may be other greater values of y not continuous with the interval $(0, \phi_\epsilon(x))$ for which the condition $| R(x, y) | < \epsilon$ is also satisfied. At a point P of non-uniform convergence of the series, the lower limit of $\phi_\epsilon(x)$ for the values of x in any neighbourhood of P is zero, whereas for a point of uniform convergence a neighbourhood of P can be found for which the lower limit of $\phi_\epsilon(x)$ is greater than zero.

The distinction between the three classes of points in the interval (a, b), viz., (i.) those at which the series is uniformly convergent, (ii.) those at which the series is non-uniformly convergent, but at which the sum-function is continuous, and (iii.) those points at which the function is discontinuous, may be illustrated by means of figures which indicate the regions of the plane (x, y) in the neighbourhood of $(x, 0)$, at which $| R(x, y) |$ is less than an arbitrarily chosen ϵ.

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[*] See the paper already referred to, *Proc. London Math. Soc.*, Vol. XXXIV.

case the end-point $x = 0$ is a point of discontinuity of $R(x, y)$, and we find that, if $\epsilon < \frac{1}{2}$, the condition $|R_n(x)| < \epsilon$ is satisfied for the space bounded by the x-axis and by the straight line

$$y = \phi_\epsilon(x) = x\left[\frac{1}{2\epsilon} - \left(\frac{1}{4\epsilon^2} - 1\right)^{\frac{1}{2}}\right].$$

The same condition is satisfied for the space between the y-axis and the straight line $y = x\left[\frac{1}{2\epsilon} + \left(\frac{1}{4\epsilon^2} - 1\right)^{\frac{1}{2}}\right]$, and thus the point $x = 0$ is a point of continuity of the function $s(x)$, although the convergence at that point is non-uniform. If $\epsilon > \frac{1}{2}$, $|R(x, y)| < \epsilon$, for the whole space between the axes; thus the measure of non-uniform convergence is $\frac{1}{2}$.

5. *Conditions for the Continuity of the Sum-Function in its whole Domain.*

The necessary and sufficient conditions can now be determined that $s(x)$ may be continuous for the whole interval (a, b). First suppose that $s(x)$ is everywhere continuous, and choose an arbitrarily small number ϵ; then, if n is sufficiently large, there are points in (a, b) at which $|R_n(x)| < \epsilon$. If P be such a point, a neighbourhood of P can be found such that at every point within it the condition $|R_n(x)| < \epsilon$ is satisfied; the size of this neighbourhood may be extended in both directions until points p, q are reached, at which $|R_n(p)| = |R_n(q)| = \epsilon$; this follows from the fact that $R_n(x)$ is a continuous function. Every such point P in (a, b) at which $|R_n(x)| < \epsilon$ may, in a similar manner, be enclosed in an interval of finite length, and in all internal points of such intervals the condition $|R_n(x)| < \epsilon$ is satisfied. Thus for any fixed value of n which is sufficiently large there exists a finite or infinite system D_n of intervals in (a, b) such that at every point which is interior to one of the intervals of D_n the condition $|R_n(x)| < \epsilon$ is satisfied; moreover, the intervals of D_n contain in their interiors all points except a, b at which the condition is fulfilled. Two intervals of the set may have a common end-point in case $|R_n(x)|$ has a maximum equal to ϵ; otherwise the intervals will be separated from one another.

Consider the systems of intervals D_n, D_{n+1}, D_{n+2}, ..., every value of n being taken from a fixed value onwards: the whole set thus formed is such that every point in (a, b), except the end-points, is interior to an infinite number of intervals of the compound set; this follows from the fact that, for any point x, a value of n, say n_1, can be found such that $|R_{n_1}(x)| < \epsilon$, $|R_{n_1+1}(x)| < \epsilon$, ... Moreover, intervals can be found with

a and b as end-points which for a sufficiently large value of m belong to D_{n+m}. The Heine-Borel theorem states that, if in (a, b) any infinite set of intervals is taken such that every point within (a, b) is an internal point of one interval at least of the set, and such that a, b are end-points of two intervals of the set, then a finite number of the intervals can be selected which is such that every internal point of (a, b) is in the interior of one at least of the finite set. If we apply this theorem to the set of intervals D_n, $D_{n+1}, \ldots$, we see that a finite number of these intervals can be chosen which contains every point within (a, b) as an internal point of one at least of the intervals. The number n can be chosen so great that $|R_n(a)|$ and $|R_n(b)|$ are both less than ϵ, and thus that the points a, b are end-points of two of the finite set of intervals. It thus appears that on the supposition that $s(x)$ is continuous at every point of (a, b), and n is a chosen number, then a finite series of numbers $n+\iota_1, n+\iota_2, \ldots, n+\iota_r$, all greater than or equal to n, can be chosen such that, for every value of x, $|R_m(x)| < \epsilon$ where m has one of the values $n+\iota_1, n+\iota_2, \ldots, n+\iota_r$; the particular value of m varies with x, but the same value of m is applicable to the whole of one of a finite number of continuous intervals. The intervals of the set will overlap, but an overlapping portion may be considered as belonging to one of the intervals only, so that (a, b) may be divided into a finite number of parts, in each of which for some value of m constant for that part, the condition $|R_m(x)| \leqslant \epsilon$ is satisfied.

It should be remarked that the set D_n for a fixed n is not necessarily a finite set, for, besides those intervals for which $|R_n(x)| < \epsilon$ and those for which $|R_n(x)| > \epsilon$, there may be points of (a, b) which are not in either set of intervals, but are limiting points of end-points of the intervals D_n; at such points $|R_n(x)| = \epsilon$. For example, if

$$|R_n(x)| = \frac{1}{n^2} + \frac{x-c}{n^3} \sin\left(\frac{1}{x-c}\right),$$

where $a < c < b$, and, if $\epsilon = 1/n^2$, the point c is a point of continuity of $R_n(x)$, and is a limiting point of end-points of those intervals for which $R_n(x)| < \epsilon$. Conversely, if, for every value of ϵ, a finite set of intervals exists, having the property described, then $s(x)$ is continuous throughout (a, b). Consider a point P of non-uniform convergence of the series: then, for a given ϵ, P is inside an interval for every point of which, for a fixed value of m, $|R_m(x)| < \epsilon$, and we have shown that this is a sufficient condition that $s(x)$ is continuous at the point P; hence every point of non-uniform convergence of the series is a point of continuity.

It has now been established that *the necessary and sufficient condition* *that a series* $u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$, *each term of which is a continuous function of* x *throughout* (a, b) *and converges at every point of this domain to a definite value* $s(x)$, be cont. *is that, corresponding to any arbitrarily chosen number* ϵ *and to an arbitrarily chosen integer* n, *we have* $|R_m(x)| < \epsilon$, *for every value of* x *in* (a, b), *where* m *has one of a finite number of values greater than or equal to* n, *and that the value of* m *depends in general on that of* x, *but is constant for points* x *which lie in one of a number of finite portions of the interval* (a, b).

This theorem, which has been established otherwise by Prof. Arzelà, shows that a certain mode of convergence of the series is the necessary and sufficient condition for the continuity of the sum; this mode has been termed by Arzelà *convergenza uniforme a tratti* ("uniform convergence by intervals"); this term is, perhaps, not altogether appropriate, because the intervals are dependent in number and length upon the arbitrarily chosen ϵ. Uniform convergence and simple uniform convergence are special cases of uniform convergence by intervals; they correspond to the case in which the finite set of intervals which correspond to an ϵ reduce to one interval, namely, the whole interval (a, b).

Conditions of Integrability of the Sum of a Series of Integrable Functions.[*]

6. Let the functions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ be no longer necessarily continuous functions, but let us suppose that each one has a proper integral in every interval contained in (a, b); that this may be the case it is necessary that the upper limits of each of $|s_1(x)|$, $|s_2(x)|$, ..., $|s_n(x)|$, ... in the interval (a, b) is finite. Denoting these upper limits by $l_1, l_2, \ldots, l_n, \ldots$, then, provided $l_1, l_2, \ldots l_n, \ldots$ have a finite upper limit L, it can be shown that $|s(x)|$ has a finite upper limit in (a, b); for, if the upper limit of $|s(x)|$ were indefinitely great, we could find a value of x such that $|s(x)| = L+a$, where a is some positive number. Now n can be taken so great that $|s(x)-s_n(x)| < \epsilon$, where ϵ is arbitrarily small; hence $|s(x)| < |s_n(x)| + \epsilon < L+\epsilon$, and ϵ can be chosen to be less than a; thus it is impossible that the upper limit of $|s(x)|$ in (a, b) is not finite. The condition just stated that $|s(x)|$ may have a finite upper limit is a sufficient one but not a necessary one; in fact we know that at the point $(x, 0)$ the function $s(x, y)$ may have an infinite discontinuity whilst $s(x)$ has only a finite discontinuity or is continuous at the point x.

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[*] §§ 6 and 7 have been rewritten December 30th, 1903.

We shall suppose, in what follows, that $|s(x)|$ has a finite upper limit in (a, b), so that, in case $s(x)$ is integrable, the integral is a proper one.

To find the conditions that $s(x)$ may be an integrable function, we shall first suppose it to be so; then those points of (a, b) at which $s(x)$ has a saltus which is $\geqslant \epsilon$ form a closed set of points with zero content. These points may be included in a finite set of intervals whose sum is arbitrarily small, say η, and the end-points of these intervals $\delta_1, \delta_2, \ldots, \delta_r$ may be so chosen that in each the saltus of $s(x)$ is $< \epsilon$. At every point of (a, b) which is not in the interior of one of the intervals $\{\delta\}$, the saltus of $s(x)$ is less than ϵ. In a similar manner the points of (a, b) at which the saltus of $s_n(x)$ is $\geqslant \epsilon$ may be included in a finite set of intervals $\delta_1^{(n)}, \delta_2^{(n)}, \ldots, \delta_p^{(n)}$, whose sum η_n is arbitrarily small, and such that at the end-points the saltus of $s_n(x)$ is less than ϵ. If both sets $\{\delta\}$, $\{\delta^{(n)}\}$ be excluded from (a, b), the remainder consists of a finite number of closed intervals, such that at every point in them both $s(x)$ and $s_n(x)$ have a saltus which is $< \epsilon$; these intervals we may denote by $\Delta_1^{(n)}, \Delta_2^{(2)}, \ldots, \Delta_s^{(n)}$, forming a finite set $\{\Delta^{(n)}\}$.

Let us consider the set $G^{(n)}$ which consists of those points of the intervals $\{\Delta^{(n)}\}$ at which $|R_n(x)| < 3\epsilon$; the set $G^{(n)}$ consists in general of a set $\{D^{(n)}\}$ of intervals, and of a set $H^{(n)}$ of points which are not in intervals for which $|R_n(x)|$ is everywhere $< 3\epsilon$; the intervals of the set $\{D^{(n)}\}$ may be finite or enumerably infinite in number, and each interval of the set may be closed, or open, at either end, according as the end-point is, or is not, a point at which $|R_n(x)| < 3\epsilon$; two intervals of the system may abut on one another, in case the common end-point does not belong to $G^{(n)}$. As regards the set of points $H^{(n)}$ and the set of intervals $\{D^{(n)}\}$, we observe that, if P be any point belonging to $G^{(n)}$, then, if no neighbourhood of P either on the right or on the left can be found, such that every point in the neighbourhood belongs to $G^{(n)}$, P is a point of the set $H^{(n)}$; if, on the other hand, a neighbourhood $(x-\alpha, x+\beta)$ of $P(x)$ exists, every point of which belongs to $G^{(n)}$, the number β is capable of having a set of values which have an upper limit β_1, and the number α is capable of having a set of values which have a lower limit α_1: in that case $(x-\alpha_1\ x+\beta_1)$ is an interval belonging to the set $\{D^{(n)}\}$; it is a closed interval in case both $x-\alpha_1$ and $x+\beta_1$ are points of $G^{(n)}$, and it is open at one or both ends in case one or both of the points $x-\alpha_1$, $x+\beta_1$ does not belong to the set $G^{(n)}$. In case x has only a neighbourhood $(x, x+\beta)$ on the right, or only a neighbourhood $(x-\alpha, x)$ on the left, every point of which belongs to $G^{(n)}$, we see, in a similar manner to the above, that a definite interval $(x, x+\beta_1)$ or $(x-\alpha_1, x)$ exists, every point of which, with the possible exception of the end-point, belongs to $G^{(n)}$, and thus that the interval belongs to $\{D^{(n)}\}$. Since P is any point whatever of $G^{(n)}$, it is now clear that the set of points $H^{(n)}$ and the set of intervals $\{D^{(n)}\}$ which together constitute $G^{(n)}$ are determinate; either $\{D^{(n)}\}$, or $\{G^{(n)}\}$, or both, may be absent for the n considered; it will, however, appear that, if n is sufficiently great, the set $\{D^{(n)}\}$ must exist.

We now see that every point x, of (a, b), which is such that $|R_n(x)| < 3\epsilon$, either lies in the interior of one of the intervals of $\{\delta\}$, or of $\{\delta^{(n)}\}$, or else belongs to one of the intervals of $\{D^{(n)}\}$, or to the set of points $H^{(n)}$.

Similar reasoning applies if, instead of n, we take $n+1, n+2, \ldots, n+m, \ldots$; if we take $\eta_n = \zeta$, $\eta_{n+1} = \frac{1}{2}\zeta$, $\eta_{n+2} = \frac{1}{2^2}\zeta$, $\ldots$, $\eta_{n+m} = \frac{1}{2^m}\zeta$, $\ldots$, where ζ is an arbitrarily small fixed number, we see that the sum of all the intervals $\{\delta^{(n)}\}$, $\{\delta^{(n+1)}\}$, $\ldots$, $\{\delta^{(n+m)}\}$, $\ldots$ is 2ζ.

It will now be shown that there is no point x which belongs to all the sets of points $H^{(n)}$, $H^{(n+1)}$, $\ldots$, $H^{(n+m)}$, $\ldots$, or to an infinite number of them; to prove this, we observe that, for any fixed x, a value of m can be found such that $|R_{n+m}(x)| < \epsilon$, for this value of m and for all greater values; taking m to have such a value, unless x lies in the interior of one of the intervals of $\{\delta\}$, $\{\delta^{(n+m)}\}$, a neighbourhood of x can be found such that, for every point $x+h$ in it, $|s(x+h)-s(x)| < \epsilon$, and also

$$|s_{n+m}(x+h)-s_{n+m}(x)| < \epsilon;$$

in this neighbourhood of x,

$$\begin{aligned} &|R_{n+m}(x+h)| \\ &\quad \leqslant |s(x+h)-s(x)| + |s(x)-s_{n+m}(x)| + |s_{n+m}(x)-s_{n+m}(x+h)| \\ &\quad < 3\epsilon. \end{aligned}$$

This holds for definite values of m; the neighbourhood in which it holds depends upon the values of m.

It has now been shown that, if x is any fixed point not in the interior of $\{\delta\}$, nor an end-point of $\{\delta\}$, a value of m can be found so great that x is either (1) interior to one of the intervals $\{\delta^{(n+m)}\}$, or (2) interior to one of the intervals of $\{D^{(n+m)}\}$, or (3) an end-point of an interval of $\{\delta^{(n+m)}\}$. In the last case, since every end-point of the $\{\delta^{(n+m)}\}$ is a point at which the saltus of $s_{n+m}(x)$ is $< 3\epsilon$, we can find a neighbourhood of x, which encroaches on the interval of which x is an end-point, but is such that the above reasoning applies to it. If x is an end-point of an interval of $\{\delta\}$, we can, in a similar manner, find an interval encroaching on the interval of $\{\delta\}$ of which it is an end-point, but such that throughout it $|R_{n+m}(x)| < 3\epsilon$. For the end-points a, b, values of m can be found such that these points are end-points of intervals through which $|R_{n+m}(x)| < 3\epsilon$, unless a or b is already an end-point of the set $\{\delta\}$. If we take the whole set of intervals which consists of (1) the intervals $\{\delta\}$ whose sum is η, of (2) all the sets $\{\delta^{(n+m)}\}$ whose sum is 2ζ, and (3) of all the sets $\{D^{(n+m)}\}$, where m has all values 0, 1, 2, 3, ..., with (4) the addition of intervals introduced as above whenever an end-point of one of the above intervals becomes a point of a $G^{(n+m)}$, we have, when these are taken altogether, an enumerable set of intervals which contains every point of (a, b) in the interior of at least one interval, except that a and b are end-points of intervals of the set. To this complex set of intervals the Heine-Borel theorem is applicable, and we deduce that a finite number of intervals of the set can be chosen which contain every point of (a, b) except a and b, in the interior of at least one interval, and such that a, b are themselves end-points of the intervals of the set. Those intervals of the finite set which belong to $\{\delta\}$ or $\{\delta^{(n)}\}$, $\{\delta^{(n+1)}\}$, ... have a sum less than $\eta+2\zeta$, and the rest of the intervals are such that for each of them there is a value of m such that $|R_{n+m}(x)| < 3\epsilon$, for every point of that one interval. Since $\eta+2\zeta$ is arbitrarily small, we see that, by excluding from (a, b) a finite set of intervals whose sum is arbitrarily small, the remainder of (a, b) is such that, for every point in it, $|R_{n+p}(x)| < 3\epsilon$, where p has one of a finite number of values. That this should hold for every ϵ is a necessary condition for the integrability of $s(x)$.

It can now be shown, conversely, that, provided this condition is satisfied for every ϵ, the function $s(x)$ must be integrable. To show this, we apply Riemann's test of integrability, according to which the whole interval $(a\ b)$, is divided into a finite number of parts $h_1, h_2, \ldots, h_s$, and the value of the sum of the products of each h into the fluctuation of the function in that part h is estimated; the condition of integrability is that this sum should have a zero limit when the number s is increased indefinitely, and at the same time the greatest of the intervals h has a zero limit. It is well known that it is sufficient to make the sub-division of (a, b) according to any law we please, provided the condition as to the greatest of the h is satisfied. Taking a given ϵ, let the intervals $(\delta_1, \delta_2, \ldots, \delta_r)$ be marked out, and also the finite set of intervals into which the rest of (a, b) can be divided, so that in each one of them, for some particular value of p, $|R_{n+p}(x)| < 3\epsilon$; let the end-points of $h_1, h_2, \ldots, h_s$ contain among them all the end-points of both these finite sets of intervals. In any h which lies within one of the δ, the product of the h into the fluctuation of $s(x)$ is less than or equal to $h(M-m)$, where M, m are the upper and lower limits of $s(x)$ in (a, b); in any h which lies within an interval for which $|R_{n+p}(x)| < 3\epsilon$, the fluctuation of $s(x)$ cannot exceed that of $s_{n+p}(x)$ by more than 6ϵ. It follows that the sum of the products required for Riemann's test is less than $(M-m)\Sigma\delta+\sum\limits_{p}\Sigma h\,\{\text{fluctuation of } s_{n+p}(x)+6\epsilon\}$, where in the double summation the first summation refers to all those of the h which are within an interval for which p is constant, and the second sum is taken for all the finite number of values of p. Now $s_{n+p}(x)$ is integrable through any interval; hence, through the interval for which p has a fixed value $\{\Sigma h \text{ fluctuation of } s_{n+p}(x)\}$ has a zero limit; also $(M-m)\eta$ is arbitrarily small; hence the product in Riemann's test is arbitrarily small as η becomes indefinitely small and s is indefinitely increased; therefore $s(x)$ is integrable in (a, b).

It has now been shown that, *if* $u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$ *converges to a definite value* $s(x)$ *at every point in* (a, b), *and if all the functions* $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ *have proper integrals in* (a, b), *the necessary and sufficient conditions that* $s(x)$ *may have a proper integral are* (1) *that the upper limit of* $s(x)$ *in* (a, b) *is finite, and* (2) *that corresponding to any arbitrarily small positive number* σ, *and to any positive integer* n, *a finite number of intervals whose sum is arbitrarily small can be excluded from* (a, b) *so that, in the remainder of* (a, b), $|R_{n+p}(x)| < \sigma$, *for every* x, *where* p *has one of a finite number of values which depend on* x, *but are such that the same* p *is applicable to all points* x *in a certain continuous interval.*

7. The conditions which have been found that $s(x)$ may be an integrable function are expressed by saying that $|s(x)|$ must have a finite upper limit, and that there must be a certain mode of convergence of the series, which Arzelà has denominated *convergenza uniforme a tratti in generale*, "uniform convergence by intervals in general." This mode of convergence is distinguished from uniform convergence by intervals, in that a finite number of intervals of arbitrarily small sum are excluded from the domain. As regards the proof which Arzelà has given of the theorem which has been just obtained, it may be remarked that it appears to depend upon the assumption that not only the points of (a, b) at which $s(x)$ has a saltus $\geqslant \epsilon$ form a set of zero content, but also that the set of points at which any of the functions $s_n(x)$, $s_{n+1}(x)$, $s_{n+m}(x)$, ... have a saltus $\geqslant \epsilon$ can be enclosed in a finite number of intervals whose sum is arbitrarily small; this appears from the statement on p. 708 and in the first paragraph of p. 709 (*loc. cit.*). This is, however, not necessarily the case, and has not been assumed to be true in the present paper. The set

of points for which any one $s_{n+m}(x)$ has a saltus $\geqslant \epsilon$ is a closed set, but it does not follow that the set of points at which any of the $s_{n+m}(x)$ has a saltus $\geqslant \epsilon$ can be enclosed in a finite number of intervals of arbitrarily small sum. The sets G_0, G_1, ..., G_m, ..., which correspond to $s_n(x)$, $s_{n+1}(x)$, ..., $s_{n+m}(x)$, ..., being each closed, we may consider the sequence of sets G_0, G_0+G_1, $G_0+G_1+G_2$, ..., $G_0+G_1+\ldots+G_m$, ... ; each of these sets is closed and is contained in the succeeding set; hence, by a theorem given by Prof. Osgood,* the limiting set, *provided it is closed*, has for content the limit of the content of $G_0+G_1+\ldots+G_m$; in our case this is zero. In the general case it is quite true that all the points at which any of the $s_{n+m}(x)$ has a saltus $\geqslant \epsilon$ can be enclosed in finite intervals whose sum is arbitrarily small, but not in a finite number of such intervals. It would thus appear that the proof of the theorem given by Arzelà cannot be accepted as valid, at all events in the form in which it has been given by him.

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* *American Journal*, Vol. xix., p. 178; see also Schönflies, *Bericht über die Mengenlehre*, p. 91; the proof given by Schönflies is, however, invalid.

ON GROUPS OF ORDER $p^\alpha q^\beta$

By W. BURNSIDE.

[Received and Read January 14th, 1904.]

IT may be convenient to the reader to summarize the results hitherto obtained with regard to groups of order $p^\alpha q^\beta$ other than those relating to particular values of p, q, α, and β. If m is the index to which p belongs, mod. q, the first result arrived at was that, if $\alpha \leqslant m$, the group is soluble.*

In my book on the *Theory of Groups* (1897) I extended this result, showing that, if $\alpha < 2m$, the group is soluble. In the same place I proved that, if the sub-groups of orders p^α and q^β are both Abelian, the group is soluble; and that all groups of order $p^\alpha q^2$ are soluble.

Of the last result another proof was given by Jordan (*Liouville's Journal*, Ser. 5, Vol. IV., 1898). Finally, in a memoir "Uber Gruppen der Ordnung $p^\alpha q^\beta$" (*Acta Mathematica*, Vol. XXVI., p. 189, 1902), Herr Frobenius has shown that when $\alpha \leqslant 2m$ the group is soluble, and also that when the group contains only p^m sub-groups of order q^β it is soluble.

In the present paper I have attacked the question of the solubility of a group of order $p^\alpha q^\beta$ by a consideration of certain properties of the group-characteristics of such a group; and I have succeeded in showing that all groups of order $p^\alpha q^\beta$ are soluble.

The first section of the paper is concerned with a property of the characteristics of certain operations in an irreducible group of linear substitutions in p^m variables, where p is prime; and it has bearings on other questions beside those with which the remainder of the paper is concerned.

My paper "On Group-Characteristics" (*Proc. London Math. Soc.*, Vol. XXXIII., p. 146) is referred to by the initials G.-C.

1. From the relations (G.-C., p. 151),

$$h_i h_j \chi_i \chi_j = \chi_1 \sum_k c_{ijk} h_k \chi_k,$$

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* Frobenius, *Berliner Sitzungsberichte* (1895), p. 190; and Burnside, *Proc. London Math. Soc.*, Vol. XXVI. (1895), p. 209.

for a given suffix i and each suffix j in turn, by eliminating the ratios of the quantities $h_j\chi_j$, there results

$$\begin{vmatrix} c_{i11}-\dfrac{h_i\chi_i}{\chi_1}, & c_{i12}, & \ldots, & c_{i1r} \\ c_{i21}, & c_{i22}-\dfrac{h_i\chi_i}{\chi_1}, & \ldots, & c_{i2r} \\ \ldots & \ldots & \ldots & \ldots \\ c_{ir1}, & c_{ir2}, & \ldots, & c_{irr}-\dfrac{h_i\chi_i}{\chi_1} \end{vmatrix} = 0.$$

Hence, since the c's are positive integers or zeros, $h_i\chi_i/\chi_1$ is an algebraic integer.*

Suppose that χ_1 is the power of a prime, p^m, so that the order of the group is divisible by p^m. Let p^a be the highest power of p which divides the order of the group, and let P be a self-conjugate operation of a sub-group of order p^a. Then h_P is relatively prime to p, and χ_P is the sum of p^m powers of ω, if ω is a primitive p^a-th root of unity, p^a being the order of P.

From $h_P\chi_P/\chi_1$ form the $p^{a-1}(p-1)$ conjugate expressions obtained on replacing ω by each primitive p^a-th root of unity. The elementary symmetric functions of these expressions will be algebraic integers, and, since they are rational, they must be rational integers. Now h_P and p^m (or χ_1) are relatively prime. Hence the elementary symmetric functions of χ_P/χ_1 and its conjugates are rational integers; and therefore χ_P/χ_1 is an algebraic integer. From this it follows at once that either (i) χ_P must be zero, or (ii) the p^m powers of ω, whose sum make up χ_P, must all be the same. In fact, if χ_P is not zero, (mod. χ_P)/χ_1 is (from its graphical representation) a proper fraction, except when $\chi_P=p^m\omega^x$, where x is some integer. But, if (mod. χ_P)/χ_1 is a proper fraction, so also is the product Π (mod. χ_P)/χ_1 formed from all the conjugates, and this is the same as $\Pi\,\chi_P/\chi_1$, which has been proved to be an integer. The result thus proved may be stated as the following:—

Theorem I.—If a group G of order $p^a s$ (s relatively prime to p) can be represented as an irreducible group of linear substitutions in p^m variables, then a self-conjugate operation P of a sub-group of order p^a of G has for its characteristic in this representation either zero or $p^m\omega$, where ω is a root of unity. In the latter case the substitution corresponding to P in the irreducible group is a self-conjugate substitution, and G has a self-conjugate sub-group containing P.

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* This result is given by Herr Frobenius.

If P, of order p^a, is a self-conjugate operation of a sub-group of order p^a of G, so also are P^p, P^{p^2}, ..., $P^{p^{a-1}}$. The characteristic of each of these operations is therefore either zero or p^m times a root of unity. If each of them is zero, so that no one of them is a self-conjugate operation of the irreducible group in p^m variables, the p^m roots of unity which make up χ_P must clearly be the different p^a-th roots of unity, each repeated p^{m-a} times. This is only possible when $a \leqslant m$; and, if $a > m$, the p^{a-m}-th power of P must be a self-conjugate operation of the irreducible group.

Consider in particular an irreducible group g of linear substitutions in p variables, and let p^a be the highest power of p which divides the order of g. If a sub-group of g of order p^a is not Abelian, it must be irreducible and will necessarily contain self-conjugate operations which are self-conjugate operations of g. If the sub-group of order p^a is Abelian, and if $a > 1$, the characteristics of all of its operations cannot be zero,[*] and therefore some must be self-conjugate operations of g. Hence:

Theorem II.—An irreducible group of linear substitutions in a prime number of variables p must either (i) contain self-conjugate operations whose orders are powers of p, or (ii) have no sub-group of order p^2.

2. Consider a group G of order $p^a q^\beta$. Let H and K be sub-groups of G of orders p^a and q^β respectively, and let P be a self-conjugate operation of H, and Q a self-conjugate operation of K, other than identity.

All the operations conjugate to P are obtained on transforming P by all the operations of K, or of any sub-group conjugate to K; and all those conjugate to Q on transforming Q by the operations of H. Hence, if PQ be transformed by any operation of H, it becomes PQ_j, where Q_j may be any one of the operations conjugate to Q; and, if PQ_j be transformed by any operation of the sub-group conjugate to K which contains Q_j self-conjugately, it becomes $P_i Q_j$, where P_i may be any one of the operations conjugate to P.

Hence the set of operations formed by multiplying any one of the operations of the conjugate set to which P belongs (say the i-th set) by any one of the operations of the conjugate set to which Q belongs (say the j-th set) all belong to one and the same conjugate set (say the k-th).

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[*] Thus, if $x_1' = x_1, \quad x_2' = \omega x_2, \quad \ldots, \quad x_p' = \omega^{p-1} x_p$

and $x_1' = \omega_1 x_1, \quad x_2' = \omega_2 x_2, \quad \ldots, \quad x_p' = \omega_p x_p$

be two of its operations, say P and P', of order p, P' not being a power of P, then at least one of the operations $P^x P'$ $(x = 0, 1, 2, \ldots, p-1)$ has a characteristic different from zero. The case in which the group contains operations of order p^2 comes under the head considered immediately above.

This is represented by the equation (G.-C., p. 148)

$$C_iC_j = c_{ijk}C_k,$$

involving the relations

$$h_ih_j = c_{ijk}h_k, \qquad c_{ijl} = 0 \quad (l \neq k).$$

If χ_i, χ_j, χ_k are the characteristics of the three sets in any irreducible representation of G, the relation (G.-C., p. 151)

$$h_ih_j\chi_i\chi_j = \chi_1 \sum_s c_{ijs}h_s\chi_s$$

reduces to

$$h_ih\,\chi_i\chi_j = c_{ijk}h_k\chi_1\chi_k,$$

i.e.,

$$\chi_i\chi_j = \chi_1\chi_k.$$

3. In every irreducible representation of G, χ_1 is a factor of the order of G (G.-C., p. 156), and must therefore be either unity, a power of p, a power of q, or a product of powers of p and q. For the identical representation χ_1 is unity, and (G.-C., p. 153)

$$\sum_t (\chi_1^t)^2 = p^\alpha q^\beta,$$

where the sum is extended to the r distinct irreducible representations of G. Hence every χ_1, except the first, cannot be divisible by p, nor can every one be divisible by q. It follows that either (i) other χ_1's besides the first must be unity, in which case the group can be represented as a cyclical group, and is therefore composite, or (ii) some χ_1's must be powers of p and others powers of q.

Consider an irreducible representation of G in which χ_1 is a power of p, say p^m. In this representation χ_i, the characteristic of the operation P considered in § 2 is either zero or $p^m\omega$, where ω is a p^a-th root of unity. In the former case χ_k, the characteristic of PQ, is, by the final equation of § 2, zero. In the latter case the substitution corresponding to P is a self-conjugate substitution of the irreducible representation of G in p^m variables, and G itself is composite. Similarly, in any irreducible representation of G in q^n variables, either χ_k is zero or G is composite.

Suppose now, if possible, that G has no representation, except the identical one, for which χ_1 is unity, and that χ_k is zero for every irreducible representation of G in which χ_1 is either a power of p or a power of q. Then the relation (G.-C., p. 153)

$$\sum_i \chi_1^i\chi_k^i = 0$$

becomes

$$1 + \sum_i{}' \chi_1^i\chi_k^i = 0,$$

where Σ' is limited to those representations for which χ_1 is divisible by pq. Since each χ_k is an algebraic integer, this equation may be written in the form $1/pq + a = 0$, a being an algebraic integer; and no such equation is true.

Hence either χ_1 must be unity for some representation other than the identical one, or χ_k must be different from zero in some representations in which χ_1 is a power of p or a power of q. In either case G must be composite; and, since the same reasoning applies to the factor groups and the sub-groups of G, G must be soluble. Hence:—

Theorem III.—Every group whose order is of the form $p^{\alpha}q^{\beta}$ is soluble.

ADDITION TO THE PRECEDING PAPER. *February 9th,* 1904.

Since the above was communicated to the Society I have arrived at a materially simpler manner of establishing a rather more general result.

Suppose that in a group G of finite order the number of operations which constitute one conjugate set (say the i-th) is the power of a prime, so that $h_i = p^{a}$. If χ_i is the corresponding characteristic in an irreducible representation of G, then $h_i\chi_i/\chi_1$ is an algebraic integer; and therefore, if h_i and χ_1 are relatively prime, *i.e.*, if χ_1 is not divisible by p, χ_i/χ_1 is an algebraic integer. Hence, as above, either χ_i is zero or $\chi_i = \chi_1\omega$, where ω is a root of unity. In the latter case every operation of the i-th conjugate set is a self-conjugate substitution in the irreducible representation under consideration and G is therefore composite.

Now consider the relation $\sum_s \chi_1^s \chi_i^s = 0$,

where the summation extends to the r distinct irreducible representations of G. If no χ_1, except the first, is unity, and if χ_i is zero whenever χ_1 is not divisible by p, this equation is of the form $1 + pa = 0$, where a is an algebraic integer, which is impossible. Hence either (i) some χ_1, other than the first, is unity, in which case G is isomorphic with a cyclical group, or (ii) some χ_i is equal to $\chi_1\omega$, in which case G has a self-conjugate sub-group containing the i-th set. In either case G is composite. Hence:—

Theorem.—If in a group of finite order the number of operations in any one conjugate set is the power of a prime, the group is composite.

From this the previous result follows immediately. For in a group of order $p^{\alpha}q^{\beta}$ there are necessarily conjugate sets, the numbers of operations in which are powers of primes. In fact the self-conjugate operations of a sub-group of order p^{α} (or q^{β}) belong to such sets.

ON THE DIFFRACTION OF LIGHT PRODUCED BY AN OPAQUE PRISM OF FINITE ANGLE

By W. H. JACKSON.

[Received November 11th, 1903.—Read November 12th, 1903.—Revised January 16th, 1904.]

THE main object of the following paper is to estimate the influence which the size of the angle of a perfectly reflecting prism exercises on the diffracted rays produced by it when a beam of parallel rays is directed perpendicularly upon its edge. A comparison is also made between the two cases of a perfectly reflecting prism and a perfectly absorbing one.

The result of the investigation is to show that the position of the bands of diffraction just outside the shadow is the same in all cases, but that in the shadow the polarising effect increases with the angle of the prism.

The general solution for the case of plane waves which forms the basis of the present calculation is given, together with all the references, in *Electric Waves*, Appendix D, by Mr. H. M. Macdonald, who proposed the present investigation and made suggestions throughout its course.

The paper is divided into four sections :—

I. A method of obtaining Macdonald's formula [equation (22)].

II. Verification of this formula and its transformation to a form suitable for numerical computation.

III. The numerical discussion and comparison of results.

IV. The analytical discussion of the integrals required.

SECTION I.—*A Method of obtaining Macdonald's Formula.*

Take cylindrical coordinates with the edge of the prism as axis of z, the prism lying in the space extending from the plane $\theta = a$ to the plane $\theta = 2\pi$, so that the angle of the prism is $2\pi - a$.

In the first case, that of light polarised in the plane of diffraction, the electric force is parallel to the edge of the prism. In the incident waves, if plane and of wave-length $2\pi/\kappa$, the electric force is equal to $Re^{i\kappa[r\cos(\theta-\theta_1)+Vt]}$, where R or I placed in front of any expression will be taken to denote its real or imaginary part respectively, and where θ_1 marks the direction of the incident rays. The total electric force may be written $RZe^{i\kappa Vt}$, where Z is a function of r and θ which satisfies the equation

$$\frac{\partial^2 Z}{\partial r^2}+\frac{1}{r}\,\frac{\partial Z}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 Z}{\partial \theta^2}+\kappa^2 Z=0. \tag{1}$$

Since the prism is perfectly reflecting, it must be considered as a perfect electric conductor, and therefore Z vanishes when $\theta = 0$ and when $\theta = a$. We may therefore write Z in the form

$$Z=\sum_1^\infty A_n J_{n\pi/a}(\kappa r)\sin n\pi\theta/a. \tag{2}$$

The form $J_{n\pi/a}(\kappa r)$ must be chosen since Z is finite when $r = 0$. Also Z may be divided into two parts Z_1, Z_2, so that

$$Z=Z_1+Z_2. \tag{3}$$

Z_1 will be defined by the equations

$$\left.\begin{array}{lll} Z_1=e^{i\kappa r\cos(\theta-\theta_1)}-e^{i\kappa r\cos(\theta+\theta_1)}, & \text{when} & 0<\theta<\pi-\theta_1 \\ Z_1=e^{i\kappa r\cos(\theta-\theta_1)}, & \text{when} & \pi-\theta_1<\theta<\pi+\theta_1 \\ Z_1=0, & \text{when} & \pi+\theta_1<\theta<a \end{array}\right\}, \tag{4}$$

and will be referred to as *the part of Z representing the incident and reflected waves.* The remaining part Z_2 is then defined by equation (3) and will be referred to as *the part of Z due to the diffracted waves.*

In the case of waves diverging from a line source determined by the coordinates (r_1, θ_1), the coefficients A occurring in equation (2) are determined by the conditions that Z must vanish at infinity, and that Z must be discontinuous in a particular way at the point (r_1, θ_1). In the case of plane waves these conditions are replaced by the conditions that Z must be everywhere finite, and that at infinity Z approaches the form representing the incident and reflected train of waves as a limiting value. This last condition may be written

$$\operatorname*{Lt}_{r=\infty} Z_2=0. \tag{5}$$

In order to find values of the constants A in equation (2) to be tested by means of this condition we must expand Z_1 in the form

$$Z_1=\sum_1^\infty B_n\sin n\pi\theta/a. \tag{6}$$

Applying Fourier's theorem, we find, from equation (4),

$$\tfrac{1}{2}aB_n = \int_0^{\pi+\theta_1} e^{i\kappa r\cos(\phi-\theta_1)} \sin\frac{n\pi\phi}{a}\,d\phi - \int_0^{\pi-\theta_1} e^{i\kappa r\cos(\phi+\theta_1)} \sin\frac{n\pi\phi}{a}\,d\phi.$$

Substituting in the first integral $\phi-\theta_1=\zeta$, and in the second $\phi+\theta_1=\zeta$,

$$\begin{aligned}\tfrac{1}{2}aB_n &= \int_{-\theta_1}^{\pi} e^{i\kappa r\cos\zeta}\sin\frac{n\pi}{a}(\zeta+\theta_1)\,d\zeta - \int_{\theta_1}^{\pi} e^{i\kappa r\cos\zeta}\sin\frac{n\pi}{a}(\zeta-\theta_1)\,d\zeta \\ &= \int_{-\theta_1}^{\pi} - \int_{\theta_1}^{\pi} e^{i\kappa r\cos\zeta}\sin\frac{n\pi\zeta}{a}\cos\frac{n\pi\theta_1}{a}\,d\zeta \\ &\qquad + \int_{-\theta_1}^{\pi} + \int_{\theta_1}^{\pi} e^{i\kappa r\cos\zeta}\cos\frac{n\pi\zeta}{a}\sin\frac{n\pi\theta_1}{a}\,d\zeta \\ &= 2\sin\frac{n\pi\theta_1}{a}\int_0^{\pi} e^{i\kappa r\cos\zeta}\cos\frac{n\pi\zeta}{a}\,d\zeta,\end{aligned}$$

whence

$$B_n = \frac{2}{a}\sin\frac{n\pi\theta_1}{a}\int_{-\pi}^{\pi} e^{i\kappa r\cos\zeta + in\pi\zeta/a}\,d\zeta. \tag{7}$$

But

$$J_n(x) = -\frac{1}{2\pi}e^{-\frac{1}{2}in\pi}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{ix\cos\zeta+in\zeta}\,d\zeta, \tag{8}$$

where $2\pi > \gamma > \pi,\ 0 > \gamma' > -\pi$.

On comparing the coefficients in equations (2) and (6), since

$$Z_1 = \frac{2}{a}\sum_1^{\infty}\int_{-\pi}^{\pi} e^{i\kappa r\cos\zeta}e^{in\pi\zeta/a}\sin\frac{n\pi\theta}{a}\sin\frac{n\pi\theta_1}{a}\,d\zeta, \tag{9}$$

it is therefore natural to assume that the total value of Z is

$$Z = -\frac{2}{a}\sum_1^{\infty}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}e^{in\pi\zeta/a}\sin\frac{n\pi\theta}{a}\sin\frac{n\pi\theta_1}{a}\,d\zeta. \tag{10}$$

To justify this step we must prove that equation (5) is satisfied, and to do this we must sum the series occurring in equation (10). Now

$$\sum_1^n e^{in\pi\zeta/a}\sin\frac{n\pi\theta}{a}\sin\frac{n\pi\theta_1}{a} = \tfrac{1}{2}\sum_1^n e^{in\pi\zeta/a}\left\{\cos\frac{n\pi}{a}(\theta-\theta_1) - \cos\frac{n\pi}{a}(\theta+\theta_1)\right\}. \tag{11}$$

But

$$\begin{aligned}1+\sum_1^n e^{in\phi}\cos n\psi &= \tfrac{1}{2}\frac{1-e^{i(n+1)(\phi+\psi)}}{1-e^{i(\phi+\psi)}} + \tfrac{1}{2}\frac{1-e^{i(n+1)(\phi-\psi)}}{1-e^{i(\phi-\psi)}} \\ &= \frac{1-e^{i\phi}\cos\psi + e^{i(n+2)\phi}\cos n\psi}{1+e^{2i\phi}-2e^{i\phi}\cos\psi} \\ &= \tfrac{1}{2}+\tfrac{1}{2}\frac{1-e^{2i\phi}+2e^{i(n+2)\phi}\cos n\psi}{1+e^{2i\phi}-2e^{i\phi}\cos\psi} \\ &= \tfrac{1}{2}+\tfrac{1}{2}\frac{\cos n\psi\,.\,e^{i(n+1)\phi} - i\sin\phi}{\cos\phi-\cos\psi};\end{aligned}$$

and therefore, from equation (11),

$$\sum_1^n e^{in\pi\zeta/a} \sin\frac{n\pi\theta}{a} \sin\frac{n\pi\theta_1}{a}$$

$$= \tfrac{1}{4}\left\{\frac{\cos\frac{n\pi}{a}(\theta-\theta_1)\, e^{i(n+1)\pi\zeta/a} - i\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} - \frac{\cos\frac{n\pi}{a}(\theta+\theta_1)\, e^{i(n+1)\pi\zeta/a} - i\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)}\right\}.$$

Multiply by $-\frac{2}{a} e^{i\kappa r\cos\zeta}$ and integrate with respect to ζ, whence

$$-\frac{2}{a}\sum_1^n \int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}\, e^{in\pi\zeta/a} \sin\frac{n\pi\theta}{a} \sin\frac{n\pi\theta_1}{a}\, d\zeta$$

$$= \frac{i}{2a}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}\left\{\frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} - \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)}\right\} d\zeta - R_n \ldots, \tag{12}$$

where

$$R_n = \frac{1}{2a}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}\, e^{i(n+1)\pi\zeta/a}\left\{\frac{\cos\frac{n\pi}{a}(\theta-\theta_1)}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} - \frac{\cos\frac{n\pi}{a}(\theta+\theta_1)}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)}\right\} d\zeta \ldots . \tag{13}$$

Proceeding now to the limit when n increases indefinitely, it is clear, owing both to the rapid oscillations of the term $e^{i(n+1)\pi\zeta/a}$ and to the decrease in its modulus, that

$$\operatorname*{Lt}_{n=\infty} R_n = 0. \tag{14}$$

Equation (12) may now be written

$$Z = \frac{i}{2a}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}\left\{\frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta_1)} - \frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta_1)}\right\} d\zeta. \tag{15}$$

Before completing the proof of this formula by showing that equation (5) is satisfied, it is convenient first to obtain the corresponding formula when the light is polarised perpendicularly to the plane of diffraction; the two cases can then be treated together, and the first part of the second section concludes the verification of both formulæ.

In the second case, that of light polarised perpendicularly to the plane of diffraction, the magnetic force is parallel to the edge of the prism, which, as in the case of the electric force in the first case, we may denote by $RZe^{i\kappa Vt}$. Since $\partial Z/\partial\theta$ vanishes when $\theta = 0$ and when $\theta = a$, we must write, instead of equation (2),

$$Z = \sum_1^\infty A_n J_{n\pi/a}(\kappa r)\cos\frac{n\pi\theta}{a}. \tag{16}$$

If, as in equations (4) and (6), we write

$$\left.\begin{array}{ll} Z_1 = e^{i\kappa r\cos(\theta-\theta_1)}+e^{i\kappa r\cos(\theta+\theta_1)}, & \text{when} \quad 0<\theta<\pi-\theta_1 \\ Z_1 = e^{i\kappa r\cos(\theta-\theta_1)}, & \text{when} \quad \pi-\theta_1<\theta<\pi+\theta_1 \\ Z_1 = 0, & \text{when} \quad \pi+\theta_1<\theta<a \end{array}\right\}, \tag{17}$$

$$Z_1 = \sum_1^\infty B_n\cos\frac{n\pi\theta}{a}, \tag{18}$$

we obtain, as in (7),

$$B_n = \frac{2}{a}\cos\frac{n\pi\theta_1}{a}\int_{-\pi}^{\pi} e^{i\kappa r\cos\zeta+in\pi\zeta/a}\,d\zeta, \tag{19}$$

whence, as in (9),

$$Z_1 = \frac{2}{a}\sum_1^\infty\int_{-\pi}^{\pi} e^{i\kappa r\cos\zeta}\,e^{in\pi\zeta/a}\cos\frac{n\pi\theta}{a}\cos\frac{n\pi\theta_1}{a}\,d\zeta. \tag{20}$$

Therefore, making the same assumption as regards Z, and summing the series as before, we find, corresponding to equation (15),

$$Z = \frac{i}{2a}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}\left\{\frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta-\theta_1)}+\frac{\cos\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta+\theta_1)}\right\}d\zeta. \tag{21}$$

We will now write equations (15) and (21) together in the form

$$Z = \frac{i}{2a}\int_{i\infty+\gamma}^{i\infty+\gamma'} e^{i\kappa r\cos\zeta}\left\{\frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta-\theta_1)}\mp\frac{\sin\frac{\pi\zeta}{a}}{\cos\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta+\theta_1)}\right\}d\zeta, \tag{22}$$

where $2\pi>\gamma>\pi$, $0>\gamma'>-\pi$.

In the first case, that of light polarised in the plane of diffraction, the upper sign must be taken and Z corresponds to the electric force parallel to the edge of the prism. In the second case, that of light polarised perpendicularly to the plane of diffraction, the lower sign must be taken and Z corresponds to the magnetic force parallel to the edge of the prism.

SECTION II.—*Verification of Equation* (22) *and its Transformation to a Form suitable for Numerical Computation.*

Substitute, in equation (22), $t = \cos \frac{1}{2}\zeta$.

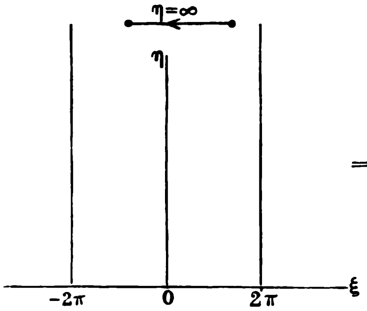

FIG. 1.

$\zeta \equiv \xi + \eta i$.

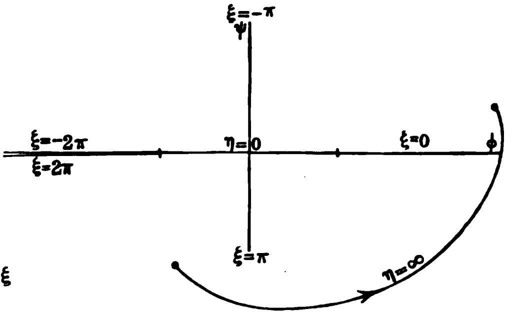

FIG. 2.

$t \equiv \phi + i\psi$.

The space in the ζ-plane, represented in Fig. 1, bounded by $\xi = \pm 2\pi$, $\eta = 0$ to ∞, corresponds to the whole of the t-plane in Fig. 2. In the t-plane the lines $\xi =$ constant are quarters of confocal hyperbolas, and the lines $\eta =$ constant are the conjugate ellipses. The path of integration from $i\infty + \gamma$ to $i\infty + \gamma'$ in the ζ-plane corresponds to a path at infinity in the t-plane extending from the third, through the fourth, to the first quadrant in that plane.

On making the substitution we obtain

$$Z = \frac{1}{a} e^{-i(\kappa r + \frac{1}{2}\pi)} \int e^{i2\kappa r t^2} F(t)\, dt, \tag{23}$$

where

$$F(t) = \frac{\sin \frac{\pi\zeta}{a}}{\sin \frac{1}{2}\zeta} \left\{ \frac{1}{\cos \frac{\pi\zeta}{a} - \cos \frac{\pi}{a}(\theta - \theta_1)} \mp \frac{1}{\cos \frac{\pi\zeta}{a} - \cos \frac{\pi}{a}(\theta + \theta_1)} \right\}. \tag{24}$$

The infinities of this function are given by

$$\left. \begin{aligned} \pm\zeta &= m \,.\, 2a + \theta - \theta_1 \\ \pm\zeta &= n \,.\, 2a + \theta + \theta_1 \end{aligned} \right\}, \tag{25}$$

where m, n are integers.

As, when ζ is wholly real, $\pm\zeta$ corresponds to the same point on the real axis of the t-plane, the number of infinities when considered as a function of t is equal to the number of values which m, n can have such that ζ lies between -2π and 2π. If $I(k)$ denote "the integral part of k," the number of infinities is

$$2+I\left(\frac{2\pi+\theta-\theta_1}{2a}\right)+I\left(\frac{2\pi-\theta+\theta_1}{2a}\right)+I\left(\frac{2\pi+\theta+\theta_1}{2a}\right)+I\left(\frac{2\pi-\theta-\theta_1}{2a}\right).$$

Since $a > \pi$, it follows that

$$I\left(\frac{2\pi-\theta-\theta_1}{2a}\right) = 0,$$

and, since $\theta_1 < a-\pi$, it follows that

$$I\left(\frac{2\pi-\theta+\theta_1}{2a}\right) = 0.$$

The maximum number of infinities is therefore four, and the minimum two.

The infinity corresponding to $m = 0$ lies on the left-hand side of the axis when $\theta-\theta_1 > \pi$, and on the right-hand side when $\theta-\theta_1 < \pi$. Similarly that corresponding to $n = 0$ lies on the left-hand side when $\theta+\theta_1 > \pi$, and on the right-hand side when $\theta+\theta_1 < \pi$.

The two infinities corresponding to $m = -1$, $n = -1$ always lie on the left-hand side. For, since the maximum value of θ is a, and that of θ_1 is $a-\pi$, both $\theta-\theta_1$ and $\theta+\theta_1$ are less than $2a-\pi$, and therefore

$$2a-\theta+\theta_1 > \pi \quad \text{and} \quad 2a-\theta-\theta_1 > \pi.$$

To calculate the value of the integral taken round a small circle enclosing an infinity of $F(t)$, let this value be $t_0 = \cos\frac{1}{2}\zeta_0$, and write

$$t = t_0+t' = \cos\tfrac{1}{2}(\zeta_0+\zeta'),$$

where ζ', t' are both small. Approximately, therefore,

$$t' = -\tfrac{1}{2}\zeta'\sin\tfrac{1}{2}\zeta_0$$

and

$$\frac{\sin\frac{\pi}{a}\zeta}{\sin\frac{\zeta}{2}\left(\cos\frac{\pi}{a}\zeta-\cos\frac{\pi}{a}(\theta\mp\theta_1)\right)} = \frac{\sin\frac{\pi}{a}\zeta_0}{-\frac{\pi}{a}\zeta'\sin\frac{1}{2}\zeta_0\sin\frac{\pi}{a}\zeta_0},$$

whence

$$F(t) = \frac{a}{2\pi t'} = \frac{a}{2\pi(t-t_0)} \text{ approx.} \tag{26}$$

This shows that the infinities of $F(t)$ are to be treated as simple poles. We may therefore deform the infinitely distant path of integration into the imaginary axis of the t-plane, provided that the term $e^{i\kappa r\cos(\theta-\theta_1)}$ be added when $\theta-\theta_1 < \pi$, and the term $\mp e^{i\kappa r\cos(\theta+\theta_1)}$ be added when

$\theta+\theta_1 < \pi$. These terms are equal to Z_1, which represents the part of Z corresponding to the incident and reflected waves [see equations (3) and (4)]. Taking them out of equation (23), we obtain

$$Z_2 = \frac{1}{a} e^{-i(\kappa r+\frac{1}{4}\pi)} \int_{-\infty i}^{\infty i} e^{i2\kappa r t^2} F(t)\, dt. \tag{27}$$

We are now able to verify that equation (5) is satisfied, since, owing to the rapid oscillations of the term $e^{i2\kappa r t^2}$, the integral approaches the limit zero as r is increased indefinitely. This completes the verification of Macdonald's formula.

The nature of the limit which this integral approaches as r increases is considered more closely in Section IV., where it is shown, equation (72), that the important part of the integral in equation (27) is due to that portion of the path which is in the neighbourhood of the origin. When t is small we have approximately $\zeta = \pi - 2t$, and therefore

$$\frac{1}{a} F(t) = \frac{1}{a} \sin\frac{\pi^2}{a} \left\{ \frac{1-\frac{2\pi}{a} t \cot\frac{\pi^2}{a}}{\cos\frac{\pi^2}{a} - \cos\frac{\pi}{a}(\theta-\theta_1) + \frac{2\pi}{a} t \sin\frac{\pi^2}{a} - \frac{1}{2}\left(\frac{2\pi}{a}\right)^2 t^2 \cos\frac{\pi^2}{a}} \right.$$

$$\left. \mp \frac{1-\frac{2\pi}{a} t \cot\frac{\pi^2}{a}}{\cos\frac{\pi^2}{a} - \cos\frac{\pi}{a}(\theta+\theta_1) + \frac{2\pi}{a} t \sin\frac{\pi^2}{a} - \frac{1}{2}\left(\frac{2\pi}{a}\right)^2 t^2 \cos\frac{\pi^2}{a}} \right\}. \tag{28}$$

Write

$$\left.\begin{aligned} \cos\frac{\pi^2}{a} - \cos\frac{\pi}{a}(\theta-\theta_1) &= \pm\frac{2\pi}{a} l \sin\frac{\pi^2}{a} \\ \cos\frac{\pi^2}{a} - \cos\frac{\pi}{a}(\theta+\theta_1) &= \pm\frac{2\pi}{a} l' \sin\frac{\pi^2}{a} \\ \lambda &= -\frac{\pi}{a}\cot\frac{\pi^2}{a} \end{aligned}\right\}, \tag{29}$$

where the signs are to be so chosen that l, l' are always positive, that is the upper signs are to be taken when $\theta \mp \theta_1 > \pi$, the lower signs when $\theta \mp \theta_1 < \pi$. On substitution in equation (28), we obtain

$$\frac{1}{a} F(t) = \frac{1}{2\pi} \left\{ \frac{1+2\lambda t}{\pm l + t + \lambda t^2} \mp \frac{1+2\lambda t}{\pm l' + t + \lambda t^2} \right\}. \tag{30}$$

For values of a considered, $0 < \lambda < 2$ [see table following equation (58)]. Under these conditions it is proved in Section IV., equation (79), that, taking $2\kappa r$ to be of order 10^6, the value of the integral in equation (27) is given approximately by

$$Z_2 = e^{-i(\kappa r - \frac{1}{4}\pi)}\left[\left\{\pm\phi(l\sqrt{2\kappa r})\right\} \mp \left\{\pm\phi(l'\sqrt{2\kappa r})\right\}\right], \tag{31}$$

where
$$\phi(m) = M(m) + iN(m) = \frac{1}{\sqrt{\pi}}\int_m^\infty e^{i(m^2 - z^2)}\,dz. \tag{32}$$

In the shadow therefore we obtain

$$Z = e^{-i(\kappa r - \frac{1}{4}\pi)}\left\{\phi(l\sqrt{2\kappa r}) \mp \phi(l'\sqrt{2\kappa r})\right\}, \tag{33}$$

and
$$I^2 = (M \mp M')^2 + (N \mp N')^2, \tag{34}$$

where I^2 denotes the intensity of the light, and M' denotes $M(l'\sqrt{2\kappa r})$, with a similar meaning for N'. Outside the shadow,

$$Z = e^{-i(\kappa r - \frac{1}{4}\pi)}\left\{-\phi(l\sqrt{2\kappa r}) \mp \phi(l'\sqrt{2\kappa r})\right\} + e^{i\kappa r\cos(\theta - \theta_1)}, \tag{35}$$

$$\begin{aligned} I^2 = &\left[\cos\left\{\kappa r[1+\cos(\theta-\theta_1)] - \tfrac{1}{4}\pi\right\} - M \mp M'\right]^2 \\ &+ \left[\sin\left\{\kappa r[1+\cos(\theta-\theta_1)] - \tfrac{1}{4}\pi\right\} - N \mp N'\right]^2, \end{aligned} \tag{36}$$

and a similar expression holds in the region of the reflected waves.

SECTION III.—*The Numerical Discussion and Comparison of Results.*

Write now
$$m = l\sqrt{2\kappa r}, \qquad m' = l'\sqrt{2\kappa r}. \tag{37}$$

1. *In the Shadow.*

Integrating by parts the expression for $\phi(m)$ in equation (32), we obtain

$$M = \frac{1}{4\sqrt{\pi}\,m^3}\left(1 - \frac{5}{4m^4} + \ldots\right), \qquad N = -\frac{1}{2\sqrt{\pi}\,m}\left(1 - \frac{3}{4m^4} + \ldots\right). \tag{38}$$

Suppose the wave-length of the light considered to be $2\pi \times 10^{-5}$ cm., and suppose the distance from the screen to be less than would occur in experimental work, say 5 cm. This gives $2\kappa r = 10^6$. On comparison with equations (29), we find that m' is a large number of order 10^3, and that as the deviation from the edge of the shadow increases the value of m rapidly rises from zero to a large number of the same magnitude.

Suppose the shadow divided into two portions by the curve $m = 5$.

Since for small values of $(\theta - \theta_1 - \pi)$, which we will write ψ, m may be written

$$m = \sqrt{\frac{\kappa r}{2}}\left(\psi - \frac{\lambda}{2}\psi^2 + \ldots\right), \tag{39}$$

the curve $m = 5$ may be replaced by the curve $\kappa r\psi^2 = 50$, which is very approximately a parabola of latus rectum $50/\kappa$.

In the first portion, near to the edge of the shadow, $m < 5$, and equation (34) reduces with an error of order 10^{-3} to the form

$$I^2 = \{M(m)\}^2 + \{N(m)\}^2. \tag{40}$$

This is the same as the expression for the intensity, when the screen is absorbing, if the first term only of the right-hand side of equation (39) be retained. [See equations (44) and (45), where references are given to the formula given by Lommel.]

A table of values of λ is given in the table following equation (58), whence it follows that the error in retaining the first term only of the expression for m is again of order 10^{-3}.

In the second portion, which is removed from the edge of the shadow, $m > 5$ and equation (34) reduces with an error less than 10^{-3} to the form

$$I^2 = \{N(m) \mp N(m')\}^2 = \frac{1}{4\pi}\left\{\frac{1}{m} \mp \frac{1}{m'}\right\}^2. \tag{41}$$

This result agrees with that obtained by Poincaré for a beam of light converging upon the edge of the screen.*

The following tables are used to construct the Figures I., II., III., placed at the end of this section. The intensity $\times 10^7$ is tabulated, and $2\kappa r$ is taken $= 10^6$.

(i.) Angle of incidence = 15°.

TABLE A.—LIGHT POLARISED IN PLANE OF DIFFRACTION.

Angle of Prism.	Deviation from Edge of Shadow.				
	5°	10°	20°	40°	60°
0°	306	58	9	1	0
60°	305	58	9	1	0
90°	303	57	9	1	0
103°	302	56	8	1	0
120°	296	54	7	0	—
150°	234	21	—	—	—

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* H. Poincaré, "Sur la polarisation par diffraction," p. 325, *Acta Mathematica*, Vol. xvi, 1892–3.

TABLE B.—LIGHT POLARISED PERPENDICULARLY TO PLANE OF DIFFRACTION.

Angle of Prism.	Deviation.				
	5°	10°	20°	40°	60°
0°	549	166	52	17	9
60°	566	176	59	22	13
90°	587	187	66	28	19
103°	603	195	72	33	26
120°	631	214	86	49	—
150°	846	389	—	—	—

(ii.) Angle of prism = 0°.

Deviation		5°	10°	15°	20°	30°	40°	60°
Absorbing Screen............		419	105	47	26	12	7	3
Reflecting Screen (Angle of Incidence = 90°)	A	383	87	35	18	6	3	1
	B	456	124	60	36	19	13	8
Reflecting Screen (Angle of Incidence = 15°)	A	306	58	20	9	3	.1	0
	B	549	166	84	52	27	17	9

It is clear from the above tables that—

For unpolarised light the effect of increasing the angle of the prism from zero is to increase both the intensity and polarisation of the diffracted light in the shadow.

The difference in phase between the two plane polarised components is inappreciable. Since the intensity at a distance from the edge of the shadow is proportional to the wave-length of the light, the diffracted rays should be tinged with red. This applies equally to both components; but the component polarised in the plane of diffraction diminishes, as the distance from the edge of the shadow increases, much more rapidly than the other component, in which the effect would therefore be more noticeable.

2. *Outside the Shadow.*

The important point is in this case to note what displacements in the position of those bands which are clearly visible, that is those just outside the shadow, are produced by an increase in the angle of the screen.

First, we will compare the case of a reflecting prism of zero angle, with an absorbing screen. Writing $\theta - \theta_1 = \pi - \psi$, we find from equations (29) and (37) that, if $a = 2\pi$,

$$m = \sqrt{2\kappa r}\sin\tfrac{1}{2}\psi, \qquad m' = \sqrt{2\kappa r}\sin(\tfrac{1}{2}\psi + \theta_1). \tag{42}$$

Since in this case $\kappa r[1+\cos(\theta-\theta_1)] = m^2$, it follows that equation (36) may be written

$$I^2 = [\cos(m^2 - \tfrac{1}{4}\pi) - M \mp M']^2 + [\sin(m^2 - \tfrac{1}{4}\pi) - N \mp N']^2. \tag{43}$$

If the terms M', N' be omitted, this is identical in form with the expression for the intensity when the screen is absorbing. Tables of the intensities in this case corresponding to integral values of $(2m^2)$ up to 100 are given by Lommel.* In his notation

$$M(m) = -\tfrac{1}{2}V_{\frac{3}{2}}(x, 0), \qquad N(m) = -\tfrac{1}{2}V_{\frac{1}{2}}(x, 0), \tag{44}$$

provided $x = 2m^2$. But his x is not the same function of ψ that $2m^2$ is; in fact, we may write it

$$x' = \kappa r\frac{\sin^2\psi}{\cos\psi}. \tag{45}$$

Expanding equations (42) and (45) in ascending powers of ψ,

$$\sqrt{\frac{x}{\kappa r}} = \psi(1 - \tfrac{1}{24}\psi^2), \qquad \sqrt{\frac{x'}{\kappa r}} = \psi(1 + \tfrac{1}{12}\psi^2). \tag{46}$$

Reversing the series,

$$\psi = \sqrt{\frac{x}{\kappa r}}\left(1 + \tfrac{1}{24}\frac{x}{\kappa r}\right), \qquad \psi' = \sqrt{\frac{x}{\kappa r}}\left(1 - \tfrac{1}{12}\frac{x}{\kappa r}\right), \tag{47}$$

where ψ refers to the angle corresponding to a value x of $2m^2$ in our notation, and ψ' to the angle in Lommel's notation corresponding to the same value x.

Taking $x = 10^2$, $2\kappa r = 10^6$, the proportionate error in writing $\psi = \psi'$ is $\frac{1}{4}(10)^{-4}$, and is therefore insensible for values of x less than 100, that is for the first eight bands; the error would be of order 10^{-3} after about 300 bands.

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* "Die Beugungserscheinungen geradlinig begrenzter Schirme," *Abh. der k. bayer Akad. der Wiss.*, II. Cl., XV. Bd., 3. Abth.

It still remains to find the effect of the terms M', N' on the values of x for which the intensity is a maximum or minimum. If the angle of incidence be 90°, we have very approximately, from equations (38) and (42),

$$N' = -\frac{10^{-8}}{2\sqrt{\pi}}(1+\tfrac{1}{8}\psi^2) = -\frac{10^{-8}}{2\sqrt{\pi}}(1+10^{-6}x), \qquad M' = \frac{10^{-9}}{4\sqrt{\pi}}. \tag{48}$$

Write now $P = \cos(m^2-\frac{1}{4}\pi)-M(m)$, $Q = \sin(m^2-\frac{1}{4}\pi)-N(m)$, (49)

and equation (43) becomes

$$I^2 = P^2+(Q \mp N')^2. \tag{50}$$

Differentiate equation (32), giving

$$\frac{d}{dm}\phi(m) = -\frac{1}{\sqrt{\pi}}+2im\,\phi(m),$$

whence $$\frac{dM}{dm} = -\left(2mN+\frac{1}{\sqrt{\pi}}\right), \quad \frac{dN}{dm} = 2mM; \tag{51}$$

and therefore $$\frac{dP}{dm} = \frac{1}{\sqrt{\pi}}-2mQ, \quad \frac{dQ}{dm} = 2mP. \tag{52}$$

Differentiate equation (50), making use of equation (52),

$$\tfrac{1}{2}\frac{dI^2}{dm} = P\left(\frac{1}{\sqrt{\pi}}-2mQ\right)+(Q\mp N')\,2mP = P\left(\frac{1}{\sqrt{\pi}}\mp 2mN'\right). \tag{53}$$

The minimax values of x are therefore still very approximately equal to the roots of the equation $P = 0$, and *the position of the bands close to the edge of the geometrical shadow is therefore the same for a thin reflecting prism as for a perfectly absorbing screen of any angle.* The correspondence is in this case solely limited by equations (47), since the approximation in equation (53) is much closer.

Secondly, when the angle of the reflecting prism is not zero the correspondence is not so close. Since equations (42) no longer hold, write

$$\mu = \sqrt{2\kappa r}\sin\tfrac{1}{2}\psi = \sqrt{2\kappa r}\,\{1+\cos(\theta-\theta_1)\}. \tag{54}$$

If now we express m, m' in terms of μ, the only alterations in the position of the bands caused by the increase in the angle of the prism are due to the changes in the position of the minimax values of the intensity regarded as a function of μ; we have, with a proportionate error of $\frac{1}{4}\psi^2$ or $\mu^2/2\kappa r$,

$$m = \sqrt{2\kappa r}\,\frac{a}{2\pi}\,\frac{\cos\dfrac{\pi}{a}(\pi-\psi)-\cos\dfrac{\pi^2}{a}}{\sin\dfrac{\pi^2}{a}} = \tfrac{1}{2}\sqrt{2\kappa r}\,\psi\left(1-\frac{\pi}{2a}\cot\frac{\pi^2}{a}\,\psi\right),$$

from equations (29) and (37).

Further, from equation (54) and the third equation in (29),

$$m = \mu\left(1+\frac{\lambda\mu}{\sqrt{2\kappa r}}\right). \tag{55}$$

Equation (36) may now be written approximately, as in (43), in the form

$$I^2 = \left[\cos(\mu^2-\tfrac{1}{4}\pi)-M\left(\mu+\frac{\lambda\mu^2}{\sqrt{2\kappa r}}\right)\right]^2 + \left[\sin(\mu^2-\tfrac{1}{4}\pi)-N\left(\mu+\frac{\lambda\mu^2}{\sqrt{2\kappa r}}\right)\mp N'\right]^2, \tag{56}$$

where N' may be treated as a constant.

Write

$$\left.\begin{aligned} P &= \cos(\mu^2-\tfrac{1}{4}\pi)-M(\mu), & p &= -\frac{\lambda\mu^2}{\sqrt{2\kappa r}}\frac{dM}{d\mu} \\ Q &= \sin(\mu^2-\tfrac{1}{4}\pi)-N(\mu), & q &= -\frac{\lambda\mu^2}{\sqrt{2\kappa r}}\frac{dN}{d\mu}\mp N' \end{aligned}\right\}, \tag{57}$$

and equation (56) takes the form

$$I^2 = (P+p)^2+(Q+q)^2. \tag{58}$$

The value of the constant λ is now of some importance. Its values for several angles of the prism are tabulated below.

Angle of Prism...	0°	30°	60°	90°	120°	150°
λ	0	·0784	·195	·385	·750	1·78

The value of λ increases indefinitely as the angle of the prism approaches 180°; but, although at first one might expect to approximate to the effect of a curved screen, a problem still unsolved, these values are unimportant owing to the fact that the angle of incidence is necessarily small, and the expression for the intensity assumes a much more complicated form owing to the superposition of the bands at the edge of the reflected light.

The equation giving the values of μ for which I^2 is a minimax is found by differentiating equation (58) to be

$$(P+p)\frac{d}{d\mu}(P+p)+(Q+q)\frac{d}{d\mu}(Q+q) = 0, \tag{59}$$

where, as in equations (52), we have

$$\frac{dP}{d\mu} = \frac{1}{\sqrt{\pi}}-2\mu Q, \qquad \frac{dQ}{d\mu} = 2\mu P. \tag{60}$$

On substitution, if we neglect $(d/d\mu)(p^2+q^2)$ as of the second order, we obtain

$$P\left(\frac{1}{\sqrt{\pi}}+\frac{dp}{d\mu}+2\mu q\right)+Q\frac{dq}{d\mu}+p\frac{dP}{d\mu}=0. \tag{61}$$

It is to be expected that roots of this equation are very nearly coincident with those of $P=0$: we may accordingly suppose any root of equation (61) to differ from the corresponding root of $P=0$ by a small quantity h, whence

$$P=h\frac{dP}{d\mu}\text{ approx.} \tag{62}$$

Substituting in (61), we find, approximately,

$$-h\frac{dP}{d\mu}=\sqrt{\pi}\left(Q\frac{dq}{d\mu}+p\frac{dP}{d\mu}\right), \tag{63}$$

which is independent of the value of N' so long as it remains sensibly constant.

Differentiating the last equation in (57), remembering that

$$\frac{dN}{d\mu}=2\mu M,$$

$$\frac{dq}{d\mu}=2\mu p-\frac{6\lambda\mu^2}{\sqrt{2\kappa r}}M;$$

whence, from (63),

$$-h\frac{dP}{d\mu}=\sqrt{\pi}\left[p\left(\frac{dP}{d\mu}+2\mu Q\right)-\frac{6\lambda\mu^2MQ}{\sqrt{2\kappa r}}\right],$$

and, from equations (57) and (60),

$$h\frac{dP}{d\mu}=\frac{\lambda\mu^2}{\sqrt{2\kappa r}}\left(\frac{dM}{d\mu}+6\sqrt{\pi}\,MQ\right), \tag{64}$$

which, written in terms of the tabulated functions,* gives

$$h=\frac{\lambda\mu^2}{\sqrt{2\kappa r}}\,\frac{(1+2\mu\sqrt{\pi}\,N+6\pi MQ)}{(1-2\mu\sqrt{\pi}\,Q)}. \tag{65}$$

For the first eight maxima and minima the coefficient of λ has a value decreasing from $2{\cdot}4\times10^{-4}$ for the first to $2{\cdot}8\times10^{-5}$ for the eighth; so that *the angle of the prism would seem to have no appreciable effect on the bands close to the edge of the geometrical shadow.* This result is in agreement with experiment. With regard to the effect of a curved or blunt surface, it is interesting to note Fresnel's observation that the bands produced by the back are the same as those made by the edge of a razor. The above calculations refer only to the position of the bands, but the alterations in intensity also are only of order $2q$, that is, of order 10^{-3}, and would therefore be inappreciable.

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* Table xxi.*a*, *l.c.*, p. 404.

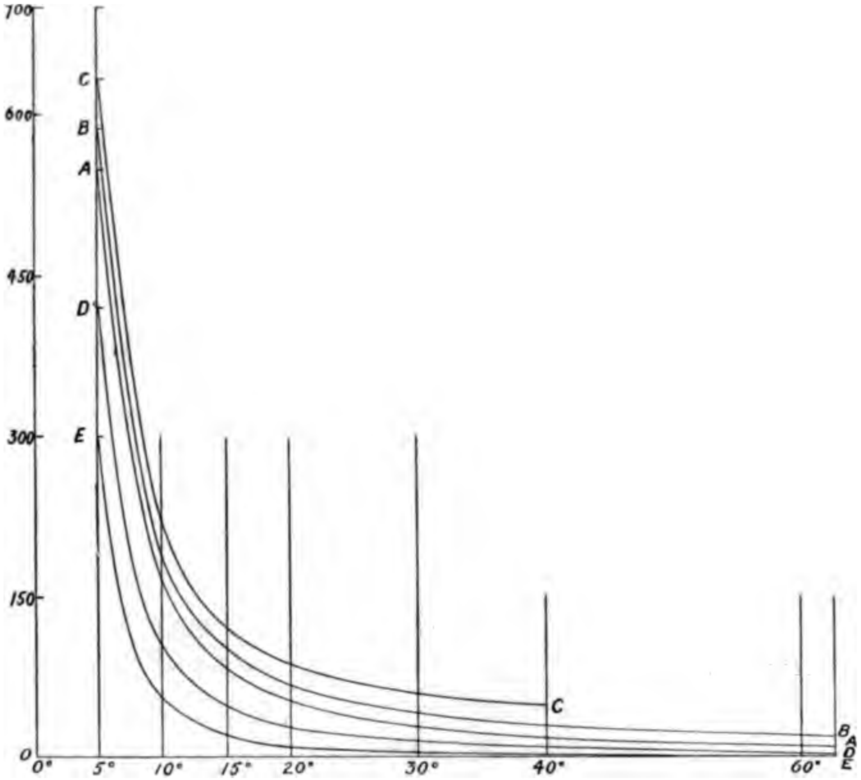

FIG. I.—Intensity curves in the shadow when the angle of incidence is 15°. The ordinates measure the intensity and the abscissæ the deviation from the edge of the shadow.

Curves A, B, C represent the intensity of light polarised perpendicularly to the plane of diffraction for perfectly reflecting prisms of angles 0°, 90°, 120° respectively. When the light is polarised in the plane of diffraction, these three cases become almost indistinguishable, and are represented by the single curve *E*.

Curve D corresponds to a perfectly absorbing prism of any angle, and is independent of the polarisation.

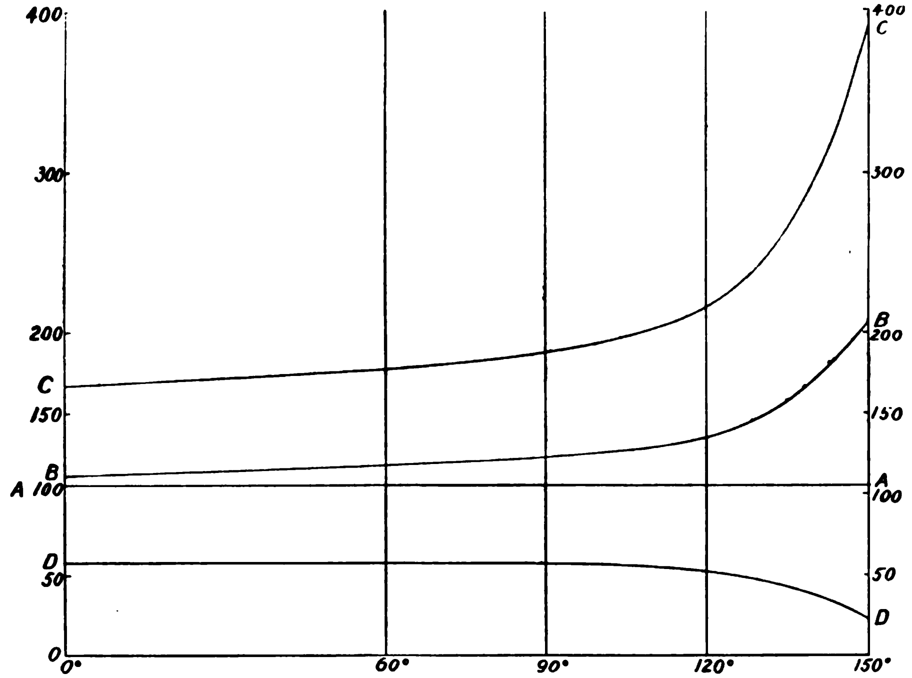

FIG. II.—The effect of the angle of the prism on the intensity; the deviation from the edge of the shadow being 10°, the angle of incidence 15°, and the distance from the edge of the screen 5 cm. The ordinates measure the intensity and the abscissæ the angle of the prism.

Curve A corresponds to the case of a perfectly absorbing prism; *Curves B, C, D* to a perfectly reflecting prism.

Curve B represents the intensity for unpolarised light.

Curves C, D represent the intensities for light polarised perpendicularly to and in the plane of diffraction respectively.

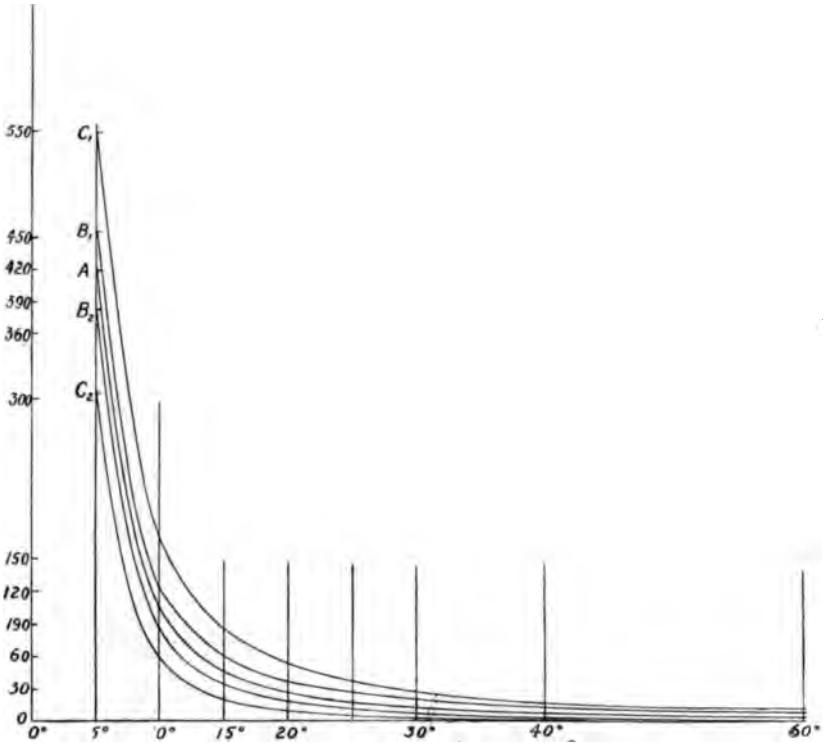

Fig. III.—The effect of the angle of incidence on the intensity when the angle of the prism is zero. The ordinates measure the intensity and the abscissæ the deviation measured from the edge of the shadow.

Curve A corresponds to any angle of incidence, the prism being perfectly absorbing.

Curves B, C correspond to angles of incidence of 90° and 15° respectively, the prism being perfectly reflecting and of zero angle. The suffixes 1, 2 denote that the light is polarised perpendicularly to or in the plane of diffraction.

SECTION IV.—*The Analytical Discussion of certain Integrals.*

It is required to discuss the nature of the limit which the integral

$$u = \int_{-i\infty}^{i\infty} e^{iat^2} F(t)\,dt \tag{66}$$

approaches when a is a large positive quantity.

In accordance with equations (24) and (30), $F(t)$ will be taken to diminish to the limit zero as the modulus of t increases, and to have no infinities except those upon the real axis in the t-plane.

When $t = \rho e^{i\phi}$, $\qquad R(iat^2) = -a\rho^2 \sin 2\phi$;

this is negative when $0 < \phi < \frac{1}{2}\pi$ or $\pi < \phi < \frac{3}{2}\pi$. The value of the integral in (66) taken over any portion of a circle of large radius having the origin as centre therefore tends to the limit zero, provided that the path lie in either the first or the third quadrant; whence the path of integration along the imaginary axis from $-i\infty$ to $i\infty$ may be replaced by the path along a straight line through the origin from $-\infty e^{\frac{1}{4}i\pi}$ to $+\infty e^{\frac{1}{4}i\pi}$.

We may now write $t = \rho e^{\frac{1}{4}i\pi}$, and equation (66) becomes

$$u = e^{\frac{1}{4}i\pi} \int_{-\infty}^{\infty} e^{-a\rho^2} F(\rho e^{\frac{1}{4}i\pi})\,d\rho. \tag{67}$$

But
$$\int_m^\infty e^{-a\rho^2}\,d\rho = \frac{e^{-am^2}}{2am} - \frac{1}{2a}\int_m^\infty e^{-a\rho^2}\,\frac{d\rho}{\rho^2},$$

where the second integral is wholly positive; whence, if m is any positive quantity, since

$$\int_{-\infty}^{\infty} e^{-a\rho^2}\,d\rho = \sqrt{\frac{\pi}{a}}, \tag{68}$$

$$\sqrt{\frac{\pi}{a}} - \int_{-m}^{m} e^{-a\rho^2}\,d\rho < \frac{e^{-am^2}}{am}. \tag{69}$$

Again, writing

$$u_1 = e^{\frac{1}{4}i\pi}\int_{-m}^{m} e^{-a\rho^2} F(\rho e^{\frac{1}{4}i\pi})\,d\rho, \qquad u = u_1 + u_2, \tag{70}$$

we have
$$|u_2| < \int_{-\infty}^{-m} + \int_m^\infty e^{-a\rho^2}\,|F(\rho e^{\frac{1}{4}i\pi})|\,d\rho,$$

and therefore
$$|u_2| < \frac{Ce^{-am^2}}{am}, \tag{71}$$

where C is the maximum value which $|F(\rho e^{\frac{1}{4}i\pi})|$ assumes.

But u_1 differs from $e^{\frac{1}{4}i\pi}F(0)\int_{-m}^{m} e^{-a\rho^2}\,d\rho$ by a quantity whose modulus is less than $D\int_{-m}^{m} e^{-a\rho^2}\,d\rho$, where D is the maximum value of $|F(\rho e^{\frac{1}{4}i\pi})-F(0)|$ for values of ρ between $\pm m$.

Therefore the error in writing

$$u = e^{\frac{1}{4}i\pi}F(0)\sqrt{\frac{\pi}{a}} \tag{72}$$

is negligible if we can find a quantity m such that both

$$\left\{1+\frac{C}{|F(0)|}\right\}\frac{e^{-am^2}}{\sqrt{\pi a m^2}} \quad \text{and} \quad \frac{D}{|F(0)|}$$

are negligible in comparison with unity.

If $a = 10^6$, choose $am^2 = 6$, whence $\dfrac{e^{-am^2}}{\sqrt{\pi a m^2}} = 10^{-3}$, $m = 2{\cdot}4\times 10^{-3}$. When a has greater values the error is correspondingly diminished.

This method of approximation amounts to expanding $F(t)$ in ascending powers of t by Taylor's theorem. Suppose m now chosen as small as is consistent with the condition that $\dfrac{e^{-am^2}}{\sqrt{\pi a m^2}}\left\{1+\dfrac{C}{|F(0)|}\right\}$ must be negligible; if the condition that $\dfrac{D}{|F(0)|}$ must also be negligible is not satisfied, some other method must be found.

In this case it is necessary to assume for $F(t)$ the form given in equation (30), to which it approximates when t is small; that is,

$$F(t) = \frac{1}{2\pi}\,\frac{1+2\lambda t+\dots}{\pm l+t+\lambda t^2+\dots}. \tag{73}$$

With an error of order t, λ being of order unity, we may write

$$F(t) = \frac{1}{2\pi}\,\frac{1}{t\pm l}. \tag{74}$$

We assume that a quantity m, itself negligible in comparison with unity, can be found such that u_2 is negligible in accordance with equations (70), (71); whence u may be replaced by $e^{\frac{1}{4}i\pi}\int_{-m}^{m} e^{-a\rho^2}F(\rho e^{\frac{1}{4}i\pi})\,d\rho$. Since the error in using equation (74) is of order t, where $|t| < m$, it is natural to assume that u may be replaced by $\dfrac{e^{\frac{1}{4}i\pi}}{2\pi}\int_{-m}^{m} e^{-a\rho^2}\dfrac{d\rho}{\rho e^{\frac{1}{4}i\pi}\pm l}$; whence, again making use of equations (70), (71), writing $t = \rho e^{\frac{1}{2}i\pi}$, and defining u_0 by the equation

$$u_0 = \frac{1}{2\pi} \int_{-\infty i}^{\infty i} e^{iat^2} \frac{dt}{t \pm l}, \tag{75}$$

the error in replacing u by u_0 is negligible.

The assumption involved in this last approximation is justified later in equation (83) and the following table, by carrying the approximation to equation (73) one stage further and calculating the corresponding error on comparison with equation (74). But, first, u_0 will be reduced to the form of Fresnel's integral.

Write
$$\frac{1}{t \pm l} = \pm \int_{-\infty}^{0} e^{u(l \pm t)} du, \tag{76}$$

where l is positive and t is wholly imaginary. We obtain, on substituting in equation (75),

$$u_0 = \pm \frac{1}{2\pi} \int_{-\infty}^{0} du \int_{-\infty i}^{\infty i} dt\, e^{ia(t \mp iu/2a)^2 + iu^2/4a + ul}$$

$$= \pm \frac{1}{2\sqrt{\pi a}} e^{ial^2 + \frac{1}{4}i\pi} \int_{-\infty}^{0} e^{i(u - 2ial)^2/4a} du.$$

Write now
$$u - 2ial = -i2\sqrt{a}z\,; \tag{77}$$

whence
$$u_0 = \pm \frac{1}{\sqrt{\pi}} e^{ial^2 - \frac{1}{4}i\pi} \int_{-i\infty}^{\sqrt{a}l} e^{-iz^2} dz = \mp \frac{1}{\sqrt{\pi}} e^{ial^2 - \frac{1}{4}i\pi} \int_{\sqrt{a}l}^{\infty} e^{-iz^2} dz.$$

Therefore, writing

$$\phi(m) = M(m) + iN(m) = \frac{1}{\sqrt{\pi}} \int_{m}^{\infty} e^{i(m^2 - z^2)} dz, \tag{78}$$

$$u_0 = \pm e^{\frac{3}{4}i\pi} \phi(\sqrt{a}l). \tag{79}$$

In order to obtain a closer approximation to equation (73) than is given by equation (74), we may write

$$\delta u_0 = u - u_0 = \frac{\lambda}{2\pi} \int_{-i\infty}^{i\infty} e^{iat^2} \left\{ \frac{2t}{\pm l + t} - \frac{t^2}{(\pm l + t)^2} \right\} dt. \tag{80}$$

The second part of this integral may be written

$$\frac{\lambda}{2\pi} \int_{-i\infty}^{i\infty} e^{iat^2} \left\{ -1 \pm \frac{2l}{\pm l + t} - \frac{l^2}{(\pm l + t)^2} \right\} dt,$$

whence we find, on integrating the last term in the bracket by parts, that

$$\delta u_0 = \frac{\lambda}{2\pi} \int_{-i\infty}^{i\infty} e^{iat^2} \left\{ 1 - \frac{2ial^2 t}{\pm l + t} \right\} dt.$$

Substitute now from equations (68) and (79), giving

$$\delta u_0 = \lambda \left\{ (1-2ial^2)\frac{e^{\frac{1}{4}i\pi}}{2\sqrt{\pi a}} + 2ial^3 e^{\frac{1}{4}i\pi}\phi(\sqrt{a}\, l) \right\}. \tag{81}$$

If we write $x = 2al^2$, this becomes

$$\delta u_0 = \lambda e^{-\frac{1}{4}i\pi} \left\{ \frac{1}{2\sqrt{\pi a}}(x+i) + x\sqrt{\frac{x}{2a}}\,(N-iM) \right\}; \tag{82}$$

whence, from equation (79),

$$\left.\begin{aligned} R(ue^{\frac{1}{4}i\pi}) &= \mp M + \lambda\left(Nx\sqrt{\frac{x}{2a}} + \frac{x}{2\sqrt{\pi a}}\right) \\ I(ue^{\frac{1}{4}i\pi}) &= \mp N + \lambda\left(\frac{1}{2\sqrt{\pi a}} - Mx\sqrt{\frac{x}{2a}}\right) \end{aligned}\right\}. \tag{83}$$

In the applications considered $0 < \frac{1}{2}\lambda < 1$ [see table following equation (58)]. Taking $a = (10)^6$, $\frac{1}{2}\lambda = 1$, the errors corresponding to M, N are given in the following table,* which shows that for our purposes they may be neglected:—

x ...	M	N
5 ...	·00020	·00014
10 ...	·00013	·00006
15 ...	·00010	·00003
25 ...	·00003	·00001

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* The values of M, N are obtained from Table XXII., *l.c.*, p. 404.

ON MANY-VALUED NEWTONIAN POTENTIALS

By A. C. DIXON.

[Received November 14th, 1903.—Read December 10th, 1903.]

1. This paper deals with a theory on the lines of Riemann's theory of the Abelian functions, but relating to space of three dimensions, instead of two. Even for a uniform potential there are known to be differences between the two cases, and the contrast is further developed when we consider many-valued potentials. For instance, on a Riemann surface there are irreducible cycles of one kind only, namely, those of one dimension. Here the cycles are divided into two kinds, those of one dimension and those of two. Again, there is nothing in this theory to play the part of the Abelian potentials of the first kind, or of a monogenic function.

2. A potential is a function of the rectangular Cartesian co-ordinates x, y, z which, except at certain special, or "singular," points is finite and continuous, and has finite and continuous differential coefficients of the first two orders, and which is reduced to zero by the operation $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, or Δ. We shall not suppose the potential to be one-valued, but generally to have at each point of space a certain constant finite number of values, which are permuted in a definite way when the point describes any assigned closed circuit. The particular permutation produced will depend on the way the circuit interlaces with certain fixed closed lines, the "branch lines." Surfaces or "doors" bounded by these branch lines may be so taken that any circuit interlacing any one of the lines will necessarily pierce the corresponding door. If the doors are reckoned as boundaries, each of the original different values of the potential becomes a one-valued potential in the bounded region. On the analogy of a Riemann surface we may suppose these different potentials to exist in different but co-extensive regions; let such regions be distinguished by colour, as there is some difficulty in conceiving them as sheets. The different colours are separated by doors. A simple small hole made in one face of one of these doors will connect two of the colours; a similar hole will connect a third with one of the two; and thus all the colours may be connected, and we arrive at a many-coloured region within which the potential is one-valued. This region will generally become

multiply connected when all the barriers that have been set up are removed; that is, it will contain closed circuits, which cannot be made infinitesimal by continuous deformation. These circuits may be of one dimension or of two.

3. Infinity will be treated as a point, and a line or surface which has no boundary in finite space will be counted as closed. This is in accordance with the known results of inversion from an arbitrary centre. Such an inversion may be used to secure that any problem under discussion shall have no special relation with the point at infinity.

4. Points on branch lines are of course singularities of the potential, but when we are dealing with potentials one-valued in a given many-coloured region these "singularities of the region" will be taken as implied. A potential may, and in fact must, have other singularities in addition. The simplest is that of the potential

$$\{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}}$$

at the point (x', y', z'), and from this others may be derived by differentiation, and by integration with respect to (x', y', z'). Two potentials ϕ, ψ will be said to have the same singularity at a point P when $\phi-\psi$ has no singularity within a sphere whose centre is P and radius some quantity ρ which is finite, but may be as small as we please. If in this case

$$\psi = C\{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}},$$

where (x', y', z') are the co-ordinates of P, we shall say that ϕ has a "pole" at P, or, to distinguish this singularity from one of higher order, a simple pole: the coefficient C is the "residue" of ϕ at P, unless P falls upon a branch line, when another definition is more convenient.

5. A question arises as to when there is a singularity at infinity. Now, if we invert from a point O, and ϕ is the value at P of a certain potential, then $\phi . OP$ is the value of the corresponding potential at Q, the inverse of P. If O is an ordinary point for ϕ, then, when P coincides with O, ϕ is finite and OP zero; so that the potential $\phi . OP$ vanishes when Q moves to infinity. But, if ϕ has a simple pole at O, with residue C, then $\phi . OP$ approaches the limit C when Q moves to infinity. I shall therefore speak of a potential as having a singularity at infinity, in any colour, when its value there differs from zero. When it has the constant value C at infinity, in any colour, it will be considered

as having a simple pole there, with residue C. The constant C itself has a pole at infinity, in each colour, and no other singularity.

6. A branch line must, in the nature of things, be closed; for, if it had an end, a small circuit enclosing it could be reduced by passing it along to the end and beyond. In like manner, if two branch lines interlace each other, both affecting one colour, each must affect all the colours that the other affects. If, for instance, a branch line A interchanging the red and blue interlaces with one, B, interchanging the red and green, then B must affect the blue, or else it would come to an end at the door that is bounded by A and separates the red and blue: similarly A must affect the green.

7. The chief problem to be solved is that of proving the existence of a potential one-valued in a given many-coloured region and having no singularity except a single simple pole at an assigned point in a particular colour. In addition to the difficulties overcome by Schwarz and Neumann for the case of a region of two dimensions there is a new one; for we cannot even in the simplest case get rid of a branch line by means of a change of variables without a material change in the operator Δ. The following method is widely applicable.

Take a complete set of doors D_1. These will be bounded by the branch lines B. Suppose, if possible, that no two doors intersect each other; but if, owing to the presence of interlacing branch lines, this is impossible, let the lines of intersection be denoted by C. Take three other complete sets of doors D_2, D_3, D_4, intersecting along C, and so that no two of the four sets have any point in common except the points on B and C. Suppose that any circuit passing round a branch line pierces the doors in the order D_1, D_2, D_3, D_4.

Again, enclose the lines B, C in two tubular surfaces, T_1, T_2, which do not meet those lines or each other. These might, for instance, be generated thus: let $2k$ be the least distance between any two of the lines B, or any two of the lines C, and take two distances r_1, r_2 so that $O < r_1 < r_2 < k$; let T_1 pass through all points whose least distance from B or C is r_1, and T_2 all points whose least distance from B or C is r_2.

Let R denote the region outside T_1, taken to be one-coloured. Then R is divided into four parts by D_1, D_2, D_3, D_4. Let (12) be the part between D_1 and D_2, (23) that between D_2 and D_3, and so on; (12), (23), (34) may each consist of separate pieces. Let (123) stand for (12)+(23), 1234) for (12)+(23)+(34), and so on. Let D_1', D_2', D_3', D_4' stand for the parts of D_1, D_2, D_3, D_4 respectively that are contained in R, and 12, 234, ... for the parts of T_1 that bound (12), (234), ...

8. Let m be the number of colours in the region as originally given. Then a function one-valued in the original region is equivalent to m functions which in ordinary space or in R are not one-valued, but are permuted by the description of closed circuits. Let S denote the permutation caused by the description of a closed circuit which pierces the doors in the order D_1, D_2, D_3, D_4. The particular permutation denoted by S will, of course, depend on the particular door of the set D_1 that is chosen.

The region R is bounded by T_1. Let us seek for m potentials $\psi_1, \psi_2, \ldots, \psi_m$, say ψ, having assigned values on T_1 and assigned singularities in R and permuted into $S\psi$ by a closed circuit piercing the doors in the order D'_1, D'_2, D'_3, D'_4. The boundary values and singularities must be assigned in such a way as to satisfy this last condition.

In (2341) construct a set of m potentials $\chi^{(1)}$ having the assigned boundary values and singularities so far as they affect this region and having any convenient values, say zero, on D'_1, D'_2. This is a case of Dirichlet's problem, and may be solved by Poincaré's method of "clearances" (*balayage*).*

In (4123) construct a set of m potentials $\chi^{(2)}$ having the given boundary values and singularities so far as they affect this region, and the values $S\chi^{(1)}$ on D'_3, $\chi^{(1)}$ on D'_4. Carry on this process, constructing $\chi^{(2n+1)}$ for (2341) with the values $\chi^{(2n)}$ on D'_1 and $S^{-1}\chi^{(2n)}$ on D'_2, then $\chi^{(2n+2)}$ for (4123) with the values $S\chi^{(2n+1)}$ on D'_3 and $\chi^{(2n+1)}$ on D'_4; $\chi^{(n)}$ throughout denotes a potential for its region having the given boundary values and singularities so far as they affect this region, and no other singularity therein.

If it so happened that the boundary values of $\chi^{(3)}$ were the same as those of $\chi^{(1)}$, we should have, throughout (2341),

$$\chi^{(1)} = \chi^{(3)} = \chi^{(5)} = \chi^{(7)} = \ldots,$$

and, throughout (4123), $\chi^{(2)} = \chi^{(4)} = \chi^{(6)} = \ldots$;

thus $\chi^{(1)}$, $\chi^{(2)}$ would be continuations of each other and would be potentials satisfying the conditions of the problem. This will not generally happen: we must prove that in any case the process is a convergent one and that the limit to which it leads when n is increased indefinitely gives the solution of the problem.

9. Let $\chi^{(n+2)} - \chi^{(n)} = \omega^{(n)}$; then $\omega^{(2n+1)}$ has no singularity in (2341)

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* *American Journal of Mathematics*, Vol. XII.

and vanishes on 2341; similarly $\omega^{(2n)}$ has no singularity in (4123) and vanishes on 4123. Now a potential without singularity in (2341) vanishing on 2341 and equal to 1 on D'_1 and D'_2 will not exceed a certain quantity on D'_3 and D'_4. Similarly a potential without singularity in (4123) vanishing on 4123 and equal to 1 on D'_3 and D'_4 will not exceed a certain quantity on D'_1 and D'_2. Let q be the greater of these superior limits; then $q < 1$. Let M be a superior limit to the values of the potentials $\omega^{(1)}$ in (2341); then qM is a superior limit to the values of $\omega^{(1)}$ on D'_3 or D'_4 and therefore to the values of $\omega^{(2)}$ on D'_3 and D'_4. Similarly q^2M is a superior limit to $\omega^{(3)}$, and so on. In general, $q^{n-1}M$ is a superior limit to the numerical values of $\omega^{(n)}$ in the region for which $\chi^{(n)}$ is constructed.

Now $$\chi^{(2n+1)}-\chi^{(1)} = \omega^{(1)}+\omega^{(3)}+\ldots+\omega^{(2n-1)}$$

and $$\chi^{(2n)}-\chi^{(2)} = \omega^{(2)}+\omega^{(4)}+\ldots+\omega^{(2n-2)}.$$

Each of these series is absolutely and uniformly convergent in its own region, and therefore $\chi^{(2n)}$ tends to a definite limit at every point of (4123), and $\chi^{(2n+1)}$ at every point of (2341), when n is increased without limit.

10. Again, these limits are potentials having the assigned singularities, and no others. For consider any particular derivative. To prove that it exists at any point, let ρ be the radius of a sphere with this point as centre and lying wholly within (2341) or (4123), as the case may be. If a potential has no singularity within this sphere and its values on the surface of the sphere do not exceed 1 numerically, then it cannot yield for this derivative at the centre a greater value than $\kappa\rho^{-p}$, where p is the order of the derivative and and κ a numerical constant. Hence the series derived by forming this particular derivative from the series

$$\omega^{(1)}+\omega^{(3)}+\omega^{(5)}+\ldots \quad \text{and} \quad \omega^{(2)}+\omega^{(4)}+\omega^{(6)}+\ldots$$

term by term are less term by term than $\kappa\rho^{-p}M\{1+q^2+q^4+\ldots\}$, which converges absolutely. Hence this derivative exists for $\chi^{(2\infty)}-\chi^{(2)}$ and for $\chi^{(2\infty+1)}-\chi^{(1)}$ at every point. The operator Δ destroys $\omega^{(1)}$, $\omega^{(2)}$, ... and therefore $\chi^{(2\infty)}-\chi^{(2)}$ and $\chi^{(2\infty+1)}-\chi^{(1)}$. Hence $\chi^{(2\infty)}$, $\chi^{(2\infty+1)}$ are potentials for their respective regions, and have the singularities of $\chi^{(2)}$, $\chi^{(1)}$ respectively, and no more, in those regions.

Again, on D'_1, $\chi^{(2n)} = \chi^{(2n+1)}$; on D'_4, $\chi^{(2n+2)} = \chi^{(2n+1)}$.

Thus, in the region (14), $\chi^{(2n)}-\chi^{(2n+1)}$ has no singularity, while it vanishes on D'_1 and 14; and on D'_4, which forms the rest of the boundary, it has the value $-\omega^{(2n)}$, which diminishes indefinitely when n increases. It follows that, in (14), $\chi^{(2\infty)} = \chi^{(2\infty+1)}$.

Again, (23) forms part of (2341) and also of (4123). The values of $\chi^{(2n)}$ and $\chi^{(2n+1)}$ on 23 are the assigned boundary values, but not in the same order; we have, in fact, on 23, $\chi^{(2n)} = S\chi^{(2n+1)}$. The same equation holds good on D_2', while, on D_3', $\chi^{(2n)} = S\chi^{(2n-1)}$ or $S\chi^{(2n+1)}-\chi^{(2n)} = S\omega^{(2n-1)}$, which tends to the limit 0 when n increases indefinitely.

Thus, in the region (23), $\chi^{(2\infty)} = S\chi^{(2\infty+1)}$.

The set of functions $\chi^{(2\infty+1)}$ which exist in (23) may be continued across the surfaces D_3', D_4' into the region (41), where they coincide with $\chi^{(2\infty)}$; these may be continued across D_1' and D_2' into (23), where they coincide with $S\chi^{(2\infty+1)}$. Hence the permutation S is produced by a closed circuit piercing the doors in the order D_3, D_4, D_1, D_2, as required. The solution will be proved complete if we can show that $\chi^{(\infty)}$ approaches the assigned values on the boundary continuously. Take any point P in the region (2341) and let η be the value at P of the potential which vanishes on 2341 and $= 1$ on D_1', D_2'. Then, at P, $\omega^{(2n+1)} \ngtr \eta M q^{2n}$ numerically, and $\chi^{(2\infty+1)}-\chi^{(1)} \ngtr \eta M(1+q^2+q^4+\dots)$ or $\chi^{(2\infty+1)}-\chi^{(1)} \ngtr \eta M/(1-q^2)$. As P approaches a point on 2341 where the assigned boundary values are continuous, $\chi^{(1)}$ approaches the assigned set of values at that point continuously and η diminishes indefinitely; hence $\chi^{(2\infty+1)}$ approaches the assigned set of values continuously, which was to be proved. Similarly for $\chi^{(2\infty)}$. The problem proposed for the region R is therefore solved.

11. The same problem is now to be solved for the region enclosed by T_2. Suppose in the first place that T_2 is a simple tube enclosing one branch line, and that a circuit round this line changes ϕ_1 into ϕ_2, ϕ_2 into ϕ_3, ..., ϕ_m into ϕ_1. Let e be any m-th root of 1. Then the circuit changes $e^{m-1}\phi_1+e^{m-2}\phi_2+\dots+\phi_m$ into $e(e^{m-1}\phi_1+e^{m-2}\phi_2+\dots+\phi_m)$. By taking the different m-th roots of 1, we can form m expressions of this kind, and when the values of these expressions are known the values of ϕ_1, ϕ_2, ..., ϕ_m can be at once deduced. The problem is therefore reduced to the following:—To find a potential for the tube having given singularities and boundary values, and such that passage round the branch line multiplies it by e. In the case when $e = 1$ this is Dirichlet's problem in its ordinary form, and we shall therefore take $e \neq 1$.

Take doors D_1, D_2, D_3, D_4* dividing the tube into four parts for which we may, without confusion, use the same notation as before, that is, (12), 12 will denote the volume and surface intercepted between D_1 and D_2, and so on; the parts of D_1, ... contained within the tube will now be called D_1', ...

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* These need not be the same as were denoted by D_1, D_2, D_3, D_4 in Art. 7, and must not, in fact, meet each other within the tube except on the branch line.

In (2341) construct a potential $\chi^{(1)}$ having the assigned boundary values and singularities, so far as they affect this region, and having any convenient values, say zero, on D'_1, D'_2. Carry on an alternating process as before, forming $\chi^{(2n)}$ for (4123) with the values $e\chi^{(2n-1)}$ on D'_3 and $\chi^{(2n-1)}$ on D'_4, $\chi^{(2n+1)}$ for (2341) with the values $\chi^{(2n)}$ on D'_1 and $e^{-1}\chi^{(2n)}$ on D'_2; each potential is to have the given boundary values and singularities so far as they affect its region. If the values thus prescribed for $\chi^{(3)}$ on D'_1 and D'_2 agree with those of $\chi^{(1)}$, then the problem is solved; for $\chi^{(1)}$ and $\chi^{(2)}$ are continuations of each other, and are the values of a potential having the desired properties; then $\chi^{(1)}=\chi^{(3)}=\chi^{(5)}=\ldots$ and $\chi^{(2)}=\chi^{(4)}=\chi^{(6)}=\ldots$.

12. In general this will not be the case, and we must prove the process a convergent one. Let $\chi^{(n+2)}-\chi^{(n)}=\omega^{(n)}$; then we need to show that, if M is a superior limit to the absolute value of $\omega^{(1)}$, a superior limit θM can be set to the absolute value of $\omega^{(3)}$, θ being a proper fraction. It is easily seen that this is not true in the excluded case when $e=1$; for then, if $\omega^{(1)}$ has the value M at a point of the branch line, $\omega^{(2)}$, $\omega^{(3)}$, ... will all have the same value at that point and and we shall have $\theta=1$.

In the region (4123), $\omega^{(2n)}$ has no singularities; it vanishes on 4123 and is equal to $e\omega^{(2n-1)}$ on D'_3, to $\omega^{(2n-1)}$ on D'_4. Hence we have, at a point P in (4123),

$$\omega^{(2n)}(P)=e\int_{D'_3}\omega^{(2n-1)}(A_3)\,F_3(A_3,P)\,da_3+\int_{D'_4}\omega^{(2n-1)}(A_4)\,F_4(A_4,P)\,da_4$$

where A_3, A_4 are points on D'_3, D'_4; da_3, da_4 the elements of area at A_3, A_4; and F_3, F_4 are definite functions whose determination is a part of Dirichlet's problem for (4123).

In the same way, in (2341),

$$\omega^{(2n+1)}(P)=\int_{D'_1}\omega^{(2n)}(A_1)\,F_1(A_1,P)\,da_1+e^{-1}\int_{D'_2}\omega^{(2n)}(A_2)\,F_2(A_2,P)\,da_2.$$

Hence

$$\begin{aligned}\omega^{(2n+1)}(P)=e\int_{D'_1}\int_{D'_3}&\omega^{(2n-1)}(A_3)\,F_3(A_3,A_1)\,F_1(A_1,P)\,da_3\,da_1\\&+\int_{D'_1}\int_{D'_4}\omega^{(2n-1)}(A_4)\,F_4(A_4,A_1)\,F_1(A_1,P)\,da_4\,da_1\\&+\int_{D'_2}\int_{D'_3}\omega^{(2n-1)}(A_3)\,F_3(A_3,A_2)\,F_2(A_2,P)\,da_3\,da_2\\&+e^{-1}\int_{D'_2}\int_{D'_4}\omega^{(2n-1)}(A_4)\,F_4(A_4,A_2)\,F_2(A_2,P)\,da_4\,da_2\end{aligned}$$

$$\begin{aligned} = \int_{D_3'} \omega^{(2n-1)}(A_3) \Big\{ e \int_{D_1'} & F_3(A_3, A_1) F_1(A_1, P)\, da_1 \\ & + \int_{D_2'} F_3(A_3, A_2) F_2(A_2, P)\, da_2 \Big\}\, da_3 \\ + \int_{D_4'} \omega^{(2n-1)}(A_4) \Big\{ \int_{D_1'} & F_4(A_4, A_1) F_1(A_1, P)\, da_1 \\ & + e^{-1} \int_{D_2'} F_4(A_4, A_2) F_2(A_2, P)\, da_2 \Big\}\, da_4. \end{aligned}$$

Here it is to be noticed that (*a*) the values of F_1, F_2, F_3, F_4 are always positive; (*b*), if 1 is substituted for $\omega^{(2n-1)}$ and also for e, the resulting expression cannot exceed 1 in absolute value. These are consequences of the theorem that the potential cannot be a maximum or minimum in empty space.

Let us consider the expressions in $\{\ \}$ and put

$$\lambda = \int_{D_1'} F_3(A_3, A_1) F_1(A_1, P)\, da_1, \qquad \mu = \int_{D_2'} F_3(A_3, A_2) F_2(A_2, P)\, da_2.$$

Since λ, μ are both positive and $|e| \neq 1$, $|e\lambda + \mu| < \lambda + \mu$, unless λ/μ vanishes or is infinite.

Now λ, as a function of P, is a potential without singularity in (2341), equal to $F_3(A_3, P)$ on D_1' and to zero on D_2' and 2341. Also $F_3(A_3, P)$ is a potential whose only singularity in (4123) is a positive infinite value at A_3 and which vanishes over the boundary of (4123). Hence λ cannot vanish or be infinite within the region (2341). The same applies to μ. On the boundary of (2341) this argument fails, but we are only concerned with the values on D_3' and D_4'. As P approaches 2341 on either D_3' or D_4', λ, μ both tend to zero, but their ratio will be finite. If the doors are inclined at finite angles along the branch line, then λ, μ will tend to zero, but in a finite ratio, when P approaches the branch line on either D_3' or D_4', unless A_3 is infinitely near the branch line, in which case λ, μ become infinite, but still in a finite ratio.

Hence λ/μ does not become either zero or infinite on D_3' or D_4'. Let κ be a superior limit to λ/μ and μ/λ. Then $|\lambda + e\mu|/(\lambda + \mu)$ cannot exceed $|1 + e\kappa|/(1 + \kappa)$, a quantity which < 1. Let θ denote this quantity or the superior limit, also < 1, of

$$\frac{\left| \left\{ \int_{D_1'} F_4(A_4, A_1) F_1(A_1, P)\, da_1 + e^{-1} \int_{D_2'} F_4(A_4, A_2) F_2(A_2, P)\, da_2 \right\} \right|}{\int_{D_1'} F_4(A_4, A_1) F_1(A_1, P)\, da_1 + \int_{D_2'} F_4(A_4, A_2) F_2(A_2, P)\, da_2}$$

for positions of P upon D_3', D_4', whichever is the greater.

Then, if M is a superior limit to $\omega^{(1)}$ upon D_3' and D_4, $M\theta$ is a superior limit to $\omega^{(3)}$, $M\theta^2$ to $\omega^{(5)}$, and so on; and generally, upon D_3' and D_4', $\omega^{(2n+1)}$ cannot exceed $M\theta^n$. Thus $\omega^{(2n+1)}$ diminishes without limit as n increases. The rest of the proof goes on as before.

[By combining the volumes (2341) and (4123) in the ordinary way* we could prove the possibility of solving Dirichlet's problem for the region (234123), which contains the whole volume of the tube and overlaps itself to the extent of (23). The same method would apply if the region were not tubular, and, in fact, to a region containing the whole or part of the space in one colour and adjoining spaces in another colour. By reducing to zero the spaces in the other colour, we should come to the case of a region bounded in part or wholly by open shells. In such a region the solution might, in fact, be found by Poincaré's clearance method: the only point at which his original proof fails is where a sphere has to be described touching the bounding surface and lying wholly outside the region.† This sphere may be replaced by any surface for the outside of which it has been proved that Dirichlet's problem has a solution. In this case let A be the shell, P a point on it, A' a shell which is part of A and contains P: then A' may take the place of the sphere in the passage mentioned, since it has just been shown that the region outside A' is *balayable*.]

13. At a point on the branch line $\chi^{(n)}$ may have different values according to the direction in which the point is approached; let g_n be the greatest value of $|\chi^{(n)}|$ at this point, a value which must occur on one of the two bounding surfaces. The values of $\chi^{(n+1)}$ here depend on those of $\chi^{(n)}$ at the point in a way that only involves the angles at which D_1', D_2', D_3', D_4' are inclined, and we have $g_{2n+1} \not> \theta g_{2n-1}$.

Hence, on the branch line, g_n, and therefore $\chi^{(n)}$, tends to zero as n increases.

14. When the proposed singularities of $\phi_1, \phi_2, \ldots, \phi_m$ include a simple pole at a point of the branch line, the residue must be the same for each of the m potentials, and the pole therefore disappears from the function

$$e^{m-1}\phi_1 + e^{m-2}\phi_2 + \cdots + \phi_m,$$

unless $e = 1$. Hence, after forming a set with the given boundary values and the other singularities, we need only add to each of the set a potential one-valued in the tube, vanishing on its surface and having the pole in question with the assigned residue.

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* As, for instance, at § 222 of Forsyth's *Theory of Functions* (1893).

† *American Journal*, Vol. XII., pp. 226, 234.

15. The problem is thus solved for a simple tube containing a single branch line, and each of the branch lines B in the many-coloured region may be thus treated. The surface T_2 encloses not only these tubes, but also the connecting lines C. We may enclose the lines C in separate tubes with ends, so that every point enclosed by T_2 lies either in a B tube or in a C tube, or in both, and so that each C tube has space in common with each of the B tubes which it connects. The C tubes are then ordinary volumes in one-coloured space, and the ordinary methods of solving Dirichlet's problem apply. The C tubes may be combined with the B tubes by the alternating method so as to give the solution for the whole many-coloured space enclosed by T_2. Then this region may be combined by the alternating method with that outside T_1, and the proof is completed. It is necessary in each case to show that the factor corresponding to q in Art. 9 is less than 1; this will be so in the first case if the two sets of tubes cut everywhere at a finite angle, as they will if, for instance, the C tubes are of one constant radius and the B tubes of another.

When the inside of T_2 is combined with the outside of T_1, we argue as follows, supposing the point infinity to belong to the outside of T_1 and not to the inside of T_2:—A potential not exceeding 1 on the surface T_2, and without singularity in the space enclosed by T_2, cannot exceed 1 on the surface T_1 which lies within T_2. A potential not exceeding 1 on the surface T_1, and without singularity outside it, must vanish at infinity, and hence on T_2 will have a superior limit less than 1. The fractional factor exists then in this case, although it would not at the corresponding step if we were dealing with the logarithmic potential in space of two dimensions.

16. The boundary has now disappeared, and we have proved the existence of a potential, one-valued in the given region, and having only assigned singularities, that is, the singularities of given potentials at specified points in specified colours.

17. The simplest singularity is a pole, and we may give the name "normal elementary potential" to a potential having no singularity in the region except that of $1/PQ$ at a point Q in a certain colour. Let $\omega(P, Q)$ denote this normal elementary potential: we wish to prove that $\omega(P, Q) = \omega(Q, P)$.

Take any two points Q, R away from the branch lines; let $\phi = \omega(P, Q)$, $\psi = \omega(P, R)$ and use the equation

$$\int\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dS = \iiint(\phi\Delta\,\psi-\psi\,\Delta\phi)\,dx\,dy\,dz.$$

Describe small spheres about the points Q, R and tubes of small section enclosing the branch lines, also a sphere of great radius enclosing these tubes and the small spheres. Then the range of the triple integration is to be the volume of the great sphere less those of the tubes and small spheres. Since ϕ, ψ are not one-valued in this volume, let a set of doors be drawn as before to separate the different colours and let Σ denote a summation over the different colours.

The volume integral vanishes, since $\Delta\phi = 0 = \Delta\psi$ throughout, while ϕ, ψ are finite. Hence

$$\Sigma\int\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dS = 0,$$

the integration being over the surfaces above mentioned. Since the sets of values of ϕ, ψ are the same, except for order, at adjacent points on the two sides of a door, the surface integrals over the two sides of the door cancel each other.

Since ϕ, ψ have no singularity outside the great sphere in any colour, the surface integrals over this sphere vanish. Similarly, the surface integrals vanish for the spheres about Q, R in the other colours, and for the tubes in colours not affected by the corresponding branch lines.

The value of the integral for the sphere about Q, the pole of ϕ, is $-4\pi\psi(Q)$ or $-4\pi\omega(Q, R)$, and, similarly, it is $4\pi\omega(R, Q)$ for the sphere about R. Thus $\omega(Q, R) = \omega(R, Q)$ if the surface integral over the tubes enclosing branch lines in the colours affected vanishes.

18. Take one of these tubes by itself. Suppose, as before, that it affects m colours, the m values of ϕ or ψ being distinguished by suffixes, and let e denote one of the m-th roots of unity. Also let us write (ϕ, ψ) for the value over the tubular surface of $\int\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dS$. Then

$$m\Sigma(\phi, \psi)$$
$$= \Sigma(\phi_1+e\phi_2+e^2\phi_3+\dots+e^{m-1}\phi_m,\ \ \psi_1+e^{-1}\psi_2+e^{-2}\psi_3+\dots+e^{-m+1}\psi_m)$$

where the summation on the right is for the m different values of e.

Since $\phi_1+\phi_2+\dots+\phi_m$, $\psi_1+\dots+\psi_m$ are potentials having no singularity within the tube, zero is the value of the term in which $e = 1$.

Also $\phi_1, \phi_2, \ldots \phi_m$ tend to equality as the branch line is approached, and the same is true of $\psi_1, \psi_2, \ldots, \psi_m$. Hence, for other values of e, $\phi_1+e\phi_2+\ldots+e^{m-1}\phi_m$ and $\psi_1+e^{-1}\psi_2+\ldots+e^{-m+1}\psi_m$ tend to zero; as, however, their derivatives may not be assumed to have finite values at points on the branch line, we cannot at once infer that

$$(\phi_1+e\phi_2+\ldots+e^{m-1}\phi_m,\ \psi_1+e^{-1}\psi_2+\ldots+e^{-m+1}\psi_m)$$

vanishes. The difficulty may be overcome as follows.

19. In Art. 15 we combined the B tubes with the C tubes, and then the region thus formed with that bounded by T_1 and outside it. We might, however, first combine this region bounded by T_1 with C tubes, so as to solve the problem for the region outside a set of tubes B_1, each of which encloses a branch line, but lies altogether within the corresponding B tube. Then the solution could be completed by combining the volume outside the tubes B_1 with that inside the B tubes. It is also possible to form a set of potentials for the region T_1, such that the mean of the values in the colours affected by a branch line shall have no singularity within the corresponding B tube other than those assigned. This statement will apply to values in a colour unaffected by the branch line on the understanding that such a value is permuted with itself, and therefore is itself the mean of the permuted values. Such a set of potentials is determined by the assigned singularities, and by the differences of the permuted values on the surface of the tubes B_1.

To show this, let sets of potentials $\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \ldots$ be constructed, as follows, each having the assigned singularities as far as they affect its region:—$\chi^{(2n)}$ is constructed without branching for the inside of the tubes B; it has on the surface of any tube the mean of the permuted values of $\chi^{(2n-1)}$; thus $\chi^{(2n)}$ means a separate set of functions for each tube, and the set for any tube contains as many different functions as there are cycles of colours interchanged by the enclosed branch line. $\chi^{(2n+1)}$ is constructed for the outside of the tubes B_1, and on the surface of a tube the differences of the permuted values are to be as assigned, while the mean of the permuted values is to be equal to $\chi^{(2n)}$; the permutations referred to are those which characterize the region as originally given: the mean of the permuted values of $\chi^{(1)}$ on any tube may be taken in any way that is convenient.

Writing again $\omega^{(n)}$ for $\chi^{(n+2)}-\chi^{(n)}$, we see that the sets ω are formed successively in the same way as the sets χ, there being no assigned singularities and the assigned differences being all zero. Let q be a proper fraction, such that a potential equal to 1 on the tubes B_1, and to zero at infinity, does not exceed q on the tubes B. Then, if M is the greatest absolute value of $\omega^{(1)}$, qM is a superior limit to $\omega^{(2)}$ and also to $\omega^{(3)}$, q^2M to $\omega^{(4)}$ and $\omega^{(5)}$, q^nM to $\omega^{(2n)}$ and $\omega^{(2n+1)}$. The existence of a fractional factor having been established, the convergency of the process follows and the rest of the proof follows the usual course.

20. If there are no assigned singularities and the assigned differences on B_1 all vanish, it follows from the above that the potentials χ constructed by this process will vanish. Further, when there are no singularities, let M be the greatest value of any of the functions $\chi^{(2\infty+1)}$, which will necessarily occur on B_1. On the tube B the functions cannot exceed qM, and therefore the mean of the permuted values cannot exceed qM on this tube or on B_1. Hence one of the functions has the value M, and at least one other must be less than qM; so that the differences rise to at least $(1-q)M$. If, then, the assigned differences do not exceed ϵ, the functions can nowhere exceed $\epsilon/(1-q)$.

Let $\Omega(P, Q)$ denote the value at P of a potential constructed thus, the only singularity being a simple pole at Q, an assigned point in an assigned colour, and the differences on B_1 being zero. Then $\omega(P, Q)-\Omega(P, Q)$ is a similar potential, but it has no singularity, and its differences on B_1 are those of $\omega(P, Q)$, which can be made as small as we please by taking the tubes B_1 thin enough. Hence $\Omega(P, Q)$ can be made as near as we please to $\omega(P, Q)$ by taking the tubes B_1 thin enough.

21. If, then, we can prove $\Omega(P, Q) = \Omega(Q, P)$, it will follow that $\omega(P, Q) = \omega(Q, P)$. As before, we find that we must prove

$$\int\left\{\Omega(P, Q)\frac{\partial}{\partial n}\Omega(P, R)-\Omega(P, R)\frac{\partial}{\partial n}\Omega(P, Q)\right\}dS = 0,$$

for a tubular surface enclosing B_1, and that this will be true if $\frac{\partial}{\partial n}\Omega(P, Q)$ and $\frac{\partial}{\partial n}\Omega(P, R)$ do not exceed finite limits, when the surface of integration tends to coincide with B_1.

To prove that the values on the boundary B_1 are finite, describe a sphere, radius a, centre K, within the tube B_1, touching it only at one point A. Take a tube B', enclosing B_1, but so that the space between them does not include Q or R: this will be possible unless Q or R lies on B_1, and B_1 may be so chosen as not to pass through either point. Let f be the least distance from K to any point of B', and M the greatest value of ξ, the difference between two of the permuted values of Ω, upon

B', which is supposed clear of the other tubes. Then

$$\frac{M(1-a/KP)}{1-a/f} \pm \xi$$

are two potentials without singularity in the region between B_1 and B', and neither of them is negative at any point of B_1 or B'. Hence at the point A, where they vanish, they must have a tendency to increase along the outward normal to B_1: that is, $\partial\xi/\partial n$, at A, is numerically less than $M/(a-a^2/f)$, a finite quantity.

Now the force in a Newtonian field cannot be a maximum in empty space: it has been proved finite all over B_1, and it is finite over B', which lies within the region where there is no singularity; therefore it is finite, that is, it has a finite superior limit, throughout the region between B' and B_1.

This completes the proof that $\omega(P, Q) = \omega(Q, P)$.

22. The theorem needs special examination when one of the poles lies upon a branch line. There is then no need to describe a sphere about it, as it is excluded by one of the tubes. Suppose ϕ to have the pole on the branch line; then the results are as before, except for the term

$$\frac{1}{m}\int\left\{(\phi_1+\ldots+\phi_m)\frac{\partial}{\partial n}(\psi_1+\ldots+\psi_m)-(\psi_1+\ldots+\psi_m)\frac{\partial}{\partial n}(\phi_1+\ldots+\phi_m)\right\}dS$$

for the tube containing the pole. The value of this is m times the residue of the mean of $\phi_1, \phi_2, \ldots, \phi_m$ multiplied by the mean of the values of ψ at the pole, which are, of course, equal. Hence, when Q is on a branch line which affects a cycle of m colours, the normal elementary function $\omega(P, Q)$ is that which has the singularity of $1/mPQ$, and on this hypothesis we have $\omega(P, Q) = \omega(Q, P)$, in all positions of P, Q.

23. A potential one-valued in the region and without singularity must vanish everywhere. This may be proved in different ways, as, for instance, by Green's theorem applied to this potential and to $\omega(P, Q)$.

From $\omega(P, Q)$ other potentials may be formed by differentiation and integration, having more complicated singularities: the potential is completely defined by its singularities, since, if two one-valued potentials had the same, their difference would be a potential one-valued in the region and without singularity; it would therefore be zero.

24. Somewhat as in Arts. 19–21, we may prove that, if F is a closed surface enclosing a branch line, and ϕ, ψ are two potentials with no

singularity inside F except those of the region, then $\Sigma \int \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS$ taken over a tube B, lying within F and enclosing the branch line, is zero. For we may construct as above two potentials ϕ_1, ψ_1 equal to ϕ, ψ on F and such that the mean value of ϕ_1 or ψ_1 has no singularity within F, while the differences between the values in the different colours vanish on the surface of a tube B_1 enclosed in B but enclosing the branch line. Then $\phi - \phi_1$, $\psi - \psi_1$ vanish on F, and their means have no singularity within F, while the differences between the values in the different colours on B_1 are those of ϕ, ψ which tend to zero as the tube B_1 is made thinner. A potential which vanishes on F and has only the singularity of the region in the space between F and B_1, whose mean has no singularity in the whole space inside F, while its different values at any point of B_1 do not differ by more than 1, cannot exceed a certain quantity, say κ, on B, and none of its first derivatives can exceed, on B, a certain quantity λ. These quantities κ, λ depend on the surface B_1, as well as on B, but do not increase when B_1 is made thinner. Hence, if on B_1 the numerical values of the differences of $\phi - \phi_1$, $\psi - \psi_1$ have a superior limit ϵ, those of ϕ, ϕ_1, ψ, ψ_1 a superior limit M, and those of the derivatives of ϕ, ψ on B a superior limit N, the difference between

$$\int_B \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS \quad \text{and} \quad \int_B \left(\phi_1 \frac{\partial \psi_1}{\partial n} - \psi_1 \frac{\partial \phi_1}{\partial n}\right) dS,$$

$$\text{or } \int_B \left\{ (\phi - \phi_1) \frac{\partial \psi}{\partial n} - (\psi - \psi_1) \frac{\partial \phi}{\partial n} + \phi_1 \frac{\partial}{\partial n} (\psi - \psi_1) - \psi_1 \frac{\partial}{\partial n} (\phi - \phi_1) \right\} dS,$$

cannot exceed $2\epsilon\beta(M\lambda + N\kappa)$, where β is the area of B. This expression diminishes indefinitely with ϵ, that is, as B_1 shrinks.

But $$\int_B \left(\phi_1 \frac{\partial \psi_1}{\partial n} - \psi_1 \frac{\partial \phi_1}{\partial n}\right) dS = 0,$$

since by our former reasoning this is its value for B_1. Hence we must have $$\int_B \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS = 0.$$

The modification in this result caused by poles of ϕ or ψ within the region gives no trouble.

25. The foregoing methods enable us to prove the existence of another class of potentials, namely, those which are increased by a definite quantity when a closed irreducible circuit is described. This quantity may be constant, in which case the functions seem analogous to those associated with a Riemann surface: such functions do not, however, take

the same place in this theory, since a constant is now a potential, not of the simplest kind. The increment may, on the other hand, be variable, in which case it is clear that it must be a potential; in the simplest cases it will be one-valued in the region.

Taking account of the m colours, we should have a set of m potentials ϕ, and a closed circuit piercing the doors D_1, D_2, D_3, D_4 in this order would change them into $S\phi+\lambda$, where S, as before, denotes a certain permutation, and λ a set of known determinate potentials, which will be zero for circuits round one particular branch line.

The process for the region R is the same as at Art. 8, except that $\chi^{(2n+1)}$ has the value $\chi^{(2n)}$ on D_1', and $S^{-1}(\chi^{(2n)}-\lambda)$ on D_2', $\chi^{(2n+2)}$ has the value $S\chi^{(2n+1)}+\lambda$ on D_3' and $\chi^{(2n+1)}$ on D_4': in assigning the singularities of $\chi^{(2n+1)}$ and $\chi^{(2n+2)}$, those of λ must be taken into account. The course of the proof is the same, the additive singularity disappearing from $\omega^{(n)}$.

The tubular region enclosed by T_2 may be treated similarly. In the B tubes (many-coloured), or in the C tubes, the potential ϕ is one-valued, but, if we call it $\phi^{(1)}$ in the B tubes and $\phi^{(2)}$ in the C tubes, we have assigned values, say λ, which do not all vanish, for $\phi^{(1)}-\phi^{(2)}$ in the parts common to the two sets of tubes. In each of the parts λ is the value of a known potential, but in two unconnected parts it may be two different potentials.

The problem is solved for the combined region by forming $\chi^{(1)}, \chi^{(3)}, \chi^{(5)}, \ldots$ for the B tubes, and $\chi^{(2)}, \chi^{(4)}, \ldots$ for the C tubes, so that the singularities and values on the surface T_2 are those prescribed for ϕ, while, on the part of the boundary of C which lies within B, we have $\chi^{(2n)} = \chi^{(2n-1)}-\lambda$, and, on the part of the boundary of B which lies within C, $\chi^{(2n+1)} = \chi^{(2n)}+\lambda$.

The success of this method is proved as before, and the combination of the tubular region with R is effected in the usual way.

26. More generally, we might investigate a set of matricial potentials, that is, a set which is linearly transformed by passage round a closed line circuit in the region. About each group of interlacing branch lines describe two simple closed surfaces F, F' in such a way that F encloses F', and F' the branch line, and that all the surfaces are clear of each other and of the branch lines.

The surfaces F enclose many-coloured regions, say R_1, for each of which Dirichlet's problem may be solved by the methods of Arts. 7–16, with a slight modification caused by the presence of a boundary.

For the space outside F', in any colour, Dirichlet's problem may also be solved. Call these spaces collectively R_2. In each of the spaces R_2 the proposed functions will be uniform, since there are no irreducible line circuits: in the several regions R_1 we shall take them to be also uniform, so that, if there are irreducible line circuits in any of the regions R_1, the matrices of the corresponding transformations are all unity; this simplifies the investigation, but there is some loss of generality. The functions will have certain prescribed singularities, such as poles, of the ordinary kind.

27. Let ϕ denote the proposed functions, and distinguish them as $\phi^{(1)}$ in R_1 and as $\phi^{(2)}$ in R_2. Then the values of $\phi^{(2)}$ in the region between F and F' are those of $M\phi^{(1)}$, where M is a certain matrix, constant in any connected part of this region, but not the same in different parts.

Form a set of potentials $\chi^{(1)}$ for R_1, having the singularities prescribed for $\phi^{(1)}$ and any convenient boundary values. Since R_1 consists of detached parts, such a set is to be formed in each of the parts.

Form a set of potentials $\chi^{(2)}$ for R_2, having the singularities of $\phi^{(2)}$ and boundary values $M\chi^{(1)}$.

Form a set $\chi^{(3)}$ for R_1, with the singularities of $\phi^{(1)}$ and boundary values $M^{-1}\chi^{(2)}$, and so on.

This process may converge, and then it will give the solution. If it does not converge, the problem may be treated by the method of *Camb. Phil. Trans.*, Vol. XIX., Part II., p. 224; it will be well to take the surfaces F, F' as analytical, so that the points of either have a (1, 1) correspondence, everywhere analytical, with the points of a spherical surface.

The matrices corresponding to the independent irreducible circuits may be prescribed arbitrarily.

28. Let η, ζ be two contragredient sets of matricial potentials, η having a pole at Q with residues λ, and ζ a pole at R with residues μ. Take $\Sigma\int\left(\eta\frac{\partial\zeta}{\partial n}-\zeta\frac{\partial\eta}{\partial n}\right)dS$ over the same surface as at Art. 17. The doors and tubes again give zero and the spheres contribute

$$4\pi\{\Sigma\mu\eta(R)-\Sigma\lambda\zeta(Q)\}.$$

Hence $$\Sigma\lambda\zeta(Q) = \Sigma\mu\eta(R).$$

As a rule the residues λ, μ may be chosen arbitrarily, so that this equation is equivalent to p^2 equations, where p is the number of rows in each matrix. But, if it happens that a set of potentials θ, with the same matrices as ζ, has no poles, we may put θ for ζ in the equation and deduce $\Sigma\lambda\theta(Q) = 0$, so that the residues λ are restricted by this condition. Such a restriction can only arise* when a set of potentials without poles, and with the same matrices as η, exists. Thus the number of such sets cogredient with η is the same as the number cogredient with ζ.

29. In the theory of the logarithmic potential, after constructing a normal elementary potential one would go on to form the conjugate by integrating, along an arc, the derivative in the direction of the normal. The analogous process in three dimensions seems, at first, to be the surface integration of the derivative in the direction of the normal. This, however, gives a result that is not invariant under inversion. It is more satisfactory to take two potentials ϕ, ψ and form from them the expression $\int\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS$; this seems to be the nearest analogue to a conjugate potential, but it is not at all of the same type.

Let O be the centre of inversion; P, P' inverse points;

$$r = OP, \quad r' = OP', \quad \phi' = (r/r')^{\frac{1}{2}}\phi, \quad \psi' = (r/r')^{\frac{1}{2}}\psi;$$

then ϕ', ψ', as functions of the point P', are potentials. We have also

$$dS' : dS = r'^2 : r^2, \qquad dn' : dn = r' : r.$$

The expression corresponding to $\int\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS$ is

$$\int\left(\phi'\frac{\partial\psi'}{\partial n'} - \psi'\frac{\partial\phi'}{\partial n'}\right)dS',$$

which has the same value; so that this double integral is unaffected by inversion.

30. The simplest case is that in which $\phi = \omega(P, Q)$, $\psi = \omega(P, R)$. Let $\Omega(Q, R, L)$ denote the integral in this case, L standing for the closed curve that bounds the surface of integration†: it follows from

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[*] Compare *Camb. Phil. Trans.*, Vol. XIX., Part II., p. 229, § 42.

[†] The logarithmic and Newtonian theories here may be compared as follows:—In three (two) dimensions take ϕ, ψ to be any potentials with poles (logarithmic points) at Q, R and form the expression $\int\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS$. In three dimensions this will be a multiple of $\Omega(Q, R, L)$. In two dimensions the coefficients of the logarithms at Q, R must be equal and opposite, in both ϕ and ψ, and the expression is therefore a multiple of $\int\frac{\partial\phi}{\partial n}dS$ or $\int\frac{\partial\psi}{\partial n}dS$ simply, since some expression of the form $A\phi + B\psi$ has a constant value.

Green's theorem that $\Omega(Q, R, L)$ is unchanged by continuous deformation of the surface of integration so long as this does not pass over one of the points Q, R.

If we invert from any point O, taking account of the parametric points Q, R, which invert into Q', R', we have

$$\omega'(P', Q') = \frac{OP}{OQ'}\,\omega(P, Q) = \frac{OQ}{OP'}\,\omega(P, Q),$$

$$\Omega'(Q', R', L') = \frac{OQ}{OR'}\,\Omega(Q, R, L) = \frac{OR}{OQ'}\,\Omega(Q, R, L).$$

Thus $\Omega(Q, R, L)$, $\omega(Q, R)$, and $1/QR$ are all changed in the same ratio.

$\Omega(Q, R, L)$ is a potential both for Q and R.

In ordinary one-coloured space $\omega(Q, R)$ is simply $1/QR$, while $\Omega(Q, R, L)$ is $1/QR$ multiplied by the solid angle formed at Q or R by a variable circle which always passes through Q and R and meets L.

Some convention must be made as to the sign of $\Omega(Q, R, L)$. Let the sense in which L is described be fixed, and suppose a man to walk along L in this sense so as to have the area of integration on his left; then dn is to be taken towards the positive side, that is, in the direction from his feet to his head. If Q_+ and Q_- are adjacent points on the positive and negative sides,

$$\Omega(Q_+, R, L) - \Omega(Q_-, R, L) = -4\pi\omega(Q, R),$$

$$\Omega(Q, R_+, L) - \Omega(Q, R_-, L) = 4\pi\omega(Q, R).$$

These equations show the effect of deforming the surface of integration so as to pass over Q or R. Ω is not affected when Q or R describes any closed path that does not interlace with L. It is changed by a quantity depending on Q, R, but not on L, when the area of integration is changed into a second having the same boundary, but not deformable into the first. The effect of interchanging Q, R or of reversing L is to change the sign of $\Omega(Q, R, L)$.

Two areas having the same boundary, but not deformable into each other, form together a cycle of two dimensions, and the integrals taken over them differ by the integral taken over this cycle. Let $\Omega_1(Q, R)$, $\Omega_2(Q, R)$, ... denote the values of the integral for the different cycles; then each of them is a potential for Q and also for R. The sign is changed in each case by the interchange of Q and R, and, if Q passes round a closed circuit, there is no effect if it does not pass through the surface of integration; every such passage causes a change to the amount of $\pm 4\pi\omega(Q, R)$ according to the direction in which it takes place. It is assumed that the cycle is a bifacial surface. If it were unifacial, the integral over it would vanish identically, since each element of the surface would be taken twice, in opposite senses.

31. In studying the topological properties of a many-coloured space it is a help to make use of certain deformations. Ordinary deformation of a surface consists in transferring from one side of it to the other a volume, part of whose boundary lies in the surface to be deformed. This volume may for our purposes include the point at infinity; so that, for instance, two spherical surfaces in one colour are equivalent, even when each is wholly outside the other, if every singularity falls within one or the other.

Suppose the complete set of doors to be drawn. The colours are then separated and the space is thus divided into parts, each of which has a boundary consisting of one or more closed surfaces; for the two sides of any door together form a closed surface. These surfaces may be swollen out, and any one of them may, if we wish, be deformed over the point at infinity. Thus the space of any colour is deformed into that filled by a body with a number of hollows inside it. The different colours border on each other at the bounding surfaces of these bodies, whether outer or inner. A space topologically equivalent to the original may be formed by joining together the different bodies, taking a part of the boundary of one and applying it to the corresponding part of the boundary of another. Some of the bounding surfaces may thus disappear altogether; in other cases two or more of them may coalesce into one: the whole space is thus deformed into that filled by a body with hollows, and a point leaving the space by passing through the boundary in any part enters again at the corresponding point elsewhere on the boundary. The boundary thus consists of pairs of corresponding faces, and these faces may have bounding lines, or they may not, according as they border on other faces or not. Beginning with one face and adding to it all those which border on it, and all which border on these, and so on, we arrive at a kind of polyhedron; the whole boundary consists of such polyhedra and of faces without boundaries. If two corresponding faces F, F' belong to different polyhedra A, A', the number of polyhedra may be reduced. Enclose A in a surface B lying wholly within the region. Cut out the space between A and B, and apply the face F to F'; let B' be the new position of B. Then, instead of the two polyhedra A, A', we have two unbounded faces B, B' and a single polyhedron. Also, if two adjacent faces of a polyhedron correspond, in such a way that as a point approaches the dividing line it approaches the corresponding point, then these faces may be applied to each other and therefore cancelled as part of the boundary.

We are thus justified in supposing the boundary to consist of unbounded faces and of polyhedra each of which can be divided into pairs of corresponding faces. The latter do not occur unless there are interlacing branch lines, for, when there are none, the surface at each hollow is divided into two faces only by a branch line, and after every step in the deformation the only polyhedra left are still two-faced; none but unbounded faces are left in the end.

32. Either one of a pair of unbounded faces, corresponding to each other, forms a cycle of two dimensions, and a path joining corresponding points of the two faces forms a cycle of one dimension. It is thus clear that when the branch lines do not interlace there are as many independent cycles of one kind as of the other. I do not propose here to discuss fully the cycles of a region which has interlacing branch lines, but will take the simplest example, namely, that of a two-coloured region in which there are two branch lines interlacing once. Let these be B, B', the doors bounded by them D, D', intersecting in the line C. The two sides of either door added together form a quadrilateral, each of whose sides is originally in the position C. These two quadrilaterals form a dihedron, which separates the two colours: if we cancel the door D' so as to connect the colours, the boundary is reduced to the two faces of D; these form a dihedron, each face of which is again quadrilateral. Denote the four vertices by α, β, γ, δ: then the four sides correspond in pairs, namely, $\alpha\beta$ to $\gamma\delta$ (not $\delta\gamma$) and $\beta\gamma$ to $\delta\alpha$ (not $\alpha\delta$). If corresponding points are united, the whole may be stretched and flattened into two coincident circular sheets, connected round the common circumference and along one radius.

It is readily seen that this surface is unifacial. There is one irreducible closed circuit, namely, $\alpha\beta\gamma$ or $\gamma\delta\alpha$; but the double of this circuit, namely, $\alpha\beta\gamma\delta\alpha$, is reducible.

33. The determination of Green's function for the space outside a plane (or spherical) lamina can be made to depend directly on that of the function ω of Art. 17 for a certain region. Let us take a two-coloured space all whose branch lines lie in one plane, and divide that plane into two parts O and I, so that when a point passes through I it changes colour, but not when it passes through O. Let $\omega(P, Q)$ be the function for this

region, and take Q_i, Q_o the two images of Q by reflexion in the plane, Q_o in the same colour, Q_i in the other.

Then $\omega(P, Q)-\omega(P, Q_i)$ is a potential with poles at Q, Q_i only, and when P is on I it vanishes ; it is therefore Green's function for space as bounded by I. Similarly $\omega(P, Q)-\omega(P, Q_o)$ is Green's function for space as bounded by O.

At p. 190, Vol. xxxv., *Proc. London Math. Soc.*, the function ω was found in terms of Lamé's functions for the case of an elliptic branch line. This might be used in problems concerning elliptic laminæ or elliptic holes in infinite plane laminæ.

ON A CALCULUS OF POINT ASSEMBLAGES

By J. D. EVERETT (*communicated by the* SECRETARIES).

[Received February 8th, 1904.*]

1. The purpose of this paper is to illustrate by examples a convenient method of investigating the properties of systematic assemblages of points; special attention being given to systems formed by the centres of equal spheres in various modes of piling.

Reference may be made to—

1. An Article by Mr. Wm. Barlow on "The Probable Nature of the Symmetry of Crystals," *Nature*, Vol. xxix., p. 186 (1883).
2. Lord Kelvin on "The Molecular Constitution of Matter," *Collected Papers*, Vol. III., pp. 414-419 (1889).
3. Professor Osborne Reynolds, "Sub-Mechanics of the Universe," pp. 83, 84 (1903).

2. *Steps, and 3-Step Groups.* — I shall use Clifford's word "step" to denote passage from one point to another, regarded as a vector drawn from the first point to the second. If A is the first point, and B the second, the step can be called AB; but I shall usually specify it by writing its three rectangular components in order, with small spaces between. Commas can be interposed instead of spaces, if desired; but I prefer to dispense with them, and to write $a\ b\ c$, $a\ -b\ -c$ for Hamilton's $ia+jb+kc$, $ia-jb-kc$.

A homogeneous assemblage of points is one in which a step that can be taken from one point can be taken from every point. As every step AB implies the existence of a reverse step BA, it follows that every step that can be taken positively can also be taken negatively. A step that can be taken in a given homogeneous assemblage will be called a "step of the assemblage" and will be spoken of as being "contained" in the assemblage.

In a homogeneous assemblage of points 3 properly selected steps suffice for stepping from any point of the assemblage to any other. We shall call 3 such steps the *defining steps* of the assemblage. The selection of the defining steps can be made in an infinite variety of ways.

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* The two papers read in December, 1903, and January, 1904, respectively, entitled "Normal and Antinormal Piling," and "Various Systems of Piling," have been condensed and rearranged so as to form a single paper.

It is usually convenient to select the shortest steps consistent with the restriction that the 3 steps must not be coplanar. Two of the 3 defining steps give by addition and subtraction all the steps in their own plane.

Since a *homogeneous assemblage* may be defined as an *assemblage which has 3 independent steps,* it may conveniently be called a 3-*step group.* In modern crystallography it is called a *space lattice,* and is associated with a division of space into equal parallelepipeds, the edges of a parallelepiped being identical with our 3 independent steps.

3. *Tiers.*—The closest arrangement of a plane layer of equal spheres is that in which each sphere is touched by six, their seven centres being the centre and six corners of a regular hexagon. A number of equal spheres thus arranged will, in the present paper, be called a *tier;* and the name will be applied either to the spheres themselves or to their centres regarded as a plane group of points.

4. *Superposition of Tiers.*—In building up a pile of equal spheres, by laying tier upon tier, it is impossible to lay a sphere of the second tier over each gap between 3 spheres of the first, the gaps being too near together. There is a choice between two alternatives, as shown in Fig. 1, where the centres of the spheres of the first tier are marked 0, and the choice for the second tier lies between the gaps marked 1 and those marked 2. Each sphere 0 is surrounded by 6 gaps 60° apart, and we must select 3 which are 120° apart.

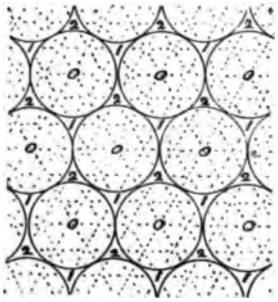

FIG. 1.

It is easy to verify, by several tests, that the operation of passing from the points 0 to the points 1 in the plane of the diagram is the same as that of passing from the points 1 to the points 2, and also the same as that of passing from the points 2 to the points 0. Hence the points 0, 1, 2, 0, are the projections of 4 successive tiers each derived in the same way from its predecessor, the projection of the fourth tier coinciding with that of the first. And we might equally well have adopted the cyclic order 0 2 1 0. Spheres arranged in either of these cyclic orders are said to form a *normal pile,* inasmuch as the structure thus obtained is that of all ordinary piles of spherical shot. The "tiers" (as defined in

Art. 3) in a pile of shot are horizontal if the pile has a triangular base, but are not horizontal if it has a square base.

All the systems of points discussed in this paper are composed of "tiers" in our special sense of the term.

5. *Choice of Axes of Coordinates.*—We shall employ axes of x, y, z equally inclined to the tier normal. The equation to the plane of any tier will therefore be $x+y+z = C$, and the constant C will have a uniform increment from tier to tier, since we shall always suppose the tiers to be equidistant.

6. *Normal Piling.*—In normal piling each sphere touches 6 in its own tier and 3 in each of the two adjacent tiers, making 12 in all.

Taking the centre of a sphere as origin, the centres of these 12 will be at the points

$$\begin{array}{cccccc} 0\;a\;a, & a\;0\;a, & a\;a\;0\,; & 0\;a\;-a, & a\;0\;-a, & a\;-a\;0, \\ 0\;-a\;-a, & -a\;0\;-a, & -a\;-a\;0\,; & 0\;-a\;a, & -a\;0\;a, & -a\;a\;0, \end{array}$$

each of which is at distance $a\sqrt{2}$ from the origin; so that $a\sqrt{2}$ is the diameter of a sphere.

The first 3 give $x+y+z = 2a$. The 3 written beneath them give $x+y+z = -2a$, and the six on the right hand give $x+y+z = 0$.

The 12 points are thus sorted into 3 tiers, and the increment of $x+y+z$ from tier to tier is $2a$. The four points $0\;0\;0$, $0\;a\;a$, $a\;0\;a$, $a\;a\;0$ are the corner points of a regular tetrahedron of edge $a\sqrt{2}$; and we select as our 3 defining steps the 3 vectors $0\;a\;a$, $a\;0\;a$, $a\;a\;0$, which lead from any point to the 3 nearest points in the next tier on one side. Reversed in sign, they lead to the three nearest points in the next tier on the other side. Their differences are the steps from any point to the 6 nearest points in its own tier, and may be called the *tier steps.* These tier steps $0\;a\;-a$, $a\;0\;-a$, $a\;-a\;0$, taken positively and negatively, will be the tier steps in every system discussed in the present paper.

6A. The origin being at a sphere centre, the sum $x+y+z$ may be any *even* multiple of a, and the values of x, y, z (as appears from the tier steps) may be any multiples of a subject to this restriction. In successive planes of points parallel to the plane of x, y the value of z will be alternately even and odd; the value of $x+y$ will therefore be alternately odd and even. Projecting two consecutive planes of points upon the plane of x, y, we shall obtain the pattern of Fig. 2; the dots being the projections of one plane of points, and the circles those of the other plane. In each plane the interval between consecutive points in a line parallel to x or y is $2a$, and in a line at 45° to either axis is $a\sqrt{2}$. Each point in one plane is also at the distance $a\sqrt{2}$ from 4 points in the other plane.

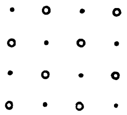

FIG. 2.

If space is partitioned into cubic cells of edge $2a$ by three sets of parallel planes, the points of a normal pile may be placed either (i.) at each cube corner, and the centre of each cube face; or (ii.) at each cube centre, and the mid-point of each cube edge.

The symmetry of the distribution shows that the division of a normal pile into tiers can be effected in 4 different ways, the tiers being parallel to any one of the four faces of the standard tetrahedron of Art. 6, or perpendicular to any one of the 4 cube diagonals.

7. *Antinormal Piling.*—Let two successive tiers of equal spheres be laid as in normal piling; and let the third be so laid that its spheres are vertically over those of the first. This can be effected by adopting either the cyclic order 0 1 0 1 ... or the cyclic order 0 2 0 2 ..., as is obvious from inspection of Fig. 1.

The increment of $x+y+z$ from any tier to the next but one is $4a$; hence the step from any sphere centre to the sphere centre vertically over it is $\frac{4a}{3}\ \frac{4a}{3}\ \frac{4a}{3}$; and by subtracting the steps $0\ a\ a$, $a\ 0\ a$, $a\ a\ 0$ from this, we obtain

$$\frac{4a}{3}\ \frac{a}{3}\ \frac{a}{3}, \quad \frac{a}{3}\ \frac{4a}{3}\ \frac{a}{3}, \quad \frac{a}{3}\ \frac{a}{3}\ \frac{4a}{3},$$

as the three shortest steps from the second of these 3 tiers to the third. The length of any one of them is $a\sqrt{2}$, and the sum of its three components $2a$, just as for the steps from the first tier to the second.

Inspection of Fig. 1 shows that, by rotating the axes of coordinates through 60° or through 180°, the expressions for the two sets of steps would be interchanged. Rotation through 120° is without effect.

7A. The whole system of sphere centres can be broken up into 2 homogeneous assemblages (or three-step groups) each having, as one of its defining steps, the step $\frac{4a}{3}\ \frac{4a}{3}\ \frac{4a}{3}$ normal to the tiers, and as its other two any two of the 3 tier steps $a\ -a\ 0$, ... One of the 2 assemblages consists of the odd tiers, and contains the following points.

In 1st tier $0\ 0\ 0,\quad a\ -a\ 0,\quad \ldots$

In 3rd tier $\frac{4a}{3}\ \frac{4a}{3}\ \frac{4a}{3},\quad \frac{a}{3}\ \frac{a}{3}\ \frac{10a}{3},\quad \ldots,$

the last being obtained by subtracting a from each of x and y, and adding $2a$ to z.

In 5th tier $\frac{8a}{3}\ \frac{8a}{3}\ \frac{8a}{3}.$

In 7th tier $4a\ 4a\ 4a,\quad 0\ 0\ 12a,\quad \ldots$

In the 3rd and 5th tiers all the coordinates will be fractional.

On lines parallel to the axes of coordinates the interval between consecutive points will be $12a$.

The other assemblage is composed of the even tiers; and, as one step from the 1st tier to the 2nd is $0\ 0\ 2a$, the interval of $12a$ will be broken up in the joint assemblage into alternate intervals of $2a$ and $10a$.

This statement remains true when the axes are rotated through 60°, so as to interchange the expressions for the two sets of steps from tier to tier. There are thus 6 directions, symmetrically distributed round the tier normal, in each of which the interval from point to point is alternately $2a$ and $10a$.

7 B. We are now in a position to make a complete chart of either of the two assemblages, plotted according to rectangular coordinates. See Fig. 3 (p. 442).

Employing squared paper, making a side of a square stand for $\frac{a}{3}$ in the plotting of x and y, and writing the value of z in figures, as a multiple of $\frac{a}{3}$, at each point thus plotted, all the points will lie at intersections of the ruled lines of the paper; and along every such line the pattern will repeat itself at intervals of $12a$, that is, of 36 chart units. In like manner the values of z will repeat themselves at intervals of 36, and the numbers to be entered will only run from 0 to 35.

The chart was made by starting from the central 0 with the horizontal tier step $3\ 0\ -3$ (that is, $a\ 0\ -a$), and the vertical tier step $0\ 3\ -3$, both carried to a distance slightly exceeding 18 in each direction. When all the points obtainable by these steps had been plotted, the chart was completed by using the step 1 1 10 $\left(\text{that is, } \frac{a}{3}\ \frac{a}{3}\ \frac{10a}{3}\right)$. As the two homogeneous assemblages are exactly alike, the chart may represent either.

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  33     30     27     24     21     18     15     12      9      6      3      0     33
23    20 · · 17 · · 14 · · 11 · ·  8 · ·  5 · ·  2 · · 35 · · 32 · · 29 · · 26 · · 23
    10 · ·  7 · ·  4 · ·  1 · · 34 · · 31 · · 28 · · 25 · · 22 · · 19 · · 16 · · 13     10
   0 · · 33 · · 30 · · 27 · · 24 · · 21 · · 18 · · 15 · · 12 · ·  9 · ·  6 · ·  3      0
26    23 · · 20 · · 17 · · 14 · · 11 · ·  8 · ·  5 · ·  2 · · 35 · · 32 · · 29 · · 26
    13 · · 10 · ·  7 · ·  4 · ·  1 · · 34 · · 31 · · 28 · · 25 · · 22 · · 19 · · 16     13
   3 · ·  0 · · 33 · · 30 · · 27 · · 24 · · 21 · · 18 · · 15 · · 12 · ·  9 · ·  6      3
29    26 · · 23 · · 20 · · 17 · · 14 · · 11 · ·  8 · ·  5 · ·  2 · · 35 . · 32 · · 29
    16 · · 13 · · 10 · ·  7 · ·  4 · ·  1 · · 34 · · 31 · · 28 · · 25 . . 22 · · 19     16
   6 · ·  3 · ·  0 · · 33 · · 30 · · 27 · · 24 · · 21 · · 18 · · 15 · · 12 · ·  9      6
32    29 · · 26 · · 23 · · 20 · · 17 · · 14 · · 11 · ·  8 · ·  5 · ·  2 · · 35 · · 32
    19 · · 16 · · 13 · · 10 · ·  7 · ·  4 · ·  1 · · 34 · · 31 · · 28 · · 25 · · 22     19
   9 · ·  6 · ·  3 · ·  0 · · 33 · · 30 · · 27 · · 24 · · 21 · · 18 · · 15 · · 12      9
35    32 · · 29 · · 26 · · 23 · · 20 · · 17 · · 14 · · 11 · ·  8 · ·  5 · ·  2 · · 35
    22 · · 19 · · 16 · · 13 · · 10 · ·  7 · ·  4 · ·  1 · · 34 · · 31 · · 28 · · 25     22
  12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33 · · 30 · · 27 · · 24 · · 21 · · 18 · · 15     12
 2    35 · · 32 · · 29 · · 26 · · 23 · · 20 · · 17 · · 14 · · 11 · ·  8 · ·  5 · ·  2
    25 · · 22 · · 19 · · 16 · · 13 · · 10 · ·  7 · ·  4 · ·  1 · · 34 · · 31 · · 28     25
  15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33 · · 30 · · 27 · · 24 · · 21 · · 18     15
 5     2 · · 35 · · 32 · · 29 · · 26 · · 23 · · 20 · · 17 · · 14 · · 11 · ·  8 · ·  5
    28 · · 25 · · 22 · · 19 · · 16 · · 13 · · 10 · ·  7 · ·  4 · ·  1 · · 34 · · 31     28
  18 · · 15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33 · · 30 · · 27 · · 24 · · 21     18
 8     5 · ·  2 · · 35 · · 32 · · 29 · · 26 · · 23 · · 20 · · 17 · · 14 · · 11 · ·  8
    31 · · 28 · · 25 · · 22 · · 19 · · 16 · · 13 · · 10 · ·  7 · ·  4 · ·  1 · · 34     31
  21 · · 18 · · 15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33 · · 30 · · 27 · · 24     21
11     8 · ·  5 · ·  2 · · 35 · · 32 · · 29 · · 26 · · 23 · · 20 · · 17 · · 14 · · 11
    34 · · 31 · · 28 · · 25 · · 22 · · 19 · · 16 · · 13 · · 10 · ·  7 · ·  4 · ·  1     34
  24 · · 21 · · 18 · · 15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33 · · 30 · · 27     24
14    11 · ·  8 · ·  5 · ·  2 · · 35 · · 32 · · 29 · · 26 · · 23 · · 20 · · 17 · · 14
     1 · · 34 · · 31 · · 28 · · 25 · · 22 · · 19 · · 16 · · 13 · · 10 · ·  7 · ·  4      1
  27 · · 24 · · 21 · · 18 · · 15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33 · · 30     27
17    14 · · 11 · ·  8 · ·  5 · ·  2 · · 35 · · 32 · · 29 · · 26 · · 23 · · 20 · · 17
     4 · ·  1 · · 34 · · 31 · · 28 · · 25 . · 22 · · 19 · · 16 · · 13 · · 10 · ·  7      4
  30 · · 27 · · 24 · · 21 · · 18 · · 15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0 · · 33     30
20    17 · · 14 · · 11 · ·  8 · ·  5 · ·  2 · · 35 · · 32 · · 29 · · 26 · · 23 · · 20
     7 · ·  4 · ·  1 · · 34 · · 31 · · 28 · · 25 · · 22 · · 19 · · 16 · · 13 · · 10      7
  33 · · 30 · · 27 · · 24 · · 21 · · 18 · · 15 · · 12 · ·  9 · ·  6 · ·  3 · ·  0     33
23    20     17     14     11      8      5      2     35     32     29     26     23
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FIG. 3.

The two assemblages cannot conveniently be shown together in one chart, because they would clash at every plotted point. For the same values of x, y there is a constant difference of 6 (that is, of $2a$) between their values of z.

In plotting a region in which the range of z is less than 6, the clashing will not occur. Fig. 4 is the plan of a region in which the values of z are 0, 1, 2, 3, the two assemblages being distinguished by different type. The two strips of the chart which contain only the entries 0 and 3 are exactly similar to normal piling. The intervening strip, containing the entries 1 and 2, is entirely different. If the chart were extended indefinitely left and right, the two kinds of strip would occur alternately.

The 12 steps of length $a\surd 2$ from each point in the joint assemblage to its nearest neighbours consist of 6 tier steps, which are the same for all tiers, and 3 steps to each adjoining tier, which are reversed in sign in passing from each tier to the next.

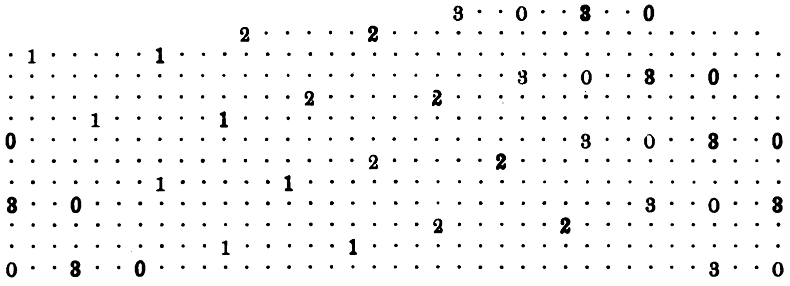

FIG. 4.

8. *Simple Cubic System.*—A system of special simplicity is that in which the defining steps are $a\ 0\ 0$, $0\ a\ 0$, $0\ 0\ a$. They are the shortest steps from tier to tier; and their differences (which are the steps in a tier) are the same as in normal piling.

Taking the origin at a point of the system, the coordinates of a point may be any multiples of a, without restriction. The increment of $x+y+z$ from tier to tier is therefore a; and the tiers are just half as far apart as in normal piling.

The asemblage can be split into two, one having $x+y+z$ an even, and the other an odd, multiple of a. As the shortest steps from any tier to the next but one are $0\ a\ a$, $a\ 0\ a$, $a\ a\ 0$, each of these two assemblages is a normal pile. The steps from any point of the simple cubic system to its six nearest neighbours are $\pm a\ 0\ 0$, $0\ \pm a\ 0$, $0\ 0\ \pm a$, their common length being a.

Equal spheres could be packed in a box on this system; and the space required would be $\surd 2$ times as great as in normal piling.

The tiers can be chosen in four different ways, just as in the case of normal piling.

9. *Anticubic System.*—If we lay two tiers as in the cubic system, and lay the third so that its points are the reflections of the points of the first tier with respect to the plane of the second, the step $\frac{2a}{3}\ \frac{2a}{3}\ \frac{2a}{3}$, which is normal to the tiers, will lead from any point of the first tier to the opposite point of the third. Subtracting from this the steps from the first to the second, which are $a\ 0\ 0$, $0\ a\ 0$, $0\ 0\ a$, we have the three

remainders $-\frac{a}{3}\ \frac{2a}{3}\ \frac{2a}{3},\ \frac{2a}{3}\ -\frac{a}{3}\ \frac{2a}{3},\ \frac{2a}{3}\ \frac{2a}{3}\ -\frac{a}{3}$; which are, accordingly, the three shortest steps from the second tier to the third. They have the same length (a) and the same sum of components (a) as the steps from the first to the second.

This system stands in the same relation to the simple cubic as antinormal to normal. It may, therefore, be called the "anti-cubic" system.

It consists of two homogeneous assemblages, each defined by the normal step $\frac{2a}{3}\ \frac{2a}{3}\ \frac{2a}{3}$ together with the usual tier steps $a\ -a\ 0\ \ldots$.

The normal step must be taken three times before fractions will disappear. We thus get the step $2a\ 2a\ 2a$, from which, by the tier steps, we deduce $6a\ 0\ 0$, ..., showing that, in each homogeneous assemblage, $6a$ is the interval between consecutive points on a line parallel to an axis of coordinates. Other steps, deducible from these, are

$$-\frac{a}{3}\ -\frac{a}{3}\ \frac{8a}{3},\quad \frac{a}{3}\ \frac{a}{3}\ \frac{10a}{3};$$

this last being identical with one of the steps in antinormal piling. We can pass from the first homogeneous assemblage to the second by the step $0\ 0\ a$.

10. *System of Cube Centres and Corners.*—In a stack of cubes of edge a the step $\frac{a}{2}\ \frac{a}{2}\ \frac{a}{2}$ leads halfway along a cube diagonal, and is therefore a step from a centre to a corner, and likewise a step from a corner to a centre. The same is true of each of the 8 steps included in the expression $\pm\frac{a}{2}\ \pm\frac{a}{2}\ \pm\frac{a}{2}$. The best selection of three defining steps is

$$-\frac{a}{2}\ \frac{a}{2}\ \frac{a}{2},\quad \frac{a}{2}\ -\frac{a}{2}\ \frac{a}{2},\quad \frac{a}{2}\ \frac{a}{2}\ -\frac{a}{2},$$

each giving $x+y+z=\frac{a}{2}$. They are the shortest steps from tier to tier. Their differences are the usual tier steps $a\ -a\ 0,\ \ldots$.

The tiers here employed are perpendicular to that cube diagonal which coincides with the step $\frac{a}{2}\ \frac{a}{2}\ \frac{a}{2}$. Hence, by symmetry, the assemblage can be divided into tiers perpendicular to any one of the four diagonals.

11. *Anti-Centre-Corner System.*—If we lay two tiers as in Art. 10, and make the third the reflection of the first by the plane of the second,

the three shortest steps from the second to the third will be

$$\frac{5a}{6}\;-\frac{a}{6}\;-\frac{a}{6},\qquad -\frac{a}{6}\;\frac{5a}{6}\;-\frac{a}{6},\qquad -\frac{a}{6}\;-\frac{a}{6}\;\frac{5a}{6}.$$

The system obtained by employing these steps alternately with the three steps $-\frac{a}{2}\;\frac{a}{2}\;\frac{a}{2}$, ... of Art. 10 may be called the "anti-centre-corner" system. It consists of two homogeneous assemblages, each defined by the normal step $\frac{a}{3}\;\frac{a}{3}\;\frac{a}{3}$ with the usual tier steps.

To get rid of fractions, we must take the normal step three times; giving $a\;a\;a$; and this, with the tier steps, gives $3a$ 0 0, ... Hence, in each homogeneous assemblage, the interval between consecutive points on a parallel to an axis of coordinates is $3a$.

To pass from the first assemblage to the second, we may combine the step $\frac{a}{2}\;\frac{a}{2}\;-\frac{a}{2}$ from the 1st tier to the 2nd with the assemblage step $-\frac{a}{3}\;-\frac{a}{3}\;-\frac{a}{3}$, giving $\frac{a}{6}\;\frac{a}{6}\;-\frac{5a}{6}$; and the fractions cannot be got rid of. Hence the clashing of values of x and y for different values of z will not occur, and both assemblages can be plotted in the same chart. See Fig 5, where they are distinguished by two different kinds of type. $\frac{a}{6}$ is here the chart-unit.

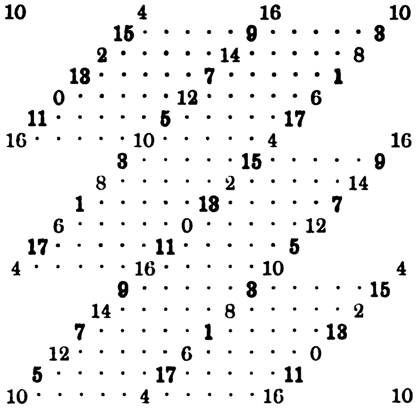

FIG. 5.

12. *Deduction of other Systems from Normal and Antinormal Piling by Compression perpendicular to Tiers.*—The following tabular statement includes all the systems thus far discussed. The first column gives the increment of $x+y+z$ from tier to tier, which is $\sqrt{3}$ times the tier distance. The second and third columns give the steps from tier to tier

referred to two different sets of axes, one set being obtained from the other by a rotation of 60° or of 180°.

$\sqrt{3}$ times tier distance.	STEPS.	STEPS.
	Axes in 1st position.	Axes in 2nd position.
$2a$	$0\ a\ a, \ldots.$	$\frac{4a}{3}\ \frac{a}{3}\ \frac{a}{3}, \ldots.$
$\frac{a}{2}$	$-\frac{a}{2}\ \frac{a}{2}\ \frac{a}{2}, \ldots.$	$\frac{5a}{6}\ -\frac{a}{6}\ -\frac{a}{6}, \ldots.$
	Axes in 2nd position.	Axes in 1st position.
a	$-\frac{a}{3}\ \frac{2a}{3}\ \frac{2a}{3}, \ldots.$	$a\ 0\ 0, \ldots.$

Comparing the steps in successive lines in either the second or the third column, we see that the first step exceeds the second by $\frac{a}{2}\ \frac{a}{2}\ \frac{a}{2}$, and exceeds the third by $\frac{a}{3}\ \frac{a}{3}\ \frac{a}{3}$. These differences represent steps normal to the tiers. The step $\frac{a}{2}\ \frac{a}{2}\ \frac{a}{2}$ taken negatively reduces the first tier distance in the ratio $\frac{1}{4}$; and the step $\frac{a}{3}\ \frac{a}{3}\ \frac{a}{3}$ taken negatively reduces it in the ratio $\frac{1}{2}$. Hence

I. Compresssion of the tier distance to $\frac{1}{4}$ of its original amount reduces normal piling to the cube-centre-corner system, and reduces antinormal piling to the anti-centre-corner system.

II. Compression of the tier distance to half its original amount reduces normal piling to the simple cubic system, and antinormal piling to the anticubic system; a rotation of 60° being at the same time introduced.

13. *Various Systems of Closest Piling.*—Closest piling can be obtained by employing the 2 sets of steps $0\ a\ a, \ldots$ and $\frac{4a}{3}\ \frac{a}{3}\ \frac{a}{3}, \ldots$ in any cyclic order of succession. Denoting one set of steps by α, and the other by β, the cycle of operations in normal piling is α, and in antinormal piling is $\alpha\beta$.

Prof. Herschel has suggested the system $\alpha\alpha\beta\beta$, which I find consists of 4 homogeneous assemblages, each having an interval $24a$ between successive points on a line parallel to an axis of coordinates. The system $\alpha\beta\beta$ consists of 3 homogeneous assemblages; and in general the number of homogeneous assemblages is equal to the number of operations in the cycle. There is no limit to this number of operations, and the number of systems of closest piling is therefore unlimited. But of all these systems normal piling is the only one that contains lines of spheres in contact other than the 3 coplanar lines which are contained in the given tiers.

In any one of the other systems, if we attempt to build up on a horizontal floor a pile with tiers inclined to the horizontal, we find that intervals of several sphere diameters occur between lines of spheres belonging to successive tiers, and the intervening regions are overhung with spheres only imperfectly supported. These statements are illustrated by Fig. 4.

14. *Barlow's 5-th System.*—If the tier distance be reduced one-half in the closest-piling system $\alpha\alpha\beta\beta$, it follows from Art. 12 that the steps from tier to tier will be those of the anticubic system, each taken twice in succession. The system thus obtained is the fifth of those specified by Barlow. It consist of four homogeneous assemblages, each containing the normal step $\frac{4a}{3}\ \frac{4a}{3}\ \frac{4a}{3}$ and the usual tier steps.

The 1st and 3rd compose one antinormal pile, and the 2nd and 4th another. The points of the 2nd and 4th assemblages lie at equal intervals on the same tier normals.

15. If a cubic network, in which the edge of each cube is $a\sqrt{2}$, is distorted till its square meshes become rhombuses of 60° and 120°, the knots (the original cube corners) will constitute a normal pile. The original cube edges have, in fact, been altered into our three fundamental vectors $0\ a\ a,\ a\ 0\ a,\ a\ a\ 0,$ which suffice for stepping from point to point over the whole assemblage.*

As a cube (or the rhomb into which it is altered) has eight corners, and eight cubes or rhombs meet at each corner, the corners are equal in number to the rhombs; and the space per point of the assemblage is the volume of a rhomb. This is $\begin{vmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{vmatrix} = 2a^3$. *The volume per point* for a homogeneous assemblage is always (as in this case) equal to *the determinant formed by the components of its steps.*

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* This representation was suggested to me by the following passage in a letter from Prof. Herschel :—" Stretch a simple cubic wire-work in the direction of any one of its cube diagonals to twice its primitive span in that direction ; and it becomes at once a 'normal-sphere arrangement of wire junctions.' " (See Art. 12.)

16. *Polyhedra whose Corners are Sphere Centres.*—The polyhedron whose corners are the centres of the 12 spheres that touch a given sphere in normal piling can be constructed by cutting off the corners of a cube by planes bisecting the cube edges. Or it can be built up of 14 pyramids having all their edges equal and their vertices at the centre of the solid, 6 of them having square bases and the other 8 being regular tetrahedra. The 6 square faces of the solid are parallel to our coordinate planes, and the 8 triangular faces are parallel (in pairs) to the 4 sets of tiers into which the spheres can be divided, or to the 4 faces of the standard tetrahedron of Art. 6. Fig. 6 shows the solid, projected on the plane of a tier; also the directions of the axes, as projected on the same plane, and the coordinates of each corner relative to the centre. The sides of the dotted triangles of Fig. 1 bisect the angles between the projections of the axes.

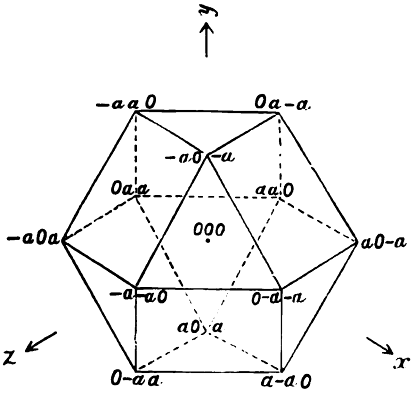

FIG. 6.—Normal pile solid, converted into antinormal by omitting dotted lines; x, y and z increase towards the back of the paper.

The solid is not symmetrical about a section midway between opposite triangular faces—the regular hexagon which forms the boundary of Fig. 6; but it will be rendered symmetrical by rotating the lower half (indicated by dotted lines) through 60° or through 180°. We thus obtain the solid whose corners are the centres of the 12 spheres that touch a given sphere in antinormal piling (see Fig. 7).

The 24 edges of the normal polyhedron are parallel (in sets of 4) to the 6 edges of the standard tetrahedron. Its 8 triangular faces are parallel (in pairs) to the 4 faces of the tetrahedron; and its 6 square faces are parallel in pairs to the 3 pairs of opposite edges of the tetrahedron.

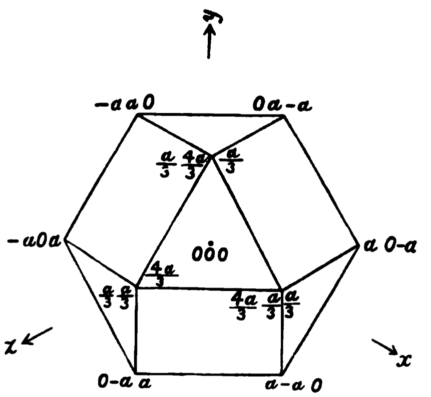
y
-aa0
0a-a
$\frac{a}{3}\frac{4a}{3}\frac{a}{3}$
-a0a
0̇00
a0-a
$\frac{4a}{3}$
$\frac{a}{3}\frac{a}{3}$
$\frac{4a}{3}\frac{a}{3}\frac{a}{3}$
z
x
0-aa
a-a0

FIG. 7.—Lower half of Antinormal Solid. [Upper half same as for normal solid (Fig. 6).]

The 24 edges of the antinormal polyhedron are parallel (in 8 sets of 4 and 6 sets of 2) to the edges of two regular tetrahedrons placed base to base. Its 8 triangular faces are parallel to their faces, and its 6 square faces to their pairs of opposite edges.

17. *Cells circumscribed to Spheres.*—The 12 planes which bisect at right angles the radii from the centre to the 12 corners of either of these solids are evidently the tangent planes at the 12 points of contact with the central sphere. Such planes divide space into equal and similar cells, which are dodecahedra. In the case of the normal pile, the cells are similarly oriented, and are rhombic dodecahedra, the 12 faces of a cell being equal rhombuses, perpendicular (in pairs) to the 6 edges of the standard tetrahedron; and the 24 edges being perpendicular (in sets of 6) to the 4 faces of the tetrahedron. It has 6 corners formed by the meeting of 3 obtuse angles, and 8 corners formed by the meeting of 4 acute angles.

Fig. 8 represents a cell as projected on the plane of a tier, one trihedral angle V being uppermost. The 3 angles marked L are also trihedral. The 3 points H are in one horizontal plane; and V is as much above this plane as L, L, L are below it. Vertical edges, which we will call HH', LL', run down from the points H and L; and from the 3 points L' edges shown by dotted lines run down to the lowest corner (say V') which is antipodal to V. As the faces are rhombuses, all the edges of the solid are equal. If the solid is bisected by a horizontal plane, each half is a typical bee's cell, except as regards the length of the 6 faces which have been cut across.

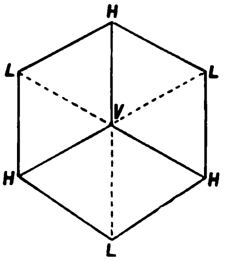

FIG. 8.

The section is a regular hexagon; and the two halves will still fit together when one of them is turned through 60° or 180°. One half will then be the reflection of the other; and, instead of the 6 rhombuses which have been cut across, we shall have 6 trapeziums of the same area. The solid as thus altered is the antinormal cell obtained by drawing tangent planes at the points of contact of spheres in antinormal piling. The 12 points of contact of a sphere are in both cases the centres of the 12 faces. Lord Kelvin has pointed out (*Phil. Mag.*, Vol. XXIV., p. 503, 1887) that the rhombic dodecahedron fulfils the conditions of equilibrium for a froth composed of equal bubbles. These conditions are also fulfilled by the antinormal dodecahedron, but with this difference, that the orientation, though the same for all cells in the same tier, is altered by 180° (or 60°) of rotation in the plane of a tier, in passing from each tier to the next.

In the cubic system of sphere packing (each sphere touching 6) the cells formed by the tangent planes are obviously cubes.

In the cube centre and corner system (each sphere touching 8) the cells are cubo-octahedra with all edges equal, and are similarly oriented. This is pointed out by Lord Kelvin in the first paper of the Royal Society *Proceedings* for 1894, where the solid is figured, and called the *orthic tetrakaidecahedron*.

In the corresponding anti-arrangement, the tangent planes do not divide space into distinct cells, but into cells which overlap.

THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS

By H. BATEMAN.

[Received January 12th, 1904.—Read January 14th, 1904.]

I. *Introduction.*

1. The solution of a partial differential equation may often be obtained in a very convenient form by means of a definite integral: thus, for example, the solution of the equation $\partial^2 V/\partial x^2 = \partial V/\partial t$ which reduces to $f(x)$ when $t = 0$ is given by

$$V = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2u\sqrt{t})e^{-u^2}du.$$

In 1902* Whittaker gave the complete solution of Laplace's equation in the form of a definite integral, and showed that the more general equation

$$\frac{\partial^2 V}{\partial x_1^2}+\ldots+\frac{\partial^2 V}{\partial x_n^2} = 0$$

could be solved in a similar manner by means of a $(n-2)$-ple definite integral.

Section II. of this paper deals with the solution of the differential equation

$$F\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} V = 0,$$

where F is a homogeneous function of the n-th degree. This is a natural extension of the solution of Laplace's equation, and the proof is based upon the one given by Whittaker in the paper just mentioned.

In Section III. the solution of the equation

$$F\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\} V = 0,$$

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* *Mathematische Annalen*, 1903, pp. 333–355.

where the function F is no longer homogeneous, is deduced; and the case when the curve $F(\xi, \eta, \zeta) = 0$ is unicursal is treated more fully.

In Section IV. an alternative solution of the equation of wave-motion is given, involving only one integration instead of two.

II. *Solution of the general Homogeneous Linear Partial Differential Equation with Constant Coefficients.*

2. In Whittaker's solution of Laplace's equation, viz.,

$$V = \int_0^{2\pi} f(x\cos t + y\sin t + iz,\ t)\, dt,$$

we may regard $\cos t$, $\sin t$, i as the coordinates of a point on the curve $\xi^2+\eta^2+\zeta^2 = 0$. Similarly, if (ξ, η, ζ) are the coordinates of a point on the curve $F(\xi, \eta, \zeta) = 0$, expressed as functions of a parameter t, then any solution of the equation

$$F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V = 0$$

can be put into the form

$$V = \int \Phi(\xi x + \eta y + \zeta z,\ t)\, dt, \tag{1}$$

where the path of integration is at present left arbitrary.

To prove this we choose for origin a point in the vicinity of which V is a regular function of x, y and z: we can then expand V as a power series in x, y, z absolutely and uniformly convergent within a certain region.

Operating on this series with $F(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, and equating to zero the coefficients of the various powers: we have when $m \geqslant n$

$$\tfrac{1}{2}(m-n+1)(m-n+2)$$

relations among the terms of order m; but when $m < n$ no such relations.

Accordingly the terms of order m are a linear combination of

$$\tfrac{1}{2}(m+1)(m+2) - \tfrac{1}{2}(m-n+1)(m-n+2),$$

i.e.

$$mn+1-\tfrac{1}{2}(n-1)(n-2),$$

linearly independent solutions when $m \geqslant n$; and of $\tfrac{1}{2}(m+1)(m+2)$ independent solutions when $m < n$.

In order to express the terms of order m in the form (1) we proceed as follows.

3. Take $mn+1-\frac{1}{2}(n-1)(n-2)$ arbitrary points on the curve; then the corresponding quantities $(\xi x+\eta y+\zeta z)^m$ will in general be linearly independent.

For, if not, there would be a linear relation between them; let it be

$$\sum_1^{mn+1-\frac{1}{2}(n-1)(n-2)} \lambda_r\{\xi_r x+\ldots\}^m = 0.$$

Leaving out one of the points, say (ξ_1, η, ζ_1), we can draw a curve of the m-th degree $f(\xi, \eta, \zeta) = 0$ through the remainder.

Operating on the syzygy with $f(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, the terms corresponding to these points disappear on account of the relation

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\{\xi x+\eta y+\zeta z\}^m = m!\, f(\xi, \eta, \zeta)$$

and we are left with the equation

$$\lambda_1 f(\xi_1, \eta_1, \zeta_1) = 0.$$

Therefore either $f(\xi_1, \eta_1, \zeta_1) = 0$, in which case all the points lie on a curve of the m-th degree and they would not have been chosen arbitrarily;* or $\lambda_1 = 0$. But λ_1 can be taken to be any one of the coefficients; accordingly all the coefficients are zero and the syzygy does not exist.

Thus we have $mn+1-\frac{1}{2}(n-1)(n-2)$ independent solutions of the equation; therefore $(\xi x+\eta y+\zeta z)^m$, being a solution of the equation, can be expressed as a linear combination of these solutions: in other words, there exists a linear relation between the m-th powers of the tangential equations of any $mn+2-\frac{1}{2}(n-1)(n-2)$ points on the curve, unless all but one of them lie on a curve of the m-th degree, in which case the syzygy does not contain a term corresponding to the last point.

We may obtain the linear relation between the m-th powers of the tangential equations of any $mn+2-\frac{1}{2}(n-1)(n-2)$ points on the curve as follows:—

Draw a C_{m+1}, $\chi(x, y, z) = 0$, through the points; this will cut the curve again in $\frac{1}{2}(n+1)(n+2)$ points, through which we may draw a C_{n-2}, say $\Theta(x, y, z) = 0$. Now, by a well known theorem of Jacobi's,* there is a linear relation between the $(m+n-2)$-th powers of the tangential equations of the points of intersection of $F = 0$ and $\chi = 0$: this relation is

$$\sum_1^{n(m+1)} k_r(x\xi_r+\ldots)^{m+n-2} = 0,$$

---

* Only $nm-\frac{1}{2}(n-1)(n-2)$ of the intersections of a C_m with a C_n can be taken at random.

where
$$\frac{\xi_r}{\dfrac{\partial\{F,\chi\}}{\partial\{\eta_r,\zeta_r\}}}=\frac{\eta_r}{\dfrac{\partial\{F,\chi\}}{\partial\{\zeta_r,\xi_r\}}}=\frac{\zeta_r}{\dfrac{\partial\{F,\chi\}}{\partial\{\xi_r,\eta_r\}}}=k_r.$$

Operating on this syzygy with $\Theta\{\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z\}$, the terms corresponding to points on $\Theta=0$ disappear, and we are left with

$$\sum_1^{mn+2-\frac{1}{2}(n-1)(n-2)} k_r\Theta_r(x\xi_r+\ldots)^m=0.$$

Similarly, when $m<n$ we may take $\frac{1}{2}(m+1)(m+2)$ points on the curve which do not lie on a curve of the m-th degree, and thus obtain $\frac{1}{2}(m+1)(m+2)$ independent solutions, and we have the corresponding theorem that when $m<n$ there is a linear relation between the m-th powers of the tangential equations of any $\frac{1}{2}(m+1)(m+2)+1$ points on the curve.*

4. Taking the two cases together, and denoting by r the number of independent solutions, we see that

$$(\xi x+\eta y+\zeta z)^m=\sum_1^r(\xi_k x+\eta_k y+\zeta_k z)^m\mu_k.$$

Moreover, the quantities μ_k are linearly independent functions of t; for, if not, $(\xi x+y\eta+\zeta z)^m$ would be expressible as a linear combination of $r-1$ solutions, and we would be able to obtain a syzygy among the tangential equations of any r points on the curve.

Accordingly

$$\int(\xi x+\eta y+\zeta z)^m f_s(t)\,dt=\sum_{l=1}^r(\xi_l x+\eta_l y+\zeta_l z)^m\int\mu_l f_s(t)\,dt$$

and the quantities $\theta_{s,l}=\int\mu_l f_s(t)\,dt$ will, in general, be linearly independent.

Choosing r linearly independent functions $f_1(t),\ \ldots,\ f_r(t)$, the determinant of the quantities $\theta_{s,l}$ will not, in general, be zero; accordingly we may choose r constants λ, so that the expressions

$$\lambda_1\theta_{1l}+\lambda_2\theta_{2l}+\ldots+\lambda_r\theta_{rl}\qquad(l=1,\ 2,\ \ldots,\ r)$$

take any r assigned values $p_1,\ \ldots,\ p_r$, and we have

$$\int(\xi x+\eta y+\zeta z)^m\sum_1^r\lambda_s f_s(t)\,dt=\sum_1^r p_l(\xi_l x+\eta_l y+\zeta_l z)^m.$$

But any solution of the m-th order can be expressed in the form

$$\sum_1^r p_l(\xi_l x+\eta_l y+\zeta_l z)^m,$$

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* This theorem is due to Serret. Compare also a paper of Clifford's: "On Syzygetic Relations between the Powers of Linear Quantics," *Collected Works*, pp. 119-129.

and can therefore be expressed in the form

$$\int (\xi x+\eta y+\zeta z)^m f(t)\, dt.$$

This being done for all values of m, our series for V takes the required form

$$V=\int \Phi \{\xi x+\eta y+\zeta z,\ t\}\, dt.$$

5. The path of integration has been left arbitrary: we might for simplicity take it to be a closed contour in the t-plane; or, again, we might express the integrand as a function of ξ, η, ζ in which case we should have

$$V=\int \phi \{\xi x+\eta y+\zeta z,\ \xi,\ \eta,\ \zeta\}\, d\xi,$$

the integral being taken along an arc of the curve $F(\xi, \eta, \zeta)=0$.

III. *Solution of Linear Differential Equations with Constant Coefficients.*

6. If we seek a solution of the equation

$$F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) V=0$$

of the form $V=e^z U$, where U is independent of z, we find that U must satisfy the equation

$$F\left\{\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ 1\right\} U=0,$$

and the corresponding solution is

$$\int e^{x(\xi/\zeta)+y(\eta/\zeta)} f(t)\, dt \quad \text{or} \quad \int e^{ux+vy} f(t)\, dt,$$

where u and v are the coordinates of a point on the curve $F\{u, v, 1\}=0$, expressed as functions of t. Thus, for example, the solution of the equation $\partial^2 V/\partial x^2=\partial V/\partial y$ is given by

$$\int e^{tx+t^2 y} f(t)\, dt,$$

where the path of integration is arbitrary.

7. When the curve $F(\xi, \eta, \zeta)=0$ is unicursal ξ, η, ζ can be expressed as rational integral functions of t; in this case we take our path of integration to be a simple closed contour round the point $t=0$; then

$$\frac{1}{2\pi i}\int_C (\xi x+\eta y+\zeta z)^m \frac{dt}{t^{s+1}}$$

is equal to the coefficient of t^s in $(\xi x+\eta y+\zeta z)^m$. Putting

$$(\xi x+\eta y+\zeta z)^m=A_0+A_1 t+\ldots+A_{mn} t^{mn},$$

there will be $mn+1-r$ linear relations among the quantities A_κ, and no more.

For, if there were $mn+2-r$ linear relations, we could express $(\xi x+\eta y+\zeta z)^m$ as a linear combination of $r-1$ independent solutions, contrary to what has been proved in Art. 3.

8. We may obtain $\frac{1}{2}(n-1)(n-2)$ relations among the quantities A as follows.

Let (α, β) be the two values of t corresponding to a double point: we have at once

$$\frac{A_0+A_1\alpha+\ldots+A_{mn}\alpha^{mn}}{\{\xi(\alpha)\}^m} = \frac{A_0+A_1\beta+\ldots+A_{mn}\beta^{mn}}{\{\xi(\beta)\}^m} = \left(x+\frac{\eta}{\xi}y+\frac{\zeta}{\xi}z\right)^m.$$

As there are $\frac{1}{2}(n-1)(n-2)$ double points on a unicursal curve, we shall get $\frac{1}{2}(n-1)(n-2)$ relations; but when $m<n$ these relations are not all independent.

9. We can evidently extend the preceding theory to equations with any number of independent variables; thus when there are four variables we associate the differential equation

$$F\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right\} V=0$$

with the surface $$F\{\alpha, \beta, \gamma, \delta\}=0.$$

Expressing $\alpha, \beta, \gamma, \delta$ as functions of two parameters u and v, we have as a solution of the equation

$$V=\iint \Phi\{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4, u, v\}\, du\, dv,$$

where the integration is taken over any portion of the surface.

IV. *Solution of the Equation of Wave-Motion by means of a Simple Definite Integral.*

10. The solution of the equation of wave-motion, which I shall write for convenience in the form

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0,$$

given by Whittaker is

$$V=\int_0^{2\pi}\int_0^{\pi} F\{x_1\cos t\cos u+x_2\cos t\sin u+x_3\sin t+ix_4,\ t, u\}\, dt\, du.$$

Now it is easy to verify that

$$V = \int_0^{2\pi} f\{x_1 \cos t + x_2 \sin t + ix_3,\ x_1 \sin t - x_2 \cos t + ix_4,\ t\}\, dt$$

is also a solution, and it can be proved that any solution of the above equation can be expressed in this form.

11. We can also reduce the number of integrations in the solution of the general equation; for consider

$$V = \int_0^{2\pi} F\{x_1 \cos t + x_2 \sin t,\ x_1 \sin t - x_2 \cos t,\ x_3,\ x_4,\ \dots,\ x_n,\ t\}\, dt.$$

This will be a solution of the equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \dots + \frac{\partial^2 V}{\partial x_n^2} = 0,$$

provided $F\{\xi,\ \eta,\ x_3,\ x_4,\ \dots,\ x_n,\ t\}$ satisfies the equations

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial x_3^2} + \dots + \frac{\partial^2 F}{\partial x_r^2} = 0, \qquad \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 F}{\partial x_{r+1}^2} + \dots + \frac{\partial^2 F}{\partial x_n^2} = 0.$$

Thus, if when $n = 4$ we take $r = 4$ and replace F by

$$\int_0^{2\pi} f\{\xi \cos u + x_3 \sin u + ix_4,\ u,\ t\}\, du,$$

we shall obtain Whittaker's solution, while when $n = 6$ we may obtain a solution in the form of a double integral by taking

$$F = \int_0^{2\pi} f\{\xi + ix_3,\ \eta \cos u + x_4 \sin u + ix_5,\ \eta \sin u - x_4 \cos u + ix_6,\ u,\ t\}\, du.$$

12. Returning to the general theorem, let us put $x_1 = \varpi \cos \phi$, $x_2 = \varpi \sin \phi$: we have then

$$V = \int_0^{2\pi} F\{\varpi \cos(t-\phi),\ \varpi \sin(t-\phi),\ x_3, x_4,\ \dots,\ x_n,\ t\}\, dt.$$

Expanding F, considered as a function of its last argument, in a Fourier series, we shall have, corresponding to a term $A \cos mt + B \sin mt$ in F, a term in V of the form $M \cos m\phi + N \sin m\phi$; so that V will not, in general,* be independent of ϕ unless F is independent of its last argument.

When V is independent of ϕ the differential equation becomes

$$\frac{\partial^2 V}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial V}{\partial \varpi} + \frac{\partial^2 V}{\partial x_3^2} + \dots + \frac{\partial^2 V}{\partial x_n^2} = 0.$$

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* This expansion is identical with the power series in $x_1, \dots, x_n$; it exists within a certain region except when the origin is a singularity of V.

Putting $\phi = 0$, a solution is given by

$$V = \int_0^{2\pi} F\{\varpi \cos t,\ \varpi \sin t,\ x_3, x_4, \ldots, x_n\}\, dt.$$

Thus when there are four independent variables the solution independent of ϕ is given by

$$V = \frac{1}{2\pi}\int_0^{2\pi} f\{x_3 + i\varpi \cos t,\ x_4 + i\varpi \sin t\}\, dt,$$

reducing to $f\{x_3, x_4\}$, when $\varpi = 0$.

[*Note added March* 12*th*, 1904.—The above expression is not the general solution: we have been tacitly assuming that V is not infinite at all points where $x_1 = 0$, $x_2 = 0$; a more general solution is given by

$$V = \int_0^{2\pi} f(x_3 + i\varpi \cos \phi,\ x_4 + i\varpi \sin \phi)\, d\phi + \int_0^c \int_0^{2\pi} \sinh^{-1}\left(\frac{a + x_3 + ix_4 \cos t}{\varpi \sin t}\right) F(a, t)\, da\, dt,$$

the last part becoming logarithmically infinite at all points where $\varpi = 0$.

The corresponding solution for Laplace's equation

$$\frac{d^2V}{d\varpi^2} + \frac{1}{\varpi}\frac{dV}{d\varpi} + \frac{d^2V}{dz^2} = 0$$

is
$$V = \int_0^{2\pi} f(z + i\varpi \cos \phi)\, d\phi + \int_0^c \sinh^{-1}\left(\frac{a+z}{\varpi}\right) F(a)\, da.$$

The form of this solution explains why V is completely determined when its value is given at all points on the axis of z.]

ELECTRIC RADIATION FROM CONDUCTORS

By H. M. MACDONALD.

[Communicated January 14th, 1904. — Received February 23rd, 1904.]

THE rate at which energy is transferred across a closed surface into the space inside it is

$$\frac{1}{8\pi}\frac{d}{dt}\iint\{F(m\gamma-n\beta)+G(n\alpha-l\gamma)+H(l\beta-m\alpha)\}\,dS$$
$$+\frac{1}{4\pi}\iint\{X(m\gamma-n\beta)+Y(n\alpha-l\gamma)+Z(l\beta-m\alpha)\}\,dS,^{*}$$

where α, β, γ, X, Y, Z, F, G, H are the components of the magnetic force, the electric force, and the electrokinetic momentum at a point on the surface, and l, m, n are the direction cosines of the normal to the surface at the point drawn outwards. The above expression does not give information as to the actual flow of energy across an element of the surface, but the flow across an element of the surface may be taken to be the corresponding element of the integral provided the results are applied to closed surfaces only. If, for simplicity, waves of one definite period are considered, it follows at once that there is no flow of energy across an element of surface which is perpendicular either to the electric force or to the magnetic force, the direction of the flow being perpendicular to both.

Now, at the surface of a perfect conductor the electric force at a point on the surface is either normal to the surface or vanishes; hence, at any instant of time the direction of the flow of energy just outside a perfect conductor is parallel to the surface of the conductor, except at points on it where the electric force vanishes. The electric force at the surface of the conductor will at any time vanish either along a definite number of closed lines on the surface or all over the surface.

The case in which the electric force vanishes along a number of closed lines and these lines are not fixed on the surface is that where there are sources outside the conductor; and the flow of the energy, except in the neighbourhood of these lines, is parallel to the surface of the conductor.

The case in which the electric force at the surface of the conductor always vanishes along a number of fixed lines on the surface is that where the electrical distribution on the surface is in a state of stationary oscillation, and in this case the electric force will vanish all over the surface once in each half period. At a time when the electric force vanishes all over the surface of the conductor there are surfaces outside it which, at previous times which are multiples of half a period, coincided with the surface of the conductor. If any one of these surfaces be followed as it advances from the conductor, it will trace out a family of surfaces, and the sufficient condition that there should be no radiation from the conductor is that the electric force tangential to each of these surfaces so traced out should vanish; for then there will be no transfer of energy across them.

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* Macdonald, *Electric Waves*, p. 72.

Choosing then curvilinear coordinates ξ, η, ζ, where the surfaces ξ are the above surfaces, and η, ζ define surfaces orthogonal to the surfaces ξ and to each other, if X, Y, Z, α, β, γ denote the components of the electric and magnetic forces normal to the surfaces ξ, η, ζ at any point, and V is the velocity of radiation, the equations satisfied by them are

$$\frac{\dot{X}}{V^2} = h_2 h_3 \left\{\frac{\partial}{\partial\eta}\left(\frac{\gamma}{h_3}\right) - \frac{\partial}{\partial\xi}\left(\frac{\beta}{h_2}\right)\right\}, \qquad \dot{\alpha} = h_2 h_3 \left\{\frac{\partial}{\partial\zeta}\left(\frac{Y}{h_2}\right) - \frac{\partial}{\partial\eta}\left(\frac{Z}{h_3}\right)\right\},$$

$$\frac{\dot{Y}}{V^2} = h_3 h_1 \left\{\frac{\partial}{\partial\zeta}\left(\frac{\alpha}{h_1}\right) - \frac{\partial}{\partial\xi}\left(\frac{\gamma}{h_3}\right)\right\}, \qquad \dot{\beta} = h_3 h_1 \left\{\frac{\partial}{\partial\xi}\left(\frac{Z}{h_3}\right) - \frac{\partial}{\partial\zeta}\left(\frac{X}{h_1}\right)\right\},$$

$$\frac{\dot{Z}}{V^2} = h_1 h_2 \left\{\frac{\partial}{\partial\xi}\left(\frac{\beta}{h_2}\right) - \frac{\partial}{\partial\eta}\left(\frac{\alpha}{h_1}\right)\right\}, \qquad \dot{\gamma} = h_1 h_2 \left\{\frac{\partial}{\partial\eta}\left(\frac{X}{h_1}\right) - \frac{\partial}{\partial\xi}\left(\frac{Y}{h_2}\right)\right\},$$

where $d\xi/h_1$, $d\eta/h_2$, $d\zeta/h_3$ are the elements of length normal to the surfaces ξ, η, ζ.

The above condition for no radiation gives $Y = 0$, $Z = 0$: hence $\alpha = 0$; β/h_2, γ/h_3 are independent of ξ; and X satisfies the equation

$$\frac{\ddot{X}}{V^2} = h_2 h_3 \left\{\frac{\partial}{\partial\eta} \frac{h_1 h_2}{h_3} \frac{\partial}{\partial\eta}\left(\frac{X}{h_1}\right) + \frac{\partial}{\partial\zeta} \frac{h_3 h_1}{h_2} \frac{\partial}{\partial\zeta}\left(\frac{X}{h_1}\right)\right\};$$

whence $X/h_2 h_3$ must be independent of ξ, and all possibilities are included in the following:—h_1/h_2 is independent of ξ and X/h_1 is independent of η; the equation satisfied by X being

$$\frac{\ddot{X}}{V^2} = h_2 h_3 \frac{\partial}{\partial\zeta} \frac{h_3 h_1}{h_2} \frac{\partial}{\partial\zeta}\left(\frac{X}{h_1}\right),$$

and this requires that h_3 should be a function of ζ only. Therefore the surfaces ζ are either concentric spheres, concentric circular cylinders, or parallel planes. The last is the important case, and it appears that, if two parallel conducting planes be joined by any number of conducting cylinders whose generators are perpendicular to the planes, there are possible systems of electrical oscillations in the unlimited space which do not radiate away energy, a result which can be easily verified.*

From this it is to be expected that, when two nearly parallel conducting planes are connected by a conducting tubular surface which cuts the planes orthogonally, the generators being arcs of circles having their centres on the line of intersection of the planes, there will be possible systems of electrical oscillations in this space from which energy will be radiated away very slowly. It may be readily shown that, if d be the mean length of the arcs of circles that bound the tube and ρ their mean radius, the corrections to be added to the expressions for the electric and magnetic forces which would hold when the planes are parallel and the tube straight depend on powers of d/ρ, the lowest power being the second, and therefore, provided d^2/ρ^2 is negligible, the rate of decay of the oscillations can be neglected.

Another case in which it is to be expected that the rate of decay of the oscillations is very small is when the amount of bending is finite but the cross section of the tube is small; for the amount of radiation can only alter slowly as the cross section is diminished, provided the line integral of magnetic force looping the tube is kept constant, whereas the amount of energy in a finite space enclosing the tube increases rapidly.

The most important case is that of a wire in the form of a closed circuit. The usual method of treatment is to evaluate the electric force tangential to the wire and in the direction of the wire, the part due to the wire being given as $B \log r_0$, where r_0 is the radius of the wire supposed to be of small circular cross section.† It is then assumed that the condition for free oscillations is $B = 0$. The expressions of the components of the electric force due to the wire are

$$X = -\frac{\iota V}{4\pi\kappa} \iint M \left[l_1 \frac{\partial^2}{\partial x^2} + m_1 \frac{\partial^2}{\partial x\, \partial y} + n_1 \frac{\partial^2}{\partial x\, \partial z} + \kappa^2 l_1 \right] \frac{e^{-\iota\kappa R}}{R}\, dS, \quad \ldots,$$

where l_1, m_1, n_1 are the direction cosines of the current at a point x_1, y_1, z_1 on the surface of the wire, M is the magnetic force at the point, R is the distance from the point x_1, y_1, z_1 to any point x, y, z, and the integrals are taken over the surface of the wire. Making the assumption, which is involved in the procedure above referred to, that l_1, m_1, n_1 are sensibly the same as the direction cosines of the circuit which is the limit of the wire when its cross section is indefinitely diminished, the expressions for the components of the electric force become

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* Cf. Lord Rayleigh, *Phil. Mag.*, August, 1897.

† Cf. Poincaré, *Les Oscillations électriques*, pp. 233, 237. Paris, 1894.

$$X = -\frac{\iota V}{4\pi\kappa}\int L\left(l_1\frac{\partial^2}{\partial x^2}+m_1\frac{\partial^2}{\partial x\,\partial y}+n_1\frac{\partial^2}{\partial x\,\partial z}+\kappa^2 l_1\right)\frac{e^{-\iota\kappa R}}{R}\,ds_1, \quad \ldots,$$

where L is the line integral of the magnetic force looping the wire and the integrals are now taken along the circuit. These expressions are equivalent to

$$X = -\frac{\iota V}{4\pi\kappa}\left\{\frac{\partial}{\partial x}\int\frac{e^{-\iota\kappa R}}{R}\frac{\partial L}{\partial s_1}\,ds_1+\kappa^2\int\frac{e^{-\iota\kappa R}}{R}\,l_1 L\,ds_1\right\}, \quad \ldots,$$

since L is single-valued; whence the electric force tangential to the wire in the direction of the circuit is

$$-\frac{\iota V}{4\pi\kappa}\left\{\frac{\partial}{\partial s}\int\frac{e^{-\iota\kappa R}}{R}\frac{\partial L}{\partial s_1}\,ds_1+\kappa^2\int\frac{e^{-\iota\kappa R}}{R}\,(ll_1+mm_1+nn_1)\,L\,ds_1\right.,$$

where l, m, n are the direction cosines of the circuit at the point corresponding to the point on the wire where the electric force is being evaluated and $\partial/\partial s$ denotes differentiation in this direction. The condition that this expression should not become infinite when r_0, the radius of the wire, is made to vanish is that L should satisfy the equation

$$\frac{\partial^2 L}{\partial s^2}+\kappa^2 L = 0,$$

which is the condition $B = 0$ given above; but, this condition being satisfied, the value of the above expression is in general a finite quantity different from zero. Hence, a circuit of Hertzian oscillators is not in general the limit of a conducting wire. For example, in the case of a circuit in the form of a circle of radius c, if θ be the angle the radius to any point makes with a fixed radius, the condition that the electric force tangential to the circuit at a point on it is not infinite gives

$$L = A\cos\kappa c\theta+B\sin\kappa c\theta,$$

and, since L is single-valued, κc must be an integer n. The value of the electric force tangential to the circuit at the point θ on it is then, from the previous expression,

$$-\frac{\iota V}{4\pi\kappa}\left\{\frac{1}{c}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{e^{-\iota\kappa R}}{R}\frac{\partial L}{\partial\theta_1}\,d\theta_1+\kappa^2 c\int_0^{2\pi}\frac{e^{-\iota\kappa R}}{R}\,L\cos(\theta-\theta_1)\,d\theta_1\right\},$$

that is $$-\frac{\iota V}{4\pi\kappa c}\int_0^{2\pi}\left\{\frac{\partial^2 L}{\partial\theta_1^2}+\kappa^2c^2L\cos(\theta-\theta_1)\right\}\frac{e^{-\iota\kappa R}}{R}\,d\theta,$$

or, writing for κc its value n,

$$\frac{\iota n V}{4\pi c}\left[\int_0^{\theta}(A\cos n\theta_1+B\sin n\theta_1)\sin\tfrac{1}{2}(\theta-\theta_1)\,e^{-2\iota n\sin\frac{1}{2}(\theta-\theta_1)}\,d\theta_1\right.$$
$$\left.+\int_{\theta}^{2\pi}(A\cos n\theta_1+B\sin n\theta_1)\sin\tfrac{1}{2}(\theta_1-\theta)\,e^{-2\iota n\sin\frac{1}{2}(\theta_1-\theta)}\,d\theta_1\right],$$

which is $\dfrac{\iota n V}{2\pi c}\displaystyle\int_0^{\pi}\{A\cos n(\theta-2\chi)+B\sin n(\theta-2\chi)\}\sin\chi e^{-2\iota n\sin\chi}\,d\chi$,

or $$\frac{nV}{4c}\{J_{2n+1}(2n)-J_{2n-1}(2n)\}\{A\cos n\theta+B\sin n\theta\},$$

where $J_n(x)$ denotes Bessel's function of order n; thus the electric force tangential to the circuit at a point on it is a finite quantity.

The radiation from a circuit of Hertzian oscillators arranged according to the law $d^2L/ds^2+\kappa^2L=0$ can be found as follows. Let a tubular surface be drawn enclosing the circuit: then, the radiation across all such tubular surfaces being the same, the rate of radiation of energy from the circuit is equal to the value of the integral

$$-\frac{1}{8\pi}\frac{d}{dt}\iint\Big\{F(m\gamma-n\beta)+G(n\alpha-l\gamma)+H(l\beta-m\alpha)\Big\}\,dS$$
$$-\frac{1}{4\pi}\iint\{X(m\gamma-n\beta)+Y(n\alpha-l\gamma)+Z(l\beta-m\alpha)\}\,dS$$

taken over any one of these tubular surfaces, and is therefore equal to the limit that this integral attains when the cross section of the tube is indefinitely diminished; the value of this limit is $-\dfrac{1}{8\pi}\displaystyle\int L_1E_1\,ds$ where L_1 is the maximum value of L and E_1 the maximum value of the electric force tangential to the circuit at a point on it, the integral being taken once round the circuit. Hence the rate of radiation of energy from the circuit is $-\dfrac{1}{8\pi}\displaystyle\int L_1E_1\,ds$.

When the circuit is a circle of radius c this becomes, from a preceding result,

$$\frac{nV}{32\pi c}\{J_{2n-1}(2n)-J_{2n+1}(2n)\}\int_0^{2\pi}\{A_1\cos n\theta+B_1\sin n\theta\}^2c\,d\theta_1$$

where A_1, B_1 are the maximum values of A and B, that is

$$\frac{nV}{32}\{J_{2n-1}(2n)-J_{2n+1}(2n)\}(A_1^2+B_1^2).$$

This result shows that the rate of radiation from a circle of Hertzian oscillators of given maximum amplitude arranged according to the above law is, for oscillations having the same number of nodes, independent of the radius of the circuit. For the oscillations of greatest wave-length $n = 1$, and the rate of radiation is

$$\frac{V}{32}\left\{J_1(2)-J_3(2)\right\}(A_1^2+B_1^2),$$

that is ·014 $V(A_1^2+B_1^2)$ approximately.

It has been seen above that a circuit of Hertzian oscillators is not the limit of a conducting wire; it also appears from the following considerations that a distribution of oscillators on the surface of the wire, each oscillator being parallel to the tangent to the circuit at the corresponding point on it, does not give an approximate solution. Taking the case of a wire in the form of a circle, the radius of the circle which is the central axis of the wire being c and the radius of the cross section being r_0, the above distribution is equivalent to a bundle of circuits of Hertzian oscillators, each circuit being on the surface of the wire and parallel to the circuit forming its central axis. Let P be a point on any one of these circuits, Q a point near to P but not in the plane of the circuit through P; then the electric force due to this circuit at Q is made up of the following—a force along PQ of the order PQ^{-1}, a force parallel to the tangent to the circuit at P of the order c^{-1}, and a force perpendicular to the tangent and parallel to the plane of the circuit of the order c^{-1}. If Q is on the surface of the wire, the distribution can be arranged so that the resultant of the first group is normal to the surface of the wire and of order r_0^{-1}, but there will remain two tangential forces each of order c^{-1}, one in the direction parallel to the tangent to the central axis of the wire at the corresponding point on it, the other perpendicular to this direction, the latter force only vanishing when Q is in the plane of the central axis of the wire. The tangential force parallel to the central axis of the wire could be made to vanish by making the period complex, but the force tangential to the cross section which is of the same order would still remain. For the oscillations of greatest wave-length the argument can be put more simply as follows:—If P is a point on the surface of the wire, O the corresponding point on the central axis of the wire, the unbalanced tangential force at P parallel to the tangent to the central axis at O is largely contributed by the oscillators situated near O_1, the extremity of the diameter through O of the circle that forms the central axis of the wire, and the unbalanced tangential force perpendicular to the tangent at O is largely contributed by the oscillators situated near O_2 and O_3, O_2O_3 being the diameter of the circle forming the central axis perpendicular to OO_1, the parts contributed by an oscillator near O_2 and one similarly placed near O_3 being additive, and the forces in both cases are of the same order.

A direct method of attempting to solve the problem of the oscillations of an electrical distribution on a wire in the form of a circle is the following. Let cylindrical coordinates r, θ, z be chosen, the axis of z being the axis of the circle that forms the central axis of the wire, and let the transformation $r+\iota z = f(\xi+\iota\eta)$ be found such that one of the surfaces η is the surface of the wire; then the equations satisfied by the components of the electric and magnetic forces are

$$\frac{\dot{X}}{V^2} = \frac{J}{r}\left[\frac{\partial}{\partial\eta}(\gamma r) - \frac{\partial}{\partial\theta}\left(\frac{\beta}{J}\right)\right], \qquad \dot{a} = \frac{J}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{Y}{J}\right) - \frac{\partial}{\partial\eta}(Zr)\right],$$

$$\frac{\dot{Y}}{V^2} = \frac{J}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{a}{J}\right) - \frac{\partial}{\partial\xi}(\gamma r)\right], \qquad \dot{\beta} = \frac{J}{r}\left[\frac{\partial}{\partial\xi}(Zr) - \frac{\partial}{\partial\theta}\left(\frac{X}{J}\right)\right],$$

$$\frac{\dot{Z}}{V^2} = J^2\left[\frac{\partial}{\partial\xi}\left(\frac{\beta}{J}\right) - \frac{\partial}{\partial\eta}\left(\frac{a}{J}\right)\right], \qquad \dot{\gamma} = J^2\left[\frac{\partial}{\partial\eta}\left(\frac{X}{J}\right) - \frac{\partial}{\partial\xi}\left(\frac{Y}{J}\right)\right],$$

where $$J^2 = \frac{\partial(\xi,\ \eta)}{\partial(r,\ z)}.$$

For the oscillations of greatest wave-length

$$a = a_1 \cos\theta, \qquad \beta = \beta_1 \cos\theta, \qquad \gamma = \gamma_1 \sin\theta,$$

$$X = X_1 \sin\theta, \qquad Y = Y_1 \sin\theta, \qquad Z = Z_1 \cos\theta,$$

and X_1, Y_1, Z_1, a_1, β_1, γ_1 satisfy the equations

$$\frac{\dot{X}_1}{V^2} = \frac{J}{r}\left[\frac{\partial}{\partial\eta}(\gamma_1 r) + \frac{B_1}{J}\right], \qquad \dot{a}_1 = \frac{J}{r}\left[\frac{Y_1}{J} - \frac{\partial}{\partial\eta}(Z_1 r)\right],$$

$$\frac{\dot{Y}_1}{V^2} = -\frac{J}{r}\left[\frac{\partial}{\partial\xi}(\gamma_1 r +)\frac{a_1}{J}\right], \qquad \dot{\beta}_1 = \frac{J}{r}\left[\frac{\partial}{\partial\xi}(Z_1 r) - \frac{X_1}{J}\right],$$

$$\frac{\dot{Z}_1}{V^2} = J^2\left[\frac{\partial}{\partial\xi}\left(\frac{\beta_1}{J}\right) - \frac{\partial}{\partial\eta}\left(\frac{a_1}{J}\right)\right], \qquad \dot{\gamma}_1 = J^2\left[\frac{\partial}{\partial\eta}\left(\frac{X_1}{J}\right) - \frac{\partial}{\partial\xi}\left(\frac{Y_1}{J}\right)\right];$$

whence $$X_1 = J\frac{\partial}{\partial\xi}(Z_1 r) - r\dot{\beta}_1, \qquad Y_1 = J\frac{\partial}{\partial\eta}(Z_1 r) + r\dot{a}_1;$$

therefore
$$J\frac{\partial}{\partial \xi}\left(\frac{\dot{Z}_1 r}{V^2}\right) - \frac{r\ddot{\beta}_1}{V^2} = \frac{J}{r}\left[\frac{\partial}{\partial \eta}(\gamma_1 r) + \frac{\beta_1}{J}\right],$$

$$J\frac{\partial}{\partial \eta}\left(\frac{\dot{Z}_1 r}{V^2}\right) + \frac{r\ddot{a}_1}{V^2} = -\frac{J}{r}\left[\frac{\partial}{\partial \xi}(\gamma_1 r) + \frac{a_1}{J}\right],$$

that is
$$a_1(\kappa^2 r^2 - 1) = J\left[\quad r\frac{\partial}{\partial \eta}\left(\frac{\dot{Z}_1 r}{V^2}\right) + \frac{\partial}{\partial \xi}(\gamma_1 r)\right],$$

$$\beta_1(\kappa^2 r^2 - 1) = J\left[-r\frac{\partial}{\partial \xi}\left(\frac{\dot{Z}_1 r}{V^2}\right) + \frac{\partial}{\partial \eta}(\gamma_1 r)\right];$$

hence
$$J^2\frac{\partial}{\partial \xi}\left\{\frac{r}{\kappa^2 r^2 - 1}\,\frac{\partial}{\partial \xi}\left(\frac{\dot{Z}_1 r}{V^2}\right)\right\} + J^2\frac{\partial}{\partial \eta}\left\{\frac{r}{\kappa^2 r^2 - 1}\,\frac{\partial}{\partial \eta}\,\frac{\dot{Z}_1 r}{V^2}\right\} + \frac{\dot{Z}_1}{V^2}$$
$$= J^2\left[\frac{\partial}{\partial \eta}(\gamma_1 r)\frac{\partial}{\partial \xi}\,\frac{1}{\kappa^2 r^2 - 1} - \frac{\partial}{\partial \xi}(\gamma_1 r)\frac{\partial}{\partial \eta}\,\frac{1}{\kappa^2 r^2 - 1}\right];$$

and, similarly,
$$J^2\frac{\partial}{\partial \xi}\left\{\frac{r}{\kappa^2 r^2 - 1}\,\frac{\partial}{\partial \xi}(\dot{\gamma}_1 r)\right\} + J^2\frac{\partial}{\partial \eta}\left\{\frac{r}{\kappa^2 r^2 - 1}\,\frac{\partial}{\partial \eta}(\dot{\gamma}_1 r)\right\} + \dot{\gamma}_1$$
$$= J^2\left[\frac{\partial}{\partial \eta}(Z_1 r)\frac{\partial}{\partial \xi}\,\frac{1}{\kappa^2 r^2 - 1} - \frac{\partial}{\partial \xi}(Z_1 r)\frac{\partial}{\partial \eta}\,\frac{1}{\kappa^2 r^2 - 1}\right].$$

Now the transformation $r + \iota z = f(\xi + \iota\eta)$ conformably represents any one of the planes through the axis of the ring on a strip of the ξ, η plane parallel to the axis of η; the breadth of this strip can be chosen to be π, and therefore it may be assumed that

$$Z_1 = \Sigma Z'_n \cos n\xi, \qquad \gamma_1 = \Sigma \gamma'_n \sin n\xi,$$

where the values of n are integers. When these expressions are substituted in the above two equations and in the boundary conditions $X = 0$, $Z = 0$ when $\eta = \eta_0$, the resulting equations are only sufficient to determine Z'_n and γ'_n, and it will remain to determine κ^2 from the condition that the expression for Z or γ converges for all values of η lying between 0 and η_0. The resulting analysis would prove complicated, and it is doubtful whether any useful result could be obtained while sufficient information for practical purposes can be obtained otherwise.

Restricting the problem to the case where the cross section of the wire is circular and of small radius r_0, approximations for the electric and magnetic forces in the immediate neighbourhood of the wire can be found. For this purpose it is convenient to adopt a different system of coordinates; choosing as coordinates the shortest distance r of a point from the central axis of the wire, the angle ϕ this shortest distance makes with the plane of the central axis, and the angle θ the plane through the axis of the ring and the shortest distance makes with a fixed plane through the axis of the ring, then X, Y, Z, α, β, γ being the components of the electric force and the magnetic force corresponding to the coordinates r, θ, ϕ, the equations satisfied by them are

$$\frac{\dot{X}}{V^2} = \frac{1}{\varpi r}\left[\frac{\partial}{\partial\theta}(\gamma r) - \frac{\partial}{\partial\phi}(\beta\varpi)\right], \qquad \dot{\alpha} = \frac{1}{\varpi r}\left[\frac{\partial}{\partial\phi}(\gamma\varpi) - \frac{\partial}{\partial\theta}(Zr)\right],$$

$$\frac{\dot{Y}}{V^2} = \frac{1}{r}\left[\frac{\partial\alpha}{\partial\phi} - \frac{\partial}{\partial r}(\gamma r)\right], \qquad \dot{\beta} = \frac{1}{r}\left[\frac{\partial}{\partial r}(Zr) - \frac{\partial\chi}{\partial\phi}\right],$$

$$\frac{\dot{Z}}{V^2} = \frac{1}{\varpi}\left[\frac{\partial}{\partial r}(\beta\varpi) - \frac{\partial\alpha}{\partial\theta}\right], \qquad \dot{\gamma} = \frac{1}{\varpi}\left[\frac{\partial\chi}{\partial\theta} - \frac{\partial}{\partial r}(\gamma\varpi)\right],$$

in which ϖ has been written for $c+r\cos\phi$. For the oscillations of greatest wave-length

$$\alpha = \alpha'\cos\theta, \qquad \beta = \beta'\sin\theta, \qquad \gamma = \gamma'\cos\theta,$$

$$X = X'\sin\theta, \qquad Y = Y'\cos\theta, \qquad Z = Z'\sin\theta,$$

whence the above equations become

$$\frac{\dot{X}'}{V^2} = -\frac{1}{\varpi r}\left[\gamma' r + \frac{\partial}{\partial\phi}(\beta'\varpi)\right], \qquad \dot{\alpha}' = \frac{1}{\varpi r}\left[\frac{\partial}{\partial\phi}(Y'\varpi) - Z'r\right],$$

$$\frac{\dot{Y}'}{V^2} = \frac{1}{r}\left[\frac{\partial\alpha'}{\partial\phi} - \frac{\partial}{\partial r}(\gamma' r)\right], \qquad \dot{\beta}' = \frac{1}{r}\left[\frac{\partial}{\partial r}(Z'r) - \frac{\partial X'}{\partial\phi}\right],$$

$$\frac{\dot{Z}'}{V^2} = \frac{1}{\varpi}\left[\frac{\partial}{\partial r}(\beta'\varpi) + \alpha'\right], \qquad \dot{\gamma}' = \frac{1}{\varpi}\left[X' - \frac{\partial}{\partial r}(Y'\varpi)\right],$$

and, eliminating X', Y', Z',

$$-\kappa^2\alpha' = \frac{1}{\varpi r}\left[\frac{\partial}{\partial\phi}\,\frac{\varpi}{r}\left\{\frac{\partial\alpha'}{\partial\phi} - \frac{\partial}{\partial r}(\gamma' r)\right\} - \frac{r}{\varpi}\left\{\frac{\partial}{\partial r}(\beta'\varpi) + \alpha'\right\}\right],$$

$$-\kappa^2\beta' = \frac{1}{r}\left[\frac{\partial}{\partial r}\,\frac{r}{\varpi}\left\{\frac{\partial}{\partial r}(\beta'\varpi) + \alpha'\right\} + \frac{\partial}{\partial\phi}\,\frac{1}{\varpi r}\left\{\gamma' r + \frac{\partial}{\partial\phi}(\beta'\varpi)\right\}\right],$$

$$-\kappa^2\gamma' = \frac{1}{\varpi}\left[-\frac{1}{\varpi r}\left\{\gamma' r + \frac{\partial}{\partial\phi}(\beta'\varpi)\right\} - \frac{\partial}{\partial r}\,\frac{\varpi}{r}\left\{\frac{\partial\alpha'}{\partial\phi} - \frac{\partial}{\partial r}(\gamma' r)\right\}\right],$$

and any one of these three equations can be replaced by the equation

$$\frac{\partial}{\partial r}(\alpha'\varpi r) + \beta' r + \frac{\partial}{\partial\phi}(\gamma'\varpi) = 0.$$

Taking the first and third of the above group, they can be written

$$(1-\kappa^2\varpi^2)\,\alpha' = \frac{\varpi}{r}\frac{\partial}{\partial\phi}\left(\frac{\varpi}{r}\frac{\partial\alpha'}{\partial\phi}\right) - \frac{\varpi}{r}\frac{\partial}{\partial\phi}\left\{\frac{\varpi}{r}\frac{\partial}{\partial r}(\gamma' r)\right\} - \frac{\partial}{\partial r}(\beta'\varpi),$$

$$(1-\kappa^2\varpi^2)\,\gamma' = \varpi\frac{\partial}{\partial r}\left\{\frac{\varpi}{r}\frac{\partial}{\partial r}(\gamma' r)\right\} - \varpi\frac{\partial}{\partial r}\left\{\frac{\varpi}{r}\frac{\partial\alpha'}{\partial\phi}\right\} - \frac{1}{r}\frac{\partial}{\partial\phi}(\beta'\varpi).$$

Writing

$$\alpha' = \sum_1^\infty \alpha_n \sin n\phi, \qquad \beta' = \sum_1^\infty \beta_n \sin n\phi, \qquad \gamma' = \gamma_0 + \sum_1^\infty \gamma_n \cos n\phi,$$

and assuming that when r/c is small α_{n+1}, β_{n+1}, γ_{n+1} are small compared with α_n, β_n, γ_n, the values of α_n, β_n, γ_n can be successively calculated to any desired degree of approximation from the above equations together with the boundary conditions $Y' = 0$, $Z' = 0$ when $r = r_0$, that is $\alpha' = 0$, $\frac{\partial}{\partial r}(\beta'\varpi) = 0$, $\frac{\partial}{\partial r}(\gamma' r) = 0$, when $r = r_0$. The most important part of γ_0 is determined by the equation

$$(1-\kappa^2 c^2)\,\gamma_0 = c^2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r\gamma_0),$$

subject to the condition that $\frac{\partial}{\partial r}(r\gamma_0) = 0$, when $r = r_0$; whence the most important part of γ_0 is given by $\gamma_0 = A/r$, and $1-\kappa^2c^2$ must be a small quantity. The most important parts of α_1, β_1, γ_1 are determined by the equations

$$\frac{\partial}{\partial r}(\alpha_1 cr) + \beta_1 r - \gamma_1 c - \gamma_0 r = 0,$$

$$-\frac{c^2}{r^2}\alpha_1 + \frac{c^2}{r^2}\frac{\partial}{\partial r}(\gamma_1 r) + \frac{c}{r}\frac{\partial}{\partial r}(\gamma_0 r) - c\frac{\partial\beta_1}{\partial r} = 0,$$

$$c^2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(\gamma_1 r) + rc\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(\gamma_0 r) + c\frac{\partial^2}{\partial r^2}(\gamma_0 r) - c^2\frac{\partial}{\partial r}\left(\frac{\alpha_1}{r}\right) + \frac{c\beta_1}{r}$$
$$= -\,2\kappa^2\gamma_0 rc,$$

subject to $\alpha_1 = 0$, $\partial\beta_1/\partial r = 0$, $\frac{\partial}{\partial r}(\gamma_1 r) = 0$, when $r = r_0$. Eliminating α_1 and γ_1, the equation to determine β_1 is

$$\frac{\partial}{\partial r}\left(r\frac{\partial\beta_1}{\partial r}\right) - \frac{\beta_1}{r} = -\frac{2}{c^2}\gamma_0 r,$$

all but the most important terms being neglected; hence

$$\beta_1 = Br - \frac{Ar}{c^2}\log r,$$

and making use of the boundary condition this becomes

$$\beta_1 = \frac{Ar}{c^2}\left(1-\log\frac{r}{r_0}\right).$$

Substituting this value of β_1 in the above equations, and rejecting all but the most important terms, they become

$$a_1 = \frac{\partial}{\partial r}(\gamma_1 r), \qquad \frac{\partial}{\partial r}\, r\frac{\partial}{\partial r}(\gamma_1 r)-\gamma_1 = \frac{A}{c};$$

whence $$\gamma_1 = C+\frac{A}{2c}\log r,$$

and therefore, determining C from the boundary condition,

$$\gamma_1 = \frac{A}{2c}\left[\log\frac{r}{r_0}-1\right], \qquad a_1 = \frac{A}{2c}\log\frac{r}{r_0}.$$

The function γ_0 has now to be determined to the same order as the order of magnitude of a_1, β_1, γ_1; assuming that a quantity multiplied by κ^2c^2-1 is more important than the same quantity multiplied by $(r/c)^2$ for the values of r considered, the equation to determine γ_0' where $\gamma_0 = A/r+\gamma_0'$ is

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(\gamma_0' r)+\left(\kappa^2-\frac{1}{c^2}\right)\frac{A}{r} = 0,$$

whence, writing

$$\kappa^2-\frac{1}{c^2} = \frac{\lambda^2}{c^2}, \qquad \gamma_0' = Dr-\frac{\lambda^2 A}{2c^2}\, r\log r,$$

and, determining D from the boundary condition,

$$\gamma_0 = \frac{A}{r}+\frac{A\lambda^2 r}{2c^2}\left[\tfrac{1}{2}-\log\frac{r}{r_0}\right].$$

The components of the magnetic force in the immediate neighbourhood of the wire are therefore approximately given by the real parts of

$$a = \frac{A}{2c}\log\frac{r}{r_0}\cos\theta\sin\phi,$$

$$\beta = \frac{Ar}{c^2}\left(1-\log\frac{r}{r_0}\right)\sin\theta\sin\phi,$$

$$\gamma = \frac{A}{r}\cos\theta+\frac{A\lambda^2 R}{2c^2}\left[\tfrac{1}{2}-\log\frac{r}{r_0}\right]\cos\theta+\frac{A}{2c}\left[\log\frac{r}{r_0}-1\right]\cos\theta\cos\phi,$$

and the components of the electric force by the real parts of

$$X = \frac{\iota VA}{\kappa cr}\sin\theta, \quad Y = -\frac{\iota V\lambda^2 A}{\kappa c^2}\log\frac{r}{r_0}\cos\theta, \quad Z = \frac{\iota VA}{2\kappa c^2}\log\frac{r}{r_0}\sin\theta\sin\phi,$$

all but the most important terms in these latter expressions being neglected. The corresponding expressions for the oscillations of shorter wave-lengths can be obtained in the same manner. It follows that for a thin wire whose central axis forms a circle the electric and magnetic forces in the immediate neighbourhood of the wire are not the same as those due to a circuit of Hertzian oscillators however thin the wire may be, but the component electric force normal to the wire at its surface is very approximately the same as that due to a circuit of Hertzian oscillators arranged so that the line integral of the magnetic force looping the circuit close to it is the real part of $2\pi A \cos\theta$. It can be inferred that whatever the form of the circuit formed by the central axis of a closed thin wire may be, provided it is of continuous curvature, the electric force normal to the wire at its surface is for the oscillations of greatest wave-length very approximately the same as that due to a circuit of Hertzian oscillators arranged so that the line integral of the magnetic force looping the circuit close to it is the real part of $2\pi A \cos 2\pi s/s_0$, where s_0 is the length of the circuit.

It remains to determine the electric and magnetic forces at a distance from the wire. Let O be a point x, y, z at a distance from the wire; then the component parallel to the axis of x of the electric force at O is given by

$$X = \frac{\iota V}{4\pi\kappa} \iint M\left[l_1 \frac{\partial^2}{\partial x^2} + m_1 \frac{\partial^2}{\partial x\, \partial y} + n_1 \frac{\partial^2}{\partial x\, \partial z} + \kappa^2 l_1\right] \frac{e^{-\iota\kappa R}}{R}\, dS,$$

where $l_1 M$, $m_1 M$, $n_1 M$ are the components of the electric flow on the surface of the wire at the point $Q\,(x', y', z')$ on it and

$$R^2 = (x-x')^2+(y-y')^2+(z-z')^2.$$

If P is the point x_1, y_1, z_1 on the central axis of the wire, PQ being perpendicular to the central axis, l_2, m_2, n_2 the direction cosines of the tangent to the central axis at P, and l_3, m_3, n_3 the direction cosines of the tangent to the wire at Q perpendicular to PQ, then, in the previous notation, $l_1 M = l_2\gamma + l_3\beta, \ldots$, and $R^2 = R_1^2 + r_0^2 + 2R_1 r_0 \sin\chi \cos(\phi-\phi')$, where R_1 is the distance OP, r_0 is the radius of the wire, χ is the angle OP makes with the tangent to the central axis at P, and $\phi-\phi'$ is the angle between the two planes drawn through the tangent at P and the points O and Q. The integrals to be evaluated are all of the fo

$$\iint (l_2\gamma + l_3\beta)\, \frac{e^{-\iota\kappa R}}{R}\, dS,$$

that is

$$\iint (l_2\gamma + l_3\beta)\, R_1^{-\frac{1}{2}} r_0^{-\frac{1}{2}} e^{\frac{1}{2}(\pi\iota)} \sum_0^\infty (2n+1)\, e^{\frac{1}{2}(n\pi\iota)} K_{n+\frac{1}{2}}(\iota\kappa R_1)\, J_{n+\frac{1}{2}}(\kappa r_0)$$
$$\times P_n\{\sin\chi \cos(\phi-\phi')\}\, r_0\, d\phi\, ds,$$

R_1 being greater than r_0. Now the order of the ratio of $K_{n+\frac{1}{2}}(\iota\kappa R_1) J_{n+\frac{1}{2}}(\kappa r_0)$ to $K_{\frac{1}{2}}(\iota\kappa R_1) J_{\frac{1}{2}}(\kappa r_0)$ is that of $(\kappa r_0)^n$ or $(r_0/R_1)^n$, whichever is the greater of these two quantities; and, if the shortest distance of the point O from the central axis of the wire is greater than $\lambda/2\pi$, where λ is the wave-length of the oscillations, the order is that of $(\kappa r_0)^n$. When the central axis is a circle of radius c, β is at most of the order r_0/c; hence, if r_0 is so small that $(\kappa r_0)^2$ can be neglected, the value of the above integral is

$$2\pi \int l_2 \gamma_0 \frac{e^{-\iota\kappa R_1}}{R_1}\, dS,$$

and the component parallel to the axis of x is the real part of

$$-\frac{\iota V}{2\kappa} \int \gamma_0 \left(l_2 \frac{\partial^2}{\partial x^2} + m_2 \frac{\partial^2}{\partial x\, \partial y} + n_2 \frac{\partial}{\partial x\, \partial z} + \kappa^2 l_2 \right) \frac{e^{\iota\kappa R_1}}{R_1}\, ds.$$

Therefore at a point whose shortest distance from the central axis of the wire is of the same order of magnitude as $\lambda/2\pi$ the electric and magnetic forces are sensibly the same as those due to a circuit of Hertzian oscillators arranged along the central axis of the wire, provided the radius r_0 of the wire is so small that $(\kappa r_0)^2$ can be neglected. It will be observed that the arrangement of Hertzian oscillators is the same as that which produces sensibly the same electric force normal to the wire at its surface. As before, it can be inferred that this result holds for a thin wire, whatever the form of the closed circuit made by it may be, provided the curvature of the circuit is continuous.

If a tubular surface be described by a sphere of radius R_1 of the same order as $\lambda/2\pi$, the centre of the sphere moving along the central axis of the wire, the amount of energy that crosses this surface in a half period is less than the amount of energy E radiated out in a half period from the circuit of Hertzian oscillators which arranged along the central axis produce sensibly the same electric and magnetic forces at points outside this surface, the amplitudes being supposed kept constant during this half period. Now E is a finite quantity depending on the line integral of the magnetic force at the surface of the wire looping the circuit, and, if this be kept constant while r_0 is diminished, E will remain sensibly constant, but the amount of energy inside the tubular surfaces increases indefinitely as r_0 is diminished; therefore the rate of decay of the oscillations can be made less than any assigned small quantity, however small, by sufficiently diminishing the radius of the wire, but it cannot be made to vanish; for, as has been seen, the limit of a conducting wire when the radius is diminished without limit has no existence.

It is perhaps unnecessary to add that this result ceases to hold when the circuit is not closed; for then the electric force in the immediate neighbourhood of a free end is indefinitely great compared with the electric force at the surface of the wire at a distance from a free end. For example, in the case of a resonator, supposed to be a thin wire with its free ends A and B close to each other, the electric forces at A and B at any instant of time are of opposite sign and very great: thus there are a very great number of lines of electric force going from the immediate neighbourhood of A to the immediate neighbourhood of B. The number of lines of force that close up in the neighbourhood of the gap each half period is therefore indefinitely great compared with the number that can close up near any node on the surface of the wire, and therefore the radiation from the gap is indefinitely great compared with that due to the remainder of the surface of the wire, and, although the latter can be neglected, the former cannot.

ON GROUP-VELOCITY

By HORACE LAMB, F.R.S.

[Received and Read, February 11th, 1904.]

BY a "group" is meant, in this connection, a long succession of waves in which the distance between successive crests, and the amplitude, vary very slightly. The well-known formula connecting the group-velocity U with the wave-velocity V, viz.,

$$U = V - \lambda \frac{dV}{d\lambda}, \tag{1}$$

was obtained originally by considering the result of superposing two infinite simple harmonic wave-trains of equal amplitude and slightly different frequency.* The extension to a more general type of group was made by Rayleigh† and Gouy.‡ The argument of these writers admits of being put very concisely. Assuming a disturbance

$$y = \Sigma C \cos(nt - kx + \epsilon), \tag{2}$$

where the summation (which may of course be replaced by an integration) embraces a series of terms in which the values of n, and therefore also of k, vary very slightly, we remark that the phase of the typical term at time $t+\Delta t$ and place $x+\Delta x$ differs from the phase at time t and place x by the amount $n\Delta t - k\Delta x$. Hence if the variations of n and k from term to term be denoted by δn and δk, the change of phase will be sensibly the same for all the terms, provided

$$\delta n \, \Delta t - \delta k \, \Delta x = 0. \tag{3}$$

The group as a whole therefore travels with the velocity

$$U = \frac{\Delta x}{\Delta t} = \frac{dn}{dk}. \tag{4}$$

Since $n = kV$, $k = 2\pi/\lambda$, this agrees with (1).

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* Rayleigh, *Proc. London Math. Soc.*, Vol. IX., p. 21 (1877); *Theory of Sound*, § 191.

† *Nature*, Vol. XXV., p. 52 (1881); *Scientific Papers*, Vol. I., p. 540.

‡ "Sur la vitesse de la lumière," *Ann. de Chim. et de Phys.*, Vol. XVI. p. 262 (1889).

Another derivation of (1) can be given which is, perhaps, more intuitive. In a medium such as we are considering, where the wave-velocity varies with the frequency, a limited initial disturbance gives rise in general to a wave-system in which the different wave-lengths, travelling with different velocities, are gradually sorted out.* If we regard the wave-length λ as a function of x and t, we have

$$\frac{\partial\lambda}{\partial t}+U\frac{\partial\lambda}{\partial x}=0, \tag{5}$$

since λ does not vary in the neighbourhood of a geometrical point travelling with velocity U; this is, in fact, the definition of U. Again, if we imagine another geometrical point to travel with the *waves*, we have

$$\frac{\partial\lambda}{\partial t}+V\frac{\partial\lambda}{\partial x}=\lambda\frac{\partial V}{\partial x}=\lambda\frac{dV}{d\lambda}\frac{\partial\lambda}{\partial x}, \tag{6}$$

the second member expressing the rate at which two consecutive wave-crests are separating from one another. Combining (5) and (6), we are led, again, to the formula (1).

It has been pointed out by the writer† that this formula admits of a simple geometrical representation: viz., if a curve be constructed with λ as abscissa and V as ordinate, the value of U for any assigned value of λ is given by the intercept which the corresponding tangent to the curve makes on the axis of ordinates.

A question of some interest arises, as to whether it is possible for the group-velocity, in any real case, to be negative.‡ This was proposed to the writer by Prof. Schuster, who pointed out that the optical formulæ relating to anomalous dispersion indicate a negative group-velocity for certain portions of the spectrum lying within the regions of special absorption. This appears readily from the curves§ by which the refractive index (μ) is exhibited as a function of the wave-length *in vacuo* (λ_0). The formula

$$\frac{V_0}{U}=\mu-\lambda_0\frac{d\mu}{d\lambda_0}, \tag{7}$$ ‖

which is easily derived from (1), shows that the group-velocity is now *in-*

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* As, for example, in the waves on deep water due to a local disturbance of finite duration (Cauchy, Poisson).

† See *Manchester Mem.*, Vol. XLIV., No. 6 (1900), where a number of examples are given.

‡ That is, to have the opposite sign to the wave-velocity.

§ See, for example, Drude, *Lehrbuch d. Optik*, p. 362.

‖ Gouy, *loc. cit.*

versely proportional to the intercept made by the tangent on the axis of ordinates, and it is evident on inspection that for certain ranges of λ_0 this intercept is, in the case referred to, negative. It is to be observed, however, that the notion of group-velocity, as hitherto developed, ceases to be applicable when the absorption is so intense that the distance within which the amplitude falls to (say) one half of its original value becomes comparable with the wave-length.

It is easy, however, to devise mechanical arrangements, free from dissipation, in which the group-velocity shall be negative. We may begin, for instance, with a long straight wire subject to a longitudinal thrust (P), instead of a tension. This is of course unstable, but we may obtain stability for waves exceeding a certain length, if we imagine, in addition, that every point of the wire is attracted towards its equilibrium position by a force varying as the distance. The equation of transverse vibration is now of the form

$$\frac{\partial^2 y}{\partial t^2}+p^2y+c^2\frac{\partial^2 y}{\partial x^2}=0, \tag{8}$$

where $c^2=P/\rho$, ρ being the density. Assuming a solution

$$y=C\cos(nt-kx), \tag{9}$$

we find
$$k^2=(p^2-n^2)/c^2, \qquad UV=-c^2, \tag{10}^*$$

provided $k<p/c$.

We may secure stability for all values of λ by further endowing the wire with a sufficient degree of stiffness. The equation of motion then assumes the form[†]

$$\frac{\partial^2 y}{\partial t^2}+p^2y+c^2\frac{\partial^2 y}{\partial x^2}+a^2\kappa^2\frac{\partial^4 y}{\partial x^4}=0, \tag{11}$$

which gives
$$n^2=p^2-k^2c^2+a^2\kappa^2k^4, \quad UV=-c^2+2\kappa^2a^2k^2. \tag{12}$$

Hence n will be real for all values of k provided $c^2<2\kappa ap$, whilst U will have the opposite sign to V for waves so long that $k^2<\frac{1}{2}c^2/\kappa^2a^2$. It may be noticed that vibrations below a certain frequency (determined by $n^2=p^2-\frac{1}{4}c^4/\kappa^2a^2$) cannot be propagated. For sufficiently long waves

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[*] The curve exhibiting the relation between V and λ has the form of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

[†] See Rayleigh, *Sound*, § 188.

the flexural term in (11) is insensible, and we fall back on the former case.

Another example, in which there may be negative group-velocity for all wave-lengths, together with thorough stability, is furnished by an arrangement employed by Lord Rayleigh* in a different connection. This consists of a tense wire attracted to its equilibrium position by a force varying as the distance, and endowed with rotatory inertia, but of negligible stiffness.† The differential equation is now of the form

$$\frac{\partial^2 y}{\partial t^2}+p^2y-a^2\frac{\partial^2 y}{\partial x^2}-\kappa^2\frac{\partial^4 y}{\partial x^2\partial t^2}=0, \tag{13}$$

where $a^2 = T/\rho$, T being the tension, and κ is a radius of gyration. This gives

$$n^2=\frac{p^2+k^2a^2}{1+\kappa^2k^2},\qquad k^2=\frac{p^2-n^2}{\kappa^2n^2-a^2}, \tag{14}$$

and thence

$$UV=-\frac{\kappa^2p^2-a^2}{(1+\kappa^2k^2)^2}=-\frac{(\kappa^2n^2-a^2)^2}{\kappa^2p^2-a^2}. \tag{15}$$

The condition for negative group-velocity is therefore $\kappa^2p^2 > a^2$. It will be observed that vibrations for which $a/\kappa < n < p$ can alone be propagated.

It is known that the velocity U, determined by (1), also expresses the rate at which energy is propagated.‡ The proof applies without qualification to such cases as we have been considering; and we infer, both from this and from the former line of argument, that when a local disturbance is produced in a medium having the property in question groups of waves will be propagated outwards, whilst the individual waves composing a group will be found to be moving inwards. It may be worth while to illustrate this by means of one of our mechanical arrangements; for simplicity we may take the one first considered. It will help to make the matter clearer if we introduce a slight degree of viscosity, so that the equation (8) is replaced by

$$\frac{\partial^2 y}{\partial t^2}+p^2y+\nu\frac{\partial y}{\partial t}+c^2\frac{\partial^2 y}{\partial x^2}=0. \tag{16}$$

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* *Phil. Mag.*, Vol. XLVI. (1898); *Scientific Papers*, Vol. IV., p. 369.

† This might be realized by a cylindrical wire with a series of close equidistant peripheral cuts extending nearly to the axis.

‡ Cf. Osborne Reynolds, *Nature*, Vol. XVI., p. 343 (1877), *Scientific Papers*, Vol. I., p. 198, for the case of water waves. The proof for the general case was given by Rayleigh in the paper cited first on p. 473 *ante*.

If we suppose that a forced vibration

$$y = C \cos nt \tag{17}$$

is maintained at the origin, the disturbance established to the right of O will be found by assuming

$$y = C e^{int+ax}, \tag{18}$$

provided that in the end we retain only the real part. Substituting in (16), we have

$$a^2 = -(p^2-n^2+i\nu)/c^2, \tag{19}$$

and therefore, on account of the assumed smallness of ν,

$$a = \pm\, ik \mp \nu/2kc^2, \tag{20}$$

where k is the positive quantity determined by (10). In order to secure finiteness for $x = \infty$, the *upper* signs must be taken, and the realized solution is

$$y = C e^{-\nu x/2kc^2} \cos(nt+kx), \tag{21}$$

representing a train in which the individual waves are travelling *inwards* towards the source of disturbance.

If we regard ν as infinitesimal, the mean energy per unit length will be double the mean kinetic energy,* and is therefore equal to $\frac{1}{2}\rho n^2 C^2$. On the other hand, the rate at which work is supplied at the origin is

$$P \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} = PknC^2 \sin^2(nt+kx), \tag{22}$$

the mean value of which is $\frac{1}{2}PknC^2$. This is equal to the energy in a length U, if

$$U = \frac{k}{n} \frac{P}{\rho} = \frac{k}{n} c^2. \tag{23}$$

This agrees with (10), since V is now equal to $-n/k$.

We may also illustrate the problem of reflection and transmission at a point where the properties of the medium are discontinuous. Referring to Lord Rayleigh's modification of the tense wire, we may suppose that to the right of the origin the equation (13) holds, whilst to the left of O

$$\frac{\partial^2 y}{\partial t^2} - a_0^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{24}$$

where $a_0^2 = T/\rho_0$. Taking the amplitude of the waves incident from the left as unity, we assume

---

* This familiar relation ceases to hold when ν has to be taken into account.

$$y = e^{i(nt-k_0x)} + A\,e^{i(nt+k_0x)} \qquad [x<0], \tag{25}$$

$$y = B\,e^{i(nt+kx)} \qquad [x>0], \tag{26}$$†

so that A denotes the amplitude of the reflected, B of the transmitted, waves. The conditions to be satisfied at the junction $x=0$ are to be found from the consideration that to the right of O we have a shearing force

$$F = \kappa^2\rho\,\frac{\partial^3 y}{\partial x\,\partial t^2}, \tag{27}$$

whilst to the left of O we have $F=0$. Since the values of y and $F+T\,\partial y/\partial x$ must evidently be continuous, we have

$$1+A = B, \tag{28}$$

$$k_0\rho_0 a_0^2(1-A) = k\rho(\kappa^2n^2-a^2)B, \tag{29}$$

whence we find

$$B = 2\bigg/\left\{1+\frac{k}{k_0}\left(\frac{\kappa^2n^2}{a^2}-1\right)\right\}. \tag{30}$$

This may be expressed in terms of n and constant quantities by means of (14) and the relation $k_0^2 = n^2/a_0^2$; thus

$$B = 2\bigg/\left\{1+\frac{a_0}{a}\sqrt{\left(\frac{p^2}{n^2}-1\right)}\sqrt{\left(\frac{\kappa^2n^2}{a^2}-1\right)}\right\}. \tag{31}$$

If we take the real parts of the expressions in (25) and (26), then, corresponding to an incident vibration

$$y = \cos(nt-k_0x), \tag{32}$$

we have a transmitted vibration

$$y = B\cos(nt+kx). \tag{33}$$

The potential energy per unit length at any point to the right of O is

$$\tfrac{1}{2}\left\{\rho p^2y^2+T\left(\frac{\partial y}{\partial x}\right)^2\right\},$$

the mean value of which is $\tfrac{1}{4}\rho(p^2+k^2a^2)B^2$. The mean total energy per unit length will be twice this. Hence, if I denote the ratio of the transmitted to the incident energy, per unit time, we shall have

---

† The necessity for the adoption of $nt+kx$ instead of the usual $nt-kx$ in the exponential might be made clearer by the introduction of a small frictional term into the equation (13). It would then appear that an assumption of the form $y = Be^{i(nt-kx)}$ would involve the existence of a source at $x=+\infty$.

$$I = \frac{\frac{1}{2}\rho\,(p^2+k^2a^2)\,U}{\frac{1}{2}\rho_0 n^2 a_0}\,B^2. \tag{34}$$

By means of (14) and (15) this can be put in a variety of forms: thus for example

$$I = \frac{k}{k_0}\left(\frac{\kappa^2 n^2}{a^2}-1\right)B^2. \tag{35}$$

Comparing this with (30), we find

$$I = (2-B)\,B. \tag{36}$$

This is in accordance with the principle of energy, for, if I' denote the ratio of the energy of the reflected to that of the incident waves, it makes

$$I+I' = (2-B)B+A^2 = 1, \tag{37}$$

by (28).

It is hardly to be expected that the notion of a negative group-velocity will have any very important physical application; but the preceding discussion may serve to emphasize the point that ideas of wave-propagation acquired in the study of air-waves (for example) need to be used with some caution when we are dealing with a dispersive medium.

ON THE IRREDUCIBILITY OF PERPETUANT TYPES

By P. W. WOOD.

[Received and Read February 11th, 1904.]

IT has been proved that*

Any perpetuant linear in each of the $(\delta+1)$ binary quantics $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, ..., $a_{\delta+1_x}^{n_{\delta+1}}$, where $n_1, n_2, \ldots, n_{\delta+1}$ are all infinite, is a linear function of products of perpetuants of lower total degrees and of symbolical products like

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \ldots (a_1 a_{r+1})^{\lambda_r} \ldots (a_1 a_{\delta+1})^{\lambda_\delta},$$

where $\lambda_1 \geqslant 2^{\delta-1}$, $\lambda_2 \geqslant 2^{\delta-2}$, ..., $\lambda_r \geqslant 2^{\delta-r}$, ..., $\lambda_\delta \geqslant 1$

and the order of the letters $a_1, a_2, \ldots, a_{\delta+1}$ is fixed beforehand.

It might happen that some of the type forms

$$(a_1 a_2)^{\lambda_1} \ldots (a_1 a_{\delta+1})^{\lambda_\delta} \quad (\lambda_1 \geqslant 2^{\delta-1}, \text{\&c.}),$$

thus retained are themselves reducible: some evidence in favour of their irreducibility is afforded by the results of Stroh† and MacMahon.‡ The theorem will be proved to be exact, if we show that there can be no linear relation connecting type forms and product forms; and this method is adopted in the present paper.

1. Consider the expression

$$M = \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_r x_r + \ldots + \mu_\delta x_\delta;$$

and suppose M is annihilated by the linear differential operator

$$O \equiv \frac{\partial}{\partial x_{r^{(\epsilon)}}} - \frac{\partial}{\partial x_{r^{(\epsilon-1)}}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(1)}}}, \tag{1}$$

so that the coefficients μ satisfy the relation

$$\mu_{r^{(\epsilon)}} - \mu_{r^{(\epsilon-1)}} + \ldots + (-)^{\epsilon-1} \mu_{r^{(1)}} = 0,$$

where $r^{(\epsilon)}, r^{(\epsilon-1)}, \ldots, r^{(2)}, r^{(1)}$ are any ϵ of the suffixes $\delta, \delta-1, \ldots, 2, 1,$ such that $r^{(\epsilon)} > r^{(\epsilon-1)} > \ldots > r^{(2)} > r^{(1)}$. Then M is expressible in the form

---

* J. H. Grace, *Proc. London Math. Soc.*, Vol. XXXV.

† E. Stroh, *Math. Ann.*, Bd. XXXVI.

‡ Major P. A. MacMahon, *Proc. London Math. Soc.*, Vol. XXVI.; *Camb. Phil. Soc. Trans.*, Vol. XIX.

$$M \equiv \nu_1 y_1 + \nu_2 y_2 + \ldots + \nu_r y_r + \ldots + \nu_{\delta-1} y_{\delta-1},$$

where the ν's are linear functions of the μ's, and each y is either one of the x's or is a sum of consecutive x's, *e.g.*,

$$y \equiv x_s + x_{s+1} + \ldots + x_{t-1} + x_t.$$

The values of the ν's and the y's may, in fact, be written down by inspection:—

$$\left.\begin{cases} y_{r^{(\theta)}} = x_{r^{(\theta)}} + x_{r^{(\theta)}+1} + \ldots + x_{r^{(\theta+1)}-1} + x_{r^{(\theta+1)}} & (\theta = 1, 2, 3, \ldots, \epsilon-1) \\ \left.\begin{aligned} y_\kappa &= x_\kappa, \quad \text{if} \quad \kappa < r^{(\epsilon)} \\ y_{\kappa-1} &= x_\kappa, \quad \text{if} \quad \kappa > r^{(\epsilon)} \end{aligned}\right\} & (\kappa \neq r^{(\theta)};\quad \theta = 1, 2, 3, \ldots, \epsilon) \end{cases}\right\}, \tag{2}$$

$$\left\{\begin{aligned} \nu_\kappa &= \mu_\kappa, \quad \text{if} \quad 1 \leqslant \kappa < r^{(1)} \\ \nu_{\kappa-1} &= \mu_\kappa, \quad \text{if} \quad r^{(\epsilon)} < \kappa \leqslant \delta \\ &\left.\begin{aligned} \nu_{r^{(\theta)}} &= \mu_{r^{(\theta)}} - \mu_{r^{(\theta-1)}} + \ldots + (-)^{\theta-1}\mu_{r^{(1)}} \\ \nu_{r^{(\theta)}+\xi} &= \mu_{r^{(\theta)}+\xi} - \nu_{r^{(\theta)}}, \qquad \text{if} \quad \xi < r^{(\theta+1)} - r^{(\theta)} \\ &= \mu_{r^{(\theta)}+\xi} - \mu_{r^{(\theta)}} + \mu_{r^{(\theta-1)}} + \ldots + (-)^{\theta}\mu_{r^{(1)}} \end{aligned}\right\} \theta = 1, 2, 3, \ldots, \epsilon-1 \end{aligned}\right\}.$$

It will be seen that $\nu_{r^{(\epsilon)}}$ is zero by virtue of the relation between the coefficients μ. The operator O is completely determined by the suffixes involved, and, since the number of operators each involving s suffixes is $\binom{\delta}{s}$, the total number of such operators is

$$\binom{\delta}{1} + \binom{\delta}{2} + \ldots + \binom{\delta}{s} + \ldots + \binom{\delta}{\delta} = 2^\delta - 1.$$

2. Let $$x_s \equiv \frac{(a_s a_{s+1})}{a_{s_z} a_{s+1_z}} \Pi, \text{ for } s = 1, 2, 3, \ldots, \delta,$$

where $\Pi = a_{1_z}, a_{2_z}, \ldots, a_{\delta+1_z}$; and $a_1, a_2, \ldots, a_s, \ldots, a_{\delta+1}$ are the symbols of binary quantics of infinite order in the variables z_1, z_2; then

$$x_s + x_{s+1} + \ldots + x_{t-1} + x_t \equiv \frac{(a_s a_{t+1})}{a_{s_z} a_{t+1_z}} \Pi.$$

Thus the y forms in (2) involve the symbolical determinants

$$\left.\begin{cases} (a_{r^{(\theta)}} a_{r^{(\theta+1)}+1}) & (\theta = 1, 2, 3, \ldots, \epsilon-1) \\ (a_\kappa a_{\kappa+1}), \text{ if } \kappa \neq r^{(\theta)} & (\theta = 1, 2, 3, \ldots, \epsilon) \end{cases}\right\}$$

and so any expression, such as M, which is annihilated by O, may be divided into two parts involving the suffixes :—

$$\left.\begin{array}{l} 1,\ 2,\ 3 \ldots r^{(1)},\quad r^{(2)}+1 \ldots r^{(3)},\quad r^{(4)}+1 \ldots r^{(5)},\qquad r^{(2\theta)}+1 \ldots r^{(2\theta+1)},\quad \ldots \\ \text{and} \\ r^{(1)}+1 \ldots r^{(2)},\quad r^{(3)}+1 \ldots r^{(4)},\quad r^{(5)}+1 \ldots r^{(6)},\qquad r^{(2\theta+1)}+1 \ldots r^{(2\theta+2)},\quad \ldots \\ \hfill \text{respectively} \end{array}\right\}, \tag{3}$$

where the dotted lines ... are to be replaced by consecutive numbers, so that there is a break in the sequence whenever we arrive at a suffix number $r^{(\theta)}$ involved in the operator O, and no number occurs in both sets.

3. Consider the expression M^{ω}, regarding the coefficients μ as symbolical letters. If M^{ω} is annihilated by O, then, by the theory of partial differential equations, if we make the preceding substitutions and multiply by the proper (infinite) number of factors $a_{1_x}, a_{2_x}, \ldots, a_{\delta+1_x}$ respectively, M^{ω} is the most general linear function of perpetuants of weight ω which are products of perpetuants of quantics of the two sets given by (3).

By taking all the $(2^{\delta}-1)$ operators like O, we get $(2^{\delta}-1)$ corresponding quantics M^{ω}, and so all possible general linear functions of perpetuant products of the different partial degrees. Of course, if we are given the forms involved in each of the two factors of a perpetuant product, we are given the two sets of suffixes (3) and thence the operator O.

The remainder of the proof depends on the result :—

> If any linear partial differential operator in any number of variables annihilates the product of two quantics p_x^m, q_x^n, then it annihilates each of these quantics p_x^m and q_x^n.

For $O\left\{p_x^m q_x^n\right\} = 0$, where O is given by (1), implies that the quantic $p_x^m q_x^n$ is a rational integral homogeneous function of the variables y given by (2); and therefore each of the quantics p_x^m and q_x^n is also a rational integral homogeneous function of the same variables, and so is annihilated by O also: and the same is clearly true of any other linear differential operator.

4. In this way we get $(2^{\delta}-1)$ different quantics

$$Q_1,\ Q_2,\ \ldots,\ Q_{(2^{\delta}-1)};$$

each of which is annihilated by the corresponding operator, and, if the quantics are to represent the most general linear functions of perpetuant products, by no other.

The order of the letters in the type theorem being arbitrary, we shall take as our type form the symbolical product

$$(a_1 a_{\delta+1})^{\lambda_\delta} (a_1 a_\delta)^{\lambda_{\delta-1}} \dots (a_1 a_{r+1})^{\lambda_r} \dots (a_1 a_2)^{\lambda_1},$$

where $\lambda_\delta \geqslant 2^{\delta-1}, \ \lambda_{\delta-1} \geqslant 2^{\delta-2}, \ \dots, \ \lambda_r \geqslant 2^{r-1}, \ \dots, \ \lambda_1 \geqslant 1.$

We have to show the impossibility of any relation such as

$$Q_1 + Q_2 + \dots + Q_{(2^\delta-1)}$$

$$\equiv (x_1+x_2+\dots+x_\delta)^{2^{\delta-1}} \dots (x_1+x_2+\dots+x_r)^{2^{r-1}} \dots x_1 \cdot f(x_1, x_2, \dots, x_\delta), \quad (4)$$

where $f(x_1, x_2, \dots, x_\delta)$ is some quantic in the variables, which is not identically zero.

We arrange the $2^\delta-1$ operators O in sets $S_1, S_2, \dots, S_r, \dots, S_{\delta-1}, S_\delta$, such that, for the values 1, 2, 3, ..., $\delta-1$ of r, S_r contains all operators beginning with $\partial/\partial x_{r+1}$ and having an even number of partial operators in all, and S_δ contains all operators having an odd number of partial operators.

Thus S_1 contains only $\partial/\partial x_2 - \partial/\partial x_1$; S_2 contains only $\partial/\partial x_3 - \partial/\partial x_2$, $\partial/\partial x_3 - \partial/\partial x_1$; and S_δ includes $\partial/\partial x_r$, $\partial/\partial x_3 - \partial/\partial x_2 + \partial/\partial x_1$, ...

It will be seen that the number of operators in the set S_r is 2^{r-1} for the values 1, 2, ..., δ of r; and we may suppose the operators of each set arranged in any sequence among themselves.

5. Operate on both sides of (4) successively with the operators of the sets $S_1, S_2, \dots, S_{\delta-1}$ in this order, and consider the effect on the right-hand side.

(1) After operating with the single operator of the set S_1, the right-hand side is

$$(x_1+x_2+\dots+x_\delta)^{2^{\delta-1}} \dots (x_1+x_2+\dots+x_r)^{2^{r-1}} \dots (x_1+x_2)^2 \, \phi(x_1, x_2, \dots, x_\delta);$$

where $\phi(x_1, x_2, \dots, x_\delta)$ is a quantic, which by § 3 is not identically zero, unless $f(x_1, x_2, \dots, x_\delta) \equiv 0$.

(2) So assume that, after operating with all the operators of the sets $S_1, S_2, \dots, S_{r-1}$ in this order, the right-hand side is

$$(x_1+x_2+\dots+x_\delta)^{2^{\delta-1}} \dots (x_1+x_2+\dots+x_r)^{2^{r-1}} \, \psi(x_1, x_2, \dots, x_\delta), \quad (5)$$

where $\psi(x_1, x_2, \dots, x_\delta)$ is a quantic which is not identically zero, unless $f(x_1, x_2, \dots, x_\delta) \equiv 0$.

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PRINTED BY C. F. HODGSON & SON,
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Issued November 12, 1903.

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PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
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Issued January 8, 1904.

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AND PUBLISHED FOR THE SOCIETY BY
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Issued February 4, 1904.

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Issued March 25, 1904.

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PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
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Issued May 5, 1904.

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Issued June 6, 1904.

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